Ordinary differential equations
Stability
Topic 7

Stability: 7.1 The concept of stability, 7.2 Two-dimensional autonomous dynamical system, 7.3 Quasilinear systems, 7.4 Linear stability, 7.5 Unidimensional mechanical systems

7.1 The concept of stability

- In general, it is not easy to solve ordinary differential equations. Fortunately, sometimes it is enough to know the behavior of the solution in the limit $t \to \pm\infty$ or to study its asymptotically.
- This field is known as qualitative dynamics
- The fundamental concept for this analysis is the stability.

- As we saw in topic 4, a system of first order differential equations is known as dynamical system. Let

$$\dot{\vec{x}} = \vec{f}(t, \vec{x}),$$

be a dynamical system, and $\vec{x}^*(t)$ its solution; that is

$$\dot{\vec{x}}^*(t) = \vec{f}(t, \vec{x}^*(t)).$$

- In order to describe the qualitative behavior of a system, it is very useful to know if and where it has an invariant set.

  If a solutions of a system always belongs to a set, then the set is invariant

  There are many types, but the most common are the equilibrium points. These are also known as stable, fixed, critic... points. Their definition is the following:

  - The point $\vec{x} = \vec{x}^*$ is a equilibrium point if:

$$\vec{f}(t, \vec{x}^*) = 0, \quad \text{(and also } \frac{\partial \vec{f}(t, \vec{x}^*)}{\partial t} = 0\text{)}.$$

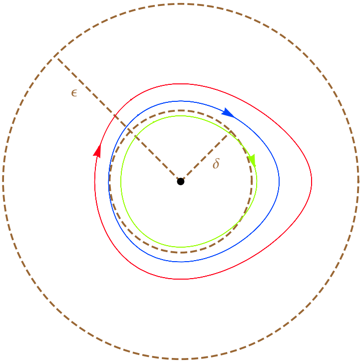
Exercise 7.1

- Find the equilibrium points of the following dynamical systems (a) $\dot{x} = ax$, $(b)\dot{x} = ax - x^3$.
- a) In this case, there is only one point $x^* = 0$.
  b) In this other case, on the other hand, there are three solutions to the equation $ax - x^3 = 0$:
  $x_1^* = 0, x_2^* = \sqrt{a}, x_3^* = -\sqrt{a}$.

- In real physical problems, there are always errors in the initial values. Therefore, it is very important to know what happens to a solution if the initial values change a bit. If the solution of the new initial values tend to the original one, then the solution is stable.
- One can examples of stability in physical systems.
  - In a pendulum it is clear that the equilibrium points are the maximum and minimum vertical positions. The lower point is stable, since it is the minimum of the potential energy (and if there is friction, is is asymptotically stable)
  - The higher point, on the other hand, it is unstable: any perturbation will make the pendulum fall from the maximum.
- In order to define stability we will use Liapunov's **criterion**

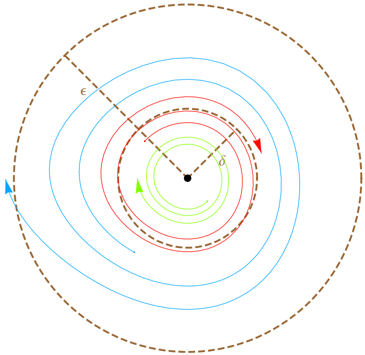
Stable points

- The solution $\vec{x}^*(t)$ is stable if for every $\epsilon > 0$ there exists a $\delta(\epsilon)) > 0$ such that:
for any solution $\vec{x}(t)$ that satisfies
$|\vec{x}(t_0) - \vec{x}^*(t_0)| < \delta(\epsilon)$ (initial condition),
then $|\vec{x}(t) - \vec{x}^*(t)| < \epsilon$ for every $t > t_0$.



