Based on JM Aguirregabiria's textbook.

1. Verhulst's equation. Examine the stability of the equilibrium points of the equation

$$\dot{x} = \epsilon x - \sigma x^2$$

for all the signs of the parameters ϵ and σ

2. Proof that a solution of the regular system Froga ezazu puntu ez-kritiko batetik abiatzen den

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y)$$

that starts at a non-critical point, cannot reach an equilibrium point at a finite value of the independent variable

3. Discuss the stability at the origin of the system

$$\dot{x} = a_{11}x + a_{12}y, \qquad \dot{y} = a_{21}x + a_{22}y$$

when $a_{11}a_{22} - a_{12}a_{21} = 0$.

4. Examine, for all signs of the parameter ϵ, the stability at the origin of the following system:

$$\dot{x} = \epsilon x + y, \qquad \dot{y} = -x + \epsilon y.$$

5. Discuss the stability of all the critical points of this system

$$\dot{x} = x - x^2 - xy, \qquad \dot{y} = 3y - xy - 2y^2.$$

6. Non-linear saddle point. Proof that this system and its linear approximation have an unstable saddle point at the origin:

$$\dot{x} = x, \qquad \dot{y} = -y + x^2.$$

Find the stable and unstable spaces for the equilibrium point in the linear approximation. Consider now the non-linear system: proof that the line $x = 0$ and the parabola $y = x^2/3$ are invariant sets; conclude that those sets are the stable manifold and the unstable manifold of he origin. PLot the phase space of both systems.

7. Examine the behavior of the origin of the following non-linear systems

(a) $\dot{x} = y + x(x^2 + y^2), \qquad \dot{y} = -x + y(x^2 + y^2),$

(b) $\dot{x} = y - x(x^2 + y^2), \qquad \dot{y} = -x - y(x^2 + y^2).$

8. Examine the stability of the origin of the following system:

$$\dot{x} = 2y - z, \qquad \dot{y} = 3x - 2z, \qquad \dot{z} = 5x - 4y.$$

9. Examine the stability of the null solution:

$$\dot{x} = -y - x^3, \qquad \dot{y} = x - y^3.$$

10. Let us suppose that for $i \neq j$, $a_{ij} = -a_{ji}$, and $a_{ii} < 0$. Proof, using Liapunov's functions $\sum_{i=1}^{n} x_i^2$, that the null solution of

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j$$

is stable.

11. For which value of the parameter α is the fixed point of the system

$$\dot{x} = \alpha x - y, \qquad \dot{y} = \alpha y - z, \qquad \dot{z} = \alpha z - x$$

stable?

12. Definite forms Proof that the function $U(x, y) = ax^2 + bxy + cy^2$ is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$. What is the necessary and sufficient for it to be negative definite?

13. Study the stability of the point $(0, 0)$ in the following system:

$$\dot{x} = -\frac{1}{2}x^3 + 2xy^2, \qquad \dot{y} = -y^3.$$

14. Use the appropriate Liapunov's function to prove the in the following system

$$\dot{x} = y - xf(x, y), \qquad \dot{y} = -x - yf(x, y)$$

the origin is asymptotically stable (unstable) if in region around the origin we have $f(x, y) > 0$ ($f(x, y) < 0$) –even if $f(0, 0)$ is null. What is the geometry of the orbits around the critical point?

15. Examine the information that the following Liapunov's functions

$$U_1 = \frac{1}{2}y^2 + (1 - \cos x), \qquad U_2 = \frac{1}{2}(x + y)^2 + x^2 + \frac{1}{2}y^2$$

give for the origin in the following system

$$\dot{x} = y, \qquad \dot{y} = -y - \sin x\,.$$

How could the stability be studied in an easier way?

16. Proof that the systems

$$\dot{x} = -y + \frac{x}{r}f(r), \qquad \dot{y} = x + \frac{y}{r}f(r), \qquad (r^2 \equiv x^2 + y^2)$$

has a periodic solution for every root of the function $f(r)$. Which is the direction of the closed trajectories? How would we study their stability? Find all the periodic solutions of the system and their stability for the special case $f(r) = r(r - 2)^2(r^2 - 4r + 3)$.

17. Find the critical points and the equation of the trajectories of the system

$$\dot{x} = -x, \qquad \dot{y} = 2x^2y^2$$

Plot the phase space.

18. Consider the equation $\ddot{x} - x + x^3 = 0$. Find the equation of the trajectories and plot the phase space. What will happen if we add a term $\gamma\dot{x}$ ($\gamma > 0$)?

19. Examine the phase space of the equation $\ddot{x} = (\cos x - 1) \sin x$.

20. Find the critical value $\lambda = \lambda_0$ of the bifurcation of the equation $\ddot{x} = x^2 - \lambda x + 9$. Plot the phase space for the cases $\lambda = 10$ and $\lambda = \lambda_0$.

21. Plot the phase space of the following system:

$$\dot{x} = x^2 - y^3, \qquad \dot{y} = 2x \left(x^2 - y\right).$$

22. Plot the phase space of the following dynamic system:

$$\dot{x} = y - y^3, \qquad \dot{y} = -x - y^2.$$

23. Find the limit cycle for the following system

$$\begin{aligned} \dot{x} &= y + \frac{x}{\sqrt{x^2 + y^2}} \left[1 - (x^2 + y^2)\right], \\ \dot{y} &= -x + \frac{y}{\sqrt{x^2 + y^2}} \left[1 - (x^2 + y^2)\right] \end{aligned}$$

and discuss its stability.

24. Find that the following equation has a limit cycle:

$$\ddot{x} + \left(x^4 - 1\right) \dot{x} + x^3 = 0.$$

25. Find the maximum Liapunov's power for the system of problem **23**.