Based on JM Aguirregabiria's textbook.

1. $\ddot{x} = y, \quad \ddot{y} = x.$

2. $\ddot{x} - 2\dot{y} + x = 0, \quad \ddot{y} - 2\dot{x} + y = e^{2t}.$

3. Calculate two independent first-integrals for the following system

$$a\dot{x} = (b-c)yz, \quad b\dot{y} = (c-a)zx, \quad c\dot{z} = (a-b)xy.$$

and the first order equation resulting using the first-integrals. (a, b and c are constants.) What is the meaning of this system in mechanics? What is the meaning of the first integrals?

4. $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}.$

5. Are the following vectors $\begin{pmatrix} 1 \\ t \end{pmatrix}$ and $\begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$ independent? Calculate their Wronskian and explain the result.

6. Liouville's formula. Remembering that the trace of a matrix is the sum of the elements of its diagonal ($\mathrm{tr}\mathbf{A} = \sum_{k=1}^{n} a_{kk}$), prove that the Wronskian of the fundamental system of a linear homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\cdot\mathbf{x}$ satisfies $W(t) = W(t_0)\exp\int_{t_0}^{t} \mathrm{tr}\mathbf{A}\, dt$

Suggestion: Start with the $n = 2$ case.

7. Systems and equations Study the first order system corresponding to a second order linear equation. Find the relation between the equation and the fundamental system and compare their Wronskians.

8. Find the linear system that admits the following fundamental matrix $\begin{pmatrix} t & t^2 \\ 0 & 0 \end{pmatrix}$.

9. Show that any matrix $\mathbf{A}$, satisfies $\det e^{\mathbf{A}} = e^{\mathrm{tr}\mathbf{A}}$.
Suggestion: Use Liouvilles's formula in $\det e^{\mathbf{A}t}$ and choose $t = 1$.

10. Asymptotic stability of the origin Let us suppose that the (constant) $n \times n$ matrices corresponding to a linear homogeneous system have n real distinct proper values. What conditions do the proper values have to satisfy so that all solutions $\mathbf{x}(t)$ obey $\lim_{t\to\infty} x_i(t) = 0$? What happens if all proper values are not real? And if all of them are not different?

11. Classify the phase-space orbits of the following linear homogeneous systems with respect to the constants a and b:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

12. Consider the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Calculate $e^{\mathbf{A}}$, $e^{\mathbf{B}}$, $e^{\mathbf{A}} \cdot e^{\mathbf{B}}$, $e^{\mathbf{B}} \cdot e^{\mathbf{A}}$ and $e^{\mathbf{A}+\mathbf{B}}$. Explain your results.

13. Baker, Campbell and Hausdorff formula.Expand the product $\mathbf{F}(t) = e^{\mathbf{A}t} \cdot \mathbf{B} \cdot e^{-\mathbf{A}t}$ in a Taylor series around $t = 0$ in order to prove

$$e^{\mathbf{A}} \cdot \mathbf{B} = \left(\mathbf{B} + [\mathbf{A}, \mathbf{B}] + \frac{1}{2}[\mathbf{A}, [\mathbf{A}, \mathbf{B}]] + \cdots + \frac{1}{n!}\underbrace{[\mathbf{A}, [\mathbf{A}, \ldots, [\mathbf{A}}_{n \text{ aldiz}}, \mathbf{B}\underbrace{] \cdots]]}_{n \text{ aldiz}} + \cdots\right) \cdot e^{\mathbf{A}}.$$

As usual, the commutator of the matrices $\mathbf{A}$ and $\mathbf{B}$ is given by $[\mathbf{A}, \mathbf{B}] \equiv \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A}$