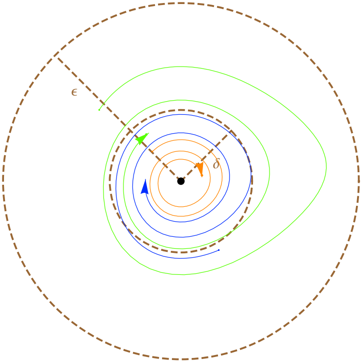
Unstable points

- The solution $\vec{x}^*(t)$ is unstable if for a $\delta > 0$ as small as we want, there exist an $\epsilon > 0$ and a $\vec{x}(t)$ for $t > t_0$ such that $|\vec{x}(t_0) - \vec{x}^*(t_0)| < \delta$ and $|\vec{x}(t) - \vec{x}^*(t)| > \epsilon$.



Asymptotically stable point

- The solution $\vec{x}^*(t)$ will be asymptotically stable if it is stable, and if there is a $\delta' > 0$ such that for all solutions whose initial values satisfy $|\vec{x}(t_0) - \vec{x}^*(t_0)| < \delta'$, then $lim_{t\to+\infty}|\vec{x}(t) - \vec{x}^*(t)| = 0$.



- The properties of an equilibrium point can be changed when a parameter changes. When that happens, there is a bifurcation
- One can use bifurcation-diagram to study bifurcations.
- The parameter that creates the change is know as bifurcation-parameter

- The bifurcation-diagrams for a first order equation are represented as follows:
  - The position of the equilibrium points is plotted as a function of the bifurcation-parameter
    - Stability is represented with a continuous line
    - Instabilities are represented by dashed lines
  - The magnitude and direction of the vector-field $\dot{x}$ is also represented

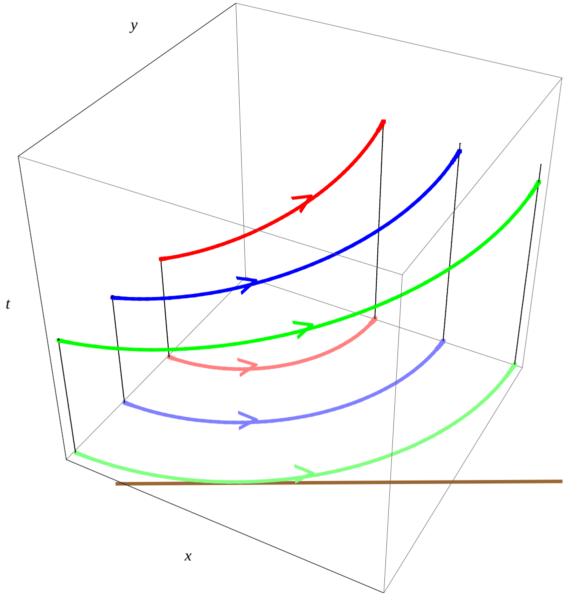
7.2 Two-dimensional autonomous dynamical systems

- We will study systems of the type

$$\dot{x} = P(x, y)$$

$$\dot{y} = Q(x, y)$$

- Its solution defines a curve-congruency (t, x, y) in space
- Since the system is autonomous, its projection defines a curve-congruency in (x, y) phase space.
- The projection of a solution $(x(t), y(t))$ onto phase space, defines a parametric curve, known as phase trajectory or phase orbit



7.3. General solution of an autonomous system and its projection onto phase space

- In order to calculate the equation of the phase trajectories ($y(x)$), taking into account that our system is autonomous, we have to integrate

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}.$$

  - From the uniqueness-existence theorem, if things are regular, phase trajectories will not intersect
  - It can also be seen that the vector field $P\hat{i} + Q\hat{j}$ is tangent to the phase trajectories
  - Using a hydrodynamic picture, the vector field would be a velocity-field, and the phase trajectories would be current lines

Unidimensional mechanical systems

- The machinery of dynamical systems is very useful in order to study unidimensional mechanical systems
- Newton's second law tells us that the equation of motion for a force that depends only on position and velocity is:
$$\ddot{x} = f(x, \dot{x}).$$
- This can be rewritten as:
$$\dot{x} = y, \qquad \dot{y} = f(x, y).$$
- It is interesting to study the case of a conservative force
$$\ddot{x} = F(x) = -V'(x).$$

The system in this case will be:

$$\dot{x} = y, \qquad \dot{y} = -V'(x).$$

7.3 Quasilinear systems

- From now on, we will consider that the dynamical systems have only a single equilibrium point

  Without loss of generality, we will suppose that that point is at the origin

$$P(0,0) = Q(0,0) = 0$$

.

