Based on JM Aguirregabiria's textbook.

1. Find the differential equations corresponding to these families of curves: (a) straight lines that go through the origin, (b) $y = C\sin 2x$.

2. Find the general solution of the following equation:

$$\frac{\left(1+y^2\right)\,dx+\left(1+x^2\right)\,dy}{\left(1-xy\right)^2}=0.$$

Discuss the domain of definition for the solutions corresponding to $y(0)=0$, $y(0)=1$ and $y(1)=1$

3. Solve $\left(ye^{xy}+2x\right)\,dx+\left(xe^{xy}-2y\right)\,dy=0,\ y(0)=2.$

4. Is this a paradox? Investigate the following equation in the points $(x,y)\neq(0,0)$:

$$\frac{x\,dy-y\,dx}{x^2+y^2}=0.$$

(a) Show that the cross derivatives are the same.
(b) Check that the equation can be written as $d\left(\arctan(y/x)\right)=0$, or, in polar coordinates, $d\varphi=0$.
(c) Calculate the line-integral corresponding to the differential equation along a circle centered at the origin.
(d) The answer you get should surprise you. Explain what is happening.

5. Find all the curves whose arc-length is proportional to the angle formed with respect to the origin. Use polar coordinates

6. Solve $\left(1-x^2\right)y'=1-y^2,\ y(1)=1.$

7. What us the condition that the equation $P(x,y)\,dx+Q(x,y)\,dy=0$ has to satisfy so that it admits an integrating factor of the form $\mu(xy)$.

8. Solve $y'=\dfrac{2x^3y-y^4}{x^4-2xy^3}.$

9. Prove that if the equation $P(x,y)\,dx+Q(x,y)\,dy=0$ is exact, and admits a non-constant integrating $\mu(x,y)$, then the general solution is $\mu(x,y)=C$.

10. Solve $2xy\,dx+\left(1-x^2-y^2\right)\,dy=0.$

11. Solve $y'-x^2y=x^5.$

12. Find the value $y(10)$, if $y'-2xy=1$ and $y(0)=1$.
Suggestion: Use the properties of the error-function (for example, in page 277 in the book)

$$\mathrm{erf(x)}\equiv\frac{2}{\sqrt{2}}\int_0^{\mathrm{x}}\mathrm{e}^{-\mathrm{u}^2}\mathrm{du}$$

13. Find the general solution for: $y^3\,dx+\left(xy-y^4\right)\,dy=0.$

Suggestion: Use the properties of the exponential-integral function (for example, in page 280 in the book)

$$\mathrm{Ei(x)}\equiv\int_{-\infty}^{\mathrm{x}}\frac{\mathrm{e}^{\mathrm{u}}}{\mathrm{u}}\mathrm{du}$$

14. RL circuit. The resistance in a circuit is R, the autoinduction is L, and it has a potential-difference $V(t)$. Calculate the intensity $I(t)$ corresponding to the initial value $I_0 = I(t_0)$. What happens if $V = V_0 \sin \omega t$?

15. Homogeneous equation. Prove that the homogeneous equation $P(x,y)\,dx + Q(x,y)\,dy = 0$ admits an integrating factor $(xP + yQ)^{-1}$, given that this last function is finite. What will be the integrating factor when $xP + yQ = 0$?

16. Isobarik equation. Homogeneous equations are scaling invariant with respect to transformations such as $(x, y) \to (ax, ay)$. We can generalize this: isobaric equations are invariant with respect to $(x, y) \to (ax, a^\lambda y)$ for some specific value of λ. In other words, for an isobaric equation $P(x,y)\,dx + Q(x,y)\,dy = 0$:

$$P(ax, a^\lambda y) = a^r\, P(x,y), \quad Q(ax, a^\lambda y) = a^{r-\lambda+1}\, Q(x,y), \quad \forall a. \tag{1}$$

Discuss the procedure to solve this kind of equations.

17. Solve $\left(y^2 - 1\right)\, dx + \left(3x^2 - 2xy\right)\, dy = 0.$

18. Orthogonal trajectories. Orthogonal trajectories are a family of curves that intersect a uniparametric family of curves $\varphi(x, y, C) = 0$ at right angles. If the differential equation of a family of curves is $F(x, y, y') = 0$, what will be the equation of the family of orthogonal trajectories?

