Based on JM Aguirregabiria's textbook

1. A point particle is falling in fluid, which opposes to its movement a viscous force proportional to its speed. Write and classify the equation of motion. What would happen were the friction proportional to the speed squared?

2. Check whether $y^2 - 2y = x^2 - x - 1$ is a solution of the following equation:

$$2y' = \frac{2x-1}{y-1}\,.$$

If it is indeed a solution, examine its domain of definition.

3. Harmonic Oscillator. Prove that the equation of motion of the harmonic oscillator

$$\ddot{x} + \omega^2 x = 0$$

admits all the following general solutions:

$$\begin{aligned} x(t) &= C_1 \cos\omega t + C_2 \sin\omega t\,, \\ x(t) &= D_1 e^{i\omega t} + D_2 e^{-i\omega t}\,, \\ x(t) &= A\cos\left(\omega t + \varphi\right)\,, \\ x(t) &= A\cos\left[\omega\left(t - t_0\right)\right]\,. \end{aligned}$$

Are all of them equivalent? Could you add any other form to that list?

4. The equation $x + yy' = 0$ admits as solutions the two functions $y_\pm = \pm\sqrt{C^2 - x^2}$ (check this statement). Which are their respective domains of definition? Is the following function also a solution?

$$y(x) = \begin{cases} \sqrt{C^2 - x^2}\,, & x < 0, \\ -\sqrt{C^2 - x^2}\,, & x \geq 0. \end{cases}$$

5. Is the expression $x^2 + Cy^2 = 1$ an implicit solution of the equation

$$y' = \frac{xy}{x^2 - 1}\ ?$$

6. Check that the family of functions

$$y = \frac{1 - Ce^{2x}}{1 + Ce^{2x}}$$

is the general solution of the equation $y' = y^2 - 1$. Find two obvious solutions of the equation by inspection, and discuss whether they are singular solutions.

7. Check that *all* solutions of the equation $xy' = y$ must obey the initial condition $y(0) = 0$. Why don't we find uniqueness? Is the function

$$y = \begin{cases} 0\,, & x \le 0, \\ x\,, & x \ge 0 \end{cases}$$

a solution of the initial value problem?

8. Show that both $y = x^2$ $(x \ge 0)$ and

$$y = \begin{cases} 0\,, & x \le C, \\ (x-C)^2\,, & x \ge 0, \end{cases}$$

are amongst the solutions of the initial value problem

$$y' = 2\sqrt{y}\,, \qquad y(0) = 0\,.$$

9. The empty vase. If we are shown an empty vase with a small hole bored in its base we cannot know when it was last full, quite obviously. Prove that our lack of knowledge is due to the inapplicability of the existence and uniqueness theorem.

Clue: the water level in the vase is given by Torricelli's law, with a constant parameter k that depends on the geometry of the vase and the hole, among other things:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -k\sqrt{h}\,.$$

10. The period of the pendulum. Use the change of variables $\sin(\varphi/2) = \sin(\alpha/2)\sin\nu$ (where ν is the new variable and φ the original one) to prove that

$$F\left[\arcsin\left(\frac{\sin(\theta/2)}{\sin(\alpha/2)}\right)\Big\backslash\frac{\alpha}{2}\right] = \sqrt{\frac{g}{l}}\,(t-t_0)$$

is the implicit solution of the equation of motion of the pendulum, $\ddot{\theta} + (g/l)\sin\theta = 0$. In order to do this, first check that the implicit solution is given by

$$\int_0^{\theta(t)} \frac{\mathrm{d}\varphi}{2\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\varphi}{2}}} = \sqrt{\frac{g}{l}}\,(t-t_0)\,,$$

when the pendulum goes through its minimum at the instant t_0, (i.e. $\theta(t_0) = 0$) and the angular amplitude is α. Then use the change of variables to identify the correct elliptic integral.

Now use the implicit solution expression involving the elliptic integral to compute the period, both exactly and to fourth order in the amplitude α