- We will suppose that the functions $P(x, y)$ and $Q(x, y)$ accept a Taylor series around the origin:

$$\dot{x} \approx a_{11}x + a_{12}y,$$

$$\dot{y} \approx a_{21}x + a_{22}y,$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix}_{(x,y)=(0,0)}$$

- The system will be quasilinear if:

$$\lim_{\sqrt{x^2+y^2}\to 0} \frac{P(x,y) - a_{11}x - a_{12}y}{\sqrt{x^2+y^2}} = 0,$$

$$\lim_{\sqrt{x^2+y^2}\to 0} \frac{Q(x,y) - a_{21}x - a_{22}y}{\sqrt{x^2+y^2}} = 0.$$

- The non-linear terms go to zero faster than the linear terms.

- The system we get once we neglect the non-linear terms is called linearized system or first approximation or linear approximation:

$$\dot{x} = a_{11}x + a_{12}y,$$

$$\dot{y} = a_{21}x + a_{22}y,$$

- The cases we will study have isolated fixed points, and therefore:

$$\mathrm{d}et\mathbf{A} \neq 0.$$

- It can be proved that the characteristic roots of the matrix $\mathbf{A}$ will be the following:

$$k_{1,2} = \frac{1}{2}\left(\mathrm{t}r\mathbf{A} \pm \Delta\right),$$

where

$$\Delta \equiv \mathrm{t}r^2\mathbf{A} - 4\mathrm{d}et\mathbf{A}.$$

7.4 Linear stability

- We will apply Liapunov's method, also known as the linear stability method. The asymptotic behaviour of a quasilinear system, and that of the linearized system, are qualitatively the same (modulo one exception)
- The dynamical system that we will mostly use in these sections is:

$$\dot{x} = -x - y - \epsilon xy,$$

$$\dot{y} = (1 + r + d)x + (1 + r)y + \epsilon(y^2 - x^2).$$

  When $\epsilon = 0$ the system is fully linear.
- For this system $\det\mathbf{A} = d$, $\mathrm{tr}\mathbf{A} = r$, the characteristic roots are $k = (r \pm \sqrt{r^2 - 4d})/2$ with corresponding eigenvectors

$$\mathbf{x} = \begin{pmatrix} -1 \\ k + 1 \end{pmatrix}$$

7.4.1 Different real characteristic roots

$$\Delta > 0 \Rightarrow k_1 > k_2$$

- In this case the eigenvectors corresponding to k_1 and k_2 will be:

$$\vec{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

- The general solution of the linear system will be:

$$\vec{x} = C_1\vec{x}_1 + C_2\vec{x}_2,$$

that is,

$$x = C_1 x_1 e^{k_1 t} + C_2 x_2 e^{k_2 t},$$

$$y = C_1 y_1 e^{k_1 t} + C_2 y_2 e^{k_2 t}.$$

- The slope of the solution curves will be:

$$\frac{y}{x} = \frac{C_1 y_1 e^{k_1 t} + C_2 y_2 e^{k_2 t}}{C_1 x_1 e^{k_1 t} + C_2 x_2 e^{k_2 t}}.$$

- There are two special solution, corresponding to the trajectories parallel to the eigenvectors:

$$C_2 = 0, \quad \frac{y}{x} = \frac{y_1}{x_1},$$

$$C_1 = 0, \quad \frac{y}{x} = \frac{y_2}{x_2}.$$

Real negative roots

$$\boxed{\Delta > 0,\ \det A > 0,\ trA < 0 \Rightarrow k_2 < k_1 < 0}$$

- In this case

$$\lim_{t\to\infty} x = \lim_{t\to\infty}\left(C_1 x_1 e^{k_1 t} + C_2 x_2 e^{k_2 t}\right) = 0$$

$$\lim_{t\to\infty} y = \lim_{t\to\infty}\left(C_1 y_1 e^{k_1 t} + C_2 y_2 e^{k_2 t}\right) = 0;$$