19. Find the orthogonal trajectories to the following paraboli $y^2 = Cx$

20. Solve $(6x + 4y + 3)\,dx + (3x + 2y + 2)\,dy = 0.$

21. Mass changing with time. A rocket whose strucutural mass is M, carries a fuel mass m. The rocket is launched from the surface of Earth in vertical direction, and it ejects gas of mass k at velocity v every second. Neglecting all forces except the weight, and if the gravitational acceleration is constant throughout all the trajectory, which will be the height and the speed when all the fuel has been consumed.

22. Relativistic mass. In general relativity, the mass of a particle travelling at velocity v is $m = m_0\left(1 - v^2/c^2\right)^{-1/2}$, where m_0 is the rest mass. (As usual, c is the velocity of light in vacuum)
(a) A particle at rest in a vacuum starts moving due to a constant gravitational field. What will it be its velocity at time t and after a long time?
(b) The change of mass for the particle is $\Delta m \equiv m - m_0$ and the magnitude ΔE is the work done by the force $F = d(mv)/dt$. Show directly that $\Delta E = \Delta m\, c^2$.
(c) In order to check our result, start from the formuli of the last subsection and the definition of force, to calculate how the mass of the particle changes with respect to velocity.

23. Solve

$$y' = \left(\frac{x - y + 3}{x - y + 1}\right)^2.$$

24. Prove that **all** homogeneous equations are separable in polar coordinates

25. Solve $y' + e^{-x}y^2 - y - e^x = 0.$

26. Find the regular solution of: $(y')^2 - 2xy' + y = 0.$

27. The straight line $x = x_0$ intersects the integral lines of the differential equation $y' + p(x)y = q(x)$. Demonstrate that the tangents at the intersecting points all cross each other in a common point. Show that in the case $y' - y/x = -x^{-3}$ the geometrical loci where all the tangents intersect is a straight line.

28. Find and plot the envelope and multiple-points corresponding to the strophoid curves $(x - C)^2(1 - y) = (1 + y)y^2$

29. Besides envelopes and multiple points, the equations

$$\phi(x,y,C)=0 \quad \frac{\partial\phi}{\partial C}(x,y,C)=0 \tag{2}$$

can also have cusps. As an example, find and plot the general solution to $9y\,(y')^2=4$. Study the solutions to (**??**) for this case. Is it a solution to the differential equation?

30. The *radious* **of the universe**. In the standard cosmological model, the scale factor R for the universe of Robertson and Walker evolves according to Friedmann's equation:

$$\dot{R}^2=\frac{1}{R}-k, \tag{3}$$

where $k=-1,0,1$. Discuss the solutions to this equation, and in special, when the Big Bang happened, and when the Universe is going to end in a Big Crunch.

31. Conditions at infinity.Sometimes, instead of using initial conditions to determine the solution, the conditions are given at infinity. As an example, find the only solution to the following equation that is bounded in the limit $x\to\infty$:

$$y'-y=\sin x.$$

32. Find all the solutions of $(y')^2+y^2=1$.

33. Write, using quadratures, the general solutions of the equation of the following type:

$$y\,f(xy)\,dx+x\,g(xy)\,dy=0.$$

34. Discuss the procedure to solve equations of the following type:

$$\frac{df}{dy}(y)\,y'+A(x)f(y)=B(x).$$

Use your method to solve $3y^2y'-xy^3=x^2$.

35. Equation of Abraham-Lorentz The equation of motion for a particle of charge q moving inside a constant electric field $\mathbf{E}$, taking into account the damping due to the emited radiation, is given by

$$m\mathbf{a}=q\mathbf{E}+\frac{2}{3}q^2\dot{\mathbf{a}}$$

in the non-relativistic approximation. Show that many solutions to this equation are not acceptable from a physics point of view, since the acceleration grows without bound. Check that in order to choose the physical solutions it is enough to impose that in the limit $q\to 0$ we should have $m\mathbf{a}=0$. Find the physical solutions.

36. Prove that all the solutions of an equation of the following type

$$\dot{x}=f(x)$$

do not oscillate. The solution of the equation $\dot{x}=\sqrt{1-x^2}$ is $x=\sin(t-t_0)$. Why is this not a contradiction to the previous statement?
Suggestion: Draw the picture of an oscillation.

37. Find all the solutions of:

$$xy'=-y+\sqrt{xy+1}.$$

38. Solve the following problem, and comment on the domain of definition of the solutions:

$$\dot{x}=\frac{1}{e^x-t}, \qquad x_0=\ln t_0.$$

39. Solve the following equation for all real values of α:

$$y^2-(y')^2=\alpha.$$