14. Glauber's formula. Suppose that both matrices $\mathbf{A}$ and $\mathbf{B}$ commute with their commutator $[\mathbf{A}, \mathbf{B}]$. Find the differential equation that $\mathbf{F}(t) = e^{\mathbf{A}t} \cdot e^{\mathbf{B}t}$ satisfies in order to prove

$$e^{\mathbf{A}} \cdot e^{\mathbf{B}} = e^{\mathbf{A}+\mathbf{B}} \cdot e^{\frac{1}{2}[\mathbf{A},\mathbf{B}]}.$$

15. RLC circuit. Find the differential equation that describes the circuit in the figure. Use as unknowns the intensity through the autoinduction and the change-in-voltage in the condensators. Calculate the fundamental matrix and the solution corresponding to null initial conditions for the following potential:

$$V(t) = \begin{cases} 1, & \text{baldin } 0 \le t \le 1, \\ 0, & \text{baldin } t > 1. \end{cases}$$

16. $\dot{x} = x + y + z, \quad \dot{y} = -2y + t, \quad \dot{z} = 2z + \sin t.$

17. $\dot{x} + 6x + 3y - 14z = 0, \quad \dot{y} - 4x - 3y + 8z = 0, \quad \dot{z} + 2x + y - 5z = \sin t.$

18. One electron is moving in a constant uniform electromagnetic field satisfying $\mathbf{E} \perp \mathbf{B}$. Show that its non-relativistic trajectory, starting from rest, is given by a cycloid.

19. $\dot{\mathbf{x}} = \begin{pmatrix} 1 & -3 & 9 \\ 0 & -5 & 18 \\ 0 & -3 & 10 \end{pmatrix} \cdot \mathbf{x}.$

20. Radioactive decay. At the beginning the isotope A is pure, but it undergoes a the double disintegration $A \to B \to C$. The isotope is C is stable. Using half-lives, calculate the relative concentration of the isotopes A, B and C at any time.

21. Study the trajectory a falling particle, when the friction is proportional to the velocity.

22. $\dot{\mathbf{x}} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \cdot \mathbf{x}.$

23. $\dot{x} + 3x + 4y = 0, \quad \dot{y} + 2x + 5y = 0.$

24. $\dot{x} = -5x - 2y, \quad \dot{y} = x - 7y.$

25. $\dot{x} - y + z = 0, \quad \dot{y} - x - y = t, \quad \dot{z} - x - z = t.$

26. How can we calculate the homogeneous linear system that has as a general solution a given fundamental matrix? Use your answer to find the system that has the following as general solution

$$\begin{aligned} x &= Ae^{3t} + Ce^{-2t}, \\ y &= \frac{3}{2}Ae^{3t} + Be^{-t} - Ce^{-2t}, \\ z &= \frac{3}{2}Ae^{3t} - Be^{-t} - Ce^{-2t}. \end{aligned}$$

27. Solve the following system

$$\begin{aligned}\dot{x} &= z\sin t - y,\\ \dot{y} &= x - z\cos t,\\ \dot{z} &= y\cos t - x\sin t.\end{aligned}$$

Can you find an easy geometric explanation to the solution?
Suggestion: Write the solution by means of matrices and using the initial condition corresponding to $t = 0$.

28. Find the general solution for the following system

$$\begin{aligned}y' &= 4\frac{xy - z}{x^2 - 1},\\ z' &= 2\frac{xz - y}{x^2 - 1}.\end{aligned}$$

29. Systems of Cauchy and Euler. Explain the method to solve a system of differential equations that can be written as follows

$$t\,\dot{\mathbf{x}} = \mathbf{A}\cdot\mathbf{x}, \qquad (\dot{\mathbf{A}} = \mathbf{0}).$$

In order to do so, use variations of the method explained to solve the equation of the same name, and apply the method to find the general solution of

$$\begin{aligned}t\,\dot{x} &= 3x - 2y,\\ t\,\dot{y} &= 2x - 2y.\end{aligned}$$

30. Solve the following system

$$\begin{aligned}y' &= \frac{z}{x},\\ z' &= -xy.\end{aligned}$$