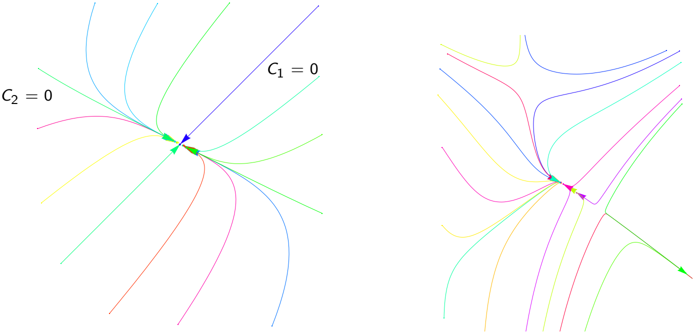
  so all the trajectories tend to the origin; therefore the equilibrium pooint is asymptotically stable

- Besides, for all curves

$$\frac{y}{x} = \lim_{t\to\infty} \frac{C_1 y_1 e^{k_1 t} + C_2 y_2 e^{k_2 t}}{C_1 x_1 e^{k_1 t} + C_2 x_2 e^{k_2 t}} = \frac{y_1}{x_1}$$

  except for $C_1 = 0$. When the geometry of the trajectories is like this, the point is called a node.

So the origin is an asymptotically stable node



7.6. The phase space for our system, for $d = 1$, $r = -5/2$ and $\epsilon = 0$ (left figure)

$\epsilon = 1$ (right figure).

Real positive roots

$$\boxed{\Delta > 0,\ \det A > 0,\ \mathrm{tr} A > 0 \Rightarrow 0 < k_2 < k_1}$$

- In this case

$$\lim_{t\to\infty} x = \lim_{t\to\infty}\left(C_1 x_1 e^{k_1 t} + C_2 x_2 e^{k_2 t}\right) = \infty$$

$$\lim_{t\to\infty} y = \lim_{t\to\infty}\left(C_1 y_1 e^{k_1 t} + C_2 y_2 e^{k_2 t}\right) = \infty;$$

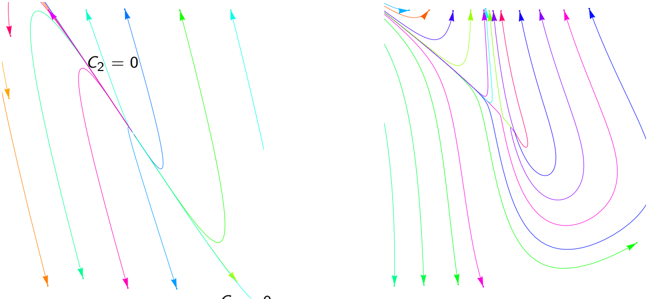
  so all trajectories go away from the origin; therefore the origin is an unstable point

- Moreover, for all curves

$$\frac{y}{x} = \lim_{t\to\infty} \frac{C_1 y_1 e^{k_1 t} + C_2 y_2 e^{k_2 t}}{C_1 x_1 e^{k_1 t} + C_2 x_2 e^{k_2 t}} = \frac{y_2}{x_2}$$

  except for $C_2 = 0$. It is a node, then

So the origin is an unstable node



7.7 The phase space for our system, for $d = 1$, $r = 5/2$ and
$\epsilon = 0$ (left figure)

$\epsilon = 1$ (right figure)

Real roots with opposite signs

$$\boxed{\Delta > 0,\ \det A < 0,\ \mathrm{tr} A > 0 \Rightarrow k_2 < 0 < k_1}$$

