Based on JM Aguirregabiria's textbook.

1. Legendre's equation. Writing Laplace's equation in cylindrical coordinates, the following equation is found

$$\left(1-x^2\right)y''-2xy'+\nu(\nu+1)y=0.$$

(a) Find the series solution around $x=0$ in the domain $(-1,1)$.

(b) Which of those solutions are polynomial?

(c) Find some **Legendre polynomials**, that is, polynomial solutions satisfying $y(1)=1$.

2. Chebyshev equation Find the solutions of the following equation in the domain $(-1,1)$

$$\left(1-x^2\right)y''-xy'+\nu^2y=0$$

which are polynomials and more precisely the ones satisfying the condition $y(1)=1$ (these are Chebyshev polynomials).

3. calculate the solution to the following order up to sixth order

$$xy''+y'+2y=0, \qquad y(1)=2, \quad y'(1)=4.$$

4. Bessel functions Prove the following properties of the Bessel function:

(a) $\dfrac{d}{dx}\left[x^\nu J_\nu(kx)\right]=kx^\nu J_{\nu-1}(kx).$

(b) $\dfrac{d}{dx}\left[x^{-\nu} J_\nu(kx)\right]=-kx^{-\nu} J_{\nu+1}(kx).$

(c) $\dfrac{d}{dx}\left[J_\nu(kx)\right]=kJ_{\nu-1}(kx)-\dfrac{\nu}{x}J_\nu(kx).$

(d) $\dfrac{d}{dx}\left[J_\nu(kx)\right]=-kJ_{\nu+1}(kx)+\dfrac{\nu}{x}J_\nu(kx).$

(e) $\dfrac{d}{dx}\left[J_\nu(kx)\right]=\dfrac{k}{2}\left[J_{\nu-1}(kx)-J_{\nu+1}(kx)\right].$

(f) $J_\nu(kx)=\dfrac{kx}{2\nu}\left[J_{\nu-1}(kx)+J_{\nu+1}(kx)\right].$

Do the Bessel functions of the second type Y_ν satisfy similar properties?

5. Bessel equation of order $1/2$. Make the change of variables : $y(x)=x^{-1/2}u(x)$ in Bessel equation. Use your result to find the solution of Bessel equation of order $1/2$.

6. $x^2y''+x(x+1)y'-y=0.$

7. $x(x-1)y''+(2x-1)y'-2y=0.$

8. $\left(x^3-x^2\right)y''+\left(2x^2-3x\right)y'-y=0.$

9. $xy'' - y = 0$.

10. $x^4 y'' + xy' + 2y = 0$.

11. $2x^2 y'' + x(2x+1)y' - y = 0$.

12. $y'' + \dfrac{4}{x}y' + \dfrac{2}{x^2}y = 0$.

13. $xy'' + (1-2x)y' + (x-1)y = 0$.

14. Laguerre equation. Find the solution of the equation

$$xy'' + (\alpha + 1 - x)y' + \nu y = 0, \qquad (\alpha \geq 0).$$

in the half-line $x > 0$. Discuss the polynomial solutions, which are proportional to the generalized Laguerre polynomials.

15. Bessel equation. Prove that the equation

$$x^2 y'' + (2c+1)xy' + \left[a^2 b^2 x^{2b} + \left(c^2 - \nu^2 b^2\right)\right] y = 0$$

can be written as a Bessel equation using a change of variables $(x, y) \to (t, u)$ given by $t \equiv ax^b$ and $u \equiv x^c y$.

16. Use the method of series to solve the following equation

$$(x-1)y'' - xy' + y = 0.$$

17. Find the first terms of the solution of this equation

$$y'' + (\cos x)\, y = 0.$$

18. $\left(x^3 - x\right) y''' + \left(9x^2 - 3\right) y'' + 18xy' + 6y = 0$.

19. It may be useful to use power series and Frobenius's series in order to obtain a particular solution of the complete equation. For example, let us consider

$$x^2 y'' - x(x+1)y' + (x+1)y = x^2.$$

Solve the homogeneous equation and use an appropriate series to find a particular solution of the complete equation. Compare the solution with the one obtained by the parameter variation method.

20. Gauss's hypergeometric equation. Find the solutions of

$$x(1-x)y'' + [\gamma - (1 + \alpha + \beta)x]\, y' - \alpha\beta y = 0$$

around $x = 0$, when $\gamma \neq 1, 0, -1, -2, \ldots$. In order to write the solution use **Gauss's hypergeometric function** $F(\alpha, \beta; \gamma; x)$, that is, the solution that satisfies the initial condition given by $F(\alpha, \beta; \gamma; 0) = 1$.
Suggestion: Use the change of variables $y = x^{1-\gamma}\, z$ to get the second solution.

21. Prove $F(\alpha, \beta; \gamma; x) = F(\beta, \alpha; \gamma; x)$ and find $F(1, \beta; \beta; x)$ and $F(\alpha, \beta; \beta; x)$.

22. Confluent hypergeometric equation. From all the solutions of null index of the following equation

$$xy'' + (\beta - x)y' - \alpha y = 0$$

find the ones satisfying $y(0) = 1$ for $\beta \neq 1, 0, -1, -2, \ldots$. This solution is know as the **Confluent hypergeometric function** or **Kummer's function**, and it is represented as $M(\alpha, \beta, x)$ or ${}_1F_1(\alpha; \beta; x)$. Discuss the cases in which Kummer's functions become polynomials, and discuss also some of the cases where we recover fundamental functions. Calculate the second linear independent solution using the change of variables $y = x^{1-\beta} z$. Explain the relation between this equation and Gauss's hypergeometric equation and how the relation can be used to obtain directly the results of the previous point. (*Suggestion*: Use the change of variables $t = \beta x$ in Gauss's equation.) Why do we get constraints β?

23. Find the general solution of

$$xy'' + xy' + y = 0.$$

24. Find all the solutions of

$$xy'' - y' + y = 0.$$

25. Write the solutions to the following equation using fundamental functions:

$$x(x-1)y'' + 3y' - 2y = 0.$$

26. Find the general solution of

$$(x^2 - x)y'' + (1 - 2x^2)y' + (4x - 2)y = 0.$$