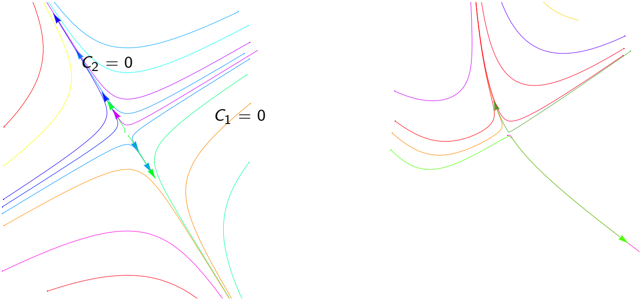
- The particular solution $C_1 = 0$ tends to the origin, following the straight line $y/x = y_2/x_2$, called the stable space
- But the particular solution $C_2 = 0$ goes away from the origin, following the line $y/x = y_1/x_1$, called the unstable space
- How do all the other solutions behave?
  - in the future infinity, they tend to the unstable space

$$\lim_{t\to\infty} y/x = y_1/x_1$$

  - in the past infinity, they tend to the stable space

$$\lim_{t\to-\infty} y/x = y_2/x_2$$

A point with this geometry is called a saddle point



7.8 The phase space for our system, for $d = r = -1$ and $\epsilon = 0$ (left figure)

$\epsilon = 1$ (right figure)

7.4.2 Complex characteristic roots

$$\Delta < 0 \Rightarrow k_+ = \alpha + i\omega, k_+ = \alpha - i\omega,$$

- The general solution for this case is:

$$x = e^{\alpha t}(C_1 x_1 \cos \omega t + C_2 x_2 \sin \omega t),$$

$$y = e^{\alpha t}(C_1 y_1 \cos \omega t + C_2 y_2 \sin \omega t).$$

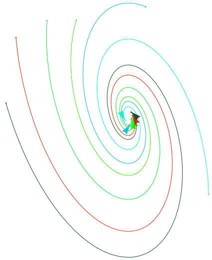
- Now we have periodic solutions, modulated by an exponential.

Complex roots with negative real part

$$\boxed{\Delta < 0,\ \mathrm{tr}A < 0 \Rightarrow \alpha < 0}$$

- $e^{\alpha t}$ is decreasing, so the solutions will tend to the origin
- but they will not follow a fixed direction, the slope will be
$$\frac{y}{x} = \frac{C_1 y_1 \cos \omega t + C_2 y_2 \sin \omega t}{C_1 x_1 \cos \omega t + C_2 x_2 \sin \omega t}$$

This kind of point is called focus or spiral point



7.9 The phase space for our system, for $d = 3$, $r = -1$ and $\epsilon = 0$

Complex roots with negative real part

$$\boxed{\Delta < 0,\ \mathrm{tr}A > 0 \Rightarrow \alpha > 0}$$

- In this case the solutions will go away from the origin
- The slope will rotate as the curves go away from the center
- Behaviour as in figure 7.9, but with the arrows pointing in the opposite direction, away from the origin

Imaginary roots

$$\boxed{\Delta < 0,\ trA = 0 \Rightarrow \alpha = 0}$$

- The orbits are periodic in this case:

$$x = e^{\alpha t}(C_1 x_1 \cos \omega t + C_2 x_2 \sin \omega t),$$

$$y = e^{\alpha t}(C_1 y_1 \cos \omega t + C_2 y_2 \sin \omega t).$$

- The point will be stable, but not asymptotically stable
- This kind of point is called center or vortex

- The influence of the non-linear term can be very important, totally changing the geometry of the point
- A point that is a center in the linear approximation can become a focus due to the non-linear term, but can continue being a center

7.10 The phase space for our system, for $d = 1$, $r = 0$ and $\epsilon = 0$ (left figure)

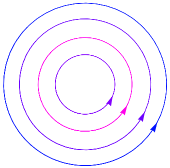
$\epsilon = 1$ (right figure)

- Another example:

$$\dot{x} = -y$$

$$\dot{y} = x - ay^n$$

- In the linear approximation, the origin is a center



7.11 Phase space of the system with $a = 0$

- It can be seen (problem 7.38) that for $n = 2$, the origin is still a center
- but for $n = 3$ the center becomes an asymptotically stable focus

7.4.3 Same real characteristic roots

$$\Delta = 0 \Rightarrow k_1 = k_2$$

- In this case
$$k_1 = k_2 = \frac{1}{2}\mathrm{t}rA.$$
- Moreover
$$\Delta = 0 = (a_{11} - a_{22}) + 4a_{12}21$$

We have two different cases

Same roots, first case

$$\boxed{a_{12} = a_{21} = 0}$$

- In this case $a \equiv a_{11} = a_{22} = 0$ and the orbits are the following straight lines

$$x = C_1 e^{at}, \; y = C_2 e^{at}.$$

- The slopes are different, and such a point is called a proper node or star node

- One example of this is given by the following system:

$$\dot{x} = -x - \epsilon xy,$$

$$\dot{y} = -y - \epsilon(y^2 - x^2).$$

7.12. Phase space for this case with $\epsilon = 0$ (left figure) and $\epsilon = 1$ (right figure).

Same roots: second case

$$|a_{12}| + |a_{21}| = 0$$

- In this case the orbits are as follows

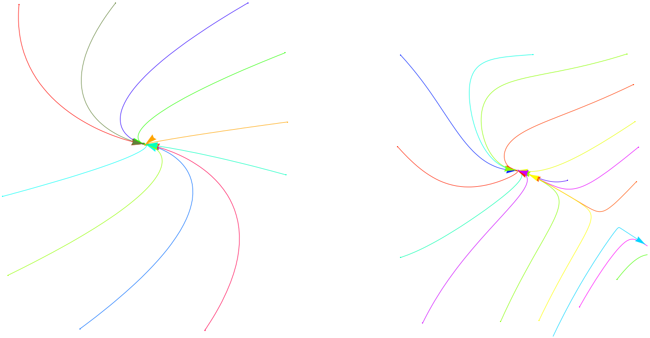
$$x = (C_1 x_1 + C_2(x_1 t + x_2))e^{kt},$$

$$y = (C_1 y_1 + C_2(y_1 t + y_2))e^{kt}.$$

- All the slopes are asymptotically the same

$$\lim_{t\to\infty} \frac{y}{x} = \frac{y_1}{x_1},$$

and this point is called a degenerate node



7.13 The phase space for our system, for $d = 1$, $r = -2$ and $\epsilon = 0$ (left figure)

$\epsilon = 1$ (right figure)

7.5 Unidimensional mechanical systems

- Let us suppose that a particle experiences a conservative force $F = -V'(x)$ indar kontserbakorrak and a friction froce $R = -\gamma \dot{x}$ $(\gamma \geq 0)$
- Its equation of motion will be:

$$\ddot{x} = -V'(x) - \frac{\gamma}{m}\dot{x}$$

- We can write this equation in a system-like form by defining $v = \dot{x}$:

$$\dot{x} = P(x, v) \equiv v$$

$$\ddot{v} = Q(x, v) \equiv -V'(x) - \frac{\gamma}{m} v$$

- If the point $x = x^*$ satisfies $V(x^*) = 0$, then the linear approximation will be

$$\dot{x} = v$$

$$\ddot{v} = -V''(x^*) - \frac{\gamma}{m} v.$$

- The characteristic roots will be

$$k_{1,2} = -\frac{\gamma}{m} \pm \sqrt{-V''(x^*) + \left(\frac{\gamma}{m}\right)^2}.$$

- Let us classify these points

Local Maximum

$$\boxed{V''(x^*) < 0 \Rightarrow k_2 < 0 < k_1}$$

- This will be a saddle point.

Inflection point or higher order extremum

$$\boxed{V''(x^*) = 0 \Rightarrow k_1 = 0, k_2 = -\gamma}$$

- In this case the linear approximation is not enough.

Local minimum

$$\boxed{V''(x^*) > 0}$$

- We have two cases here
  - $\gamma^2 < 4V''(x^*)$ (friction too big)
    The roots are complex, and with negative real part: the equilibrium point will be an asymptotically stable focus
  - $\gamma^2 > 4V''(x^*)$ (small friction)
    In this case the roots are real and negative: the equilibrium point will be an asymptotically stable node