Ordinary differential equations
Topic 6

Series solutions of ordinary differential equations

6.1 Power series: Revision, 6.2 Series solutions, 6.3 Ordinary points, 6.5 Method of Frobenius

6.1 Power series: Revision

- We will solve second order homogeneous equations using power series
- We will denote Power series as follows:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

- The convergence radius will be:

$$\rho(x_0) = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

  - When the limit exists, or it is $+\infty$, the series is absolutely and uniformly convergent in the radius $|x - x_0| < \rho$, but divergent for $|x - x_0| > \rho$.

- Thus, when $\rho = 0$ the series is convergent at most at x_0.
- On the other hand, if $\rho = +\infty$, then the series is convergent everywhere.

Exercise 6.1

- Can you give another way of calculating the radius of convergence?
- $$\rho(x_0) = \lim_{n\to\infty} |a_n|^{-1/n}.$$

- Let us consider the following two series:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

  and let us assume that they are convergent for $|x - x_0| < \rho$.
- Their linear combination goes to the sum of the corresponding combination:

$$\alpha f(x) + \beta g(x) = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x - x_0)^n \ ,$$

  where α and β are constant.
- The formal product of two series also goes to another series:

$$f(x)g(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right] \left[\sum_{n=0}^{\infty} b_n (x - x_0)^n\right] =$$

$$\sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} a_k b_{n-k}\right] (x - x_0)^n.$$

- One can calculate the case $f(x)/g(x)$ in a similar manner, for the points $g(x_0) \neq 0$, but it is not easy, in general, to get an expression for the coefficients
- The series can be differentiated indefinitely in the circle $|x - x_0| < \rho$ and its coefficients can be calculated term by term:

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

  where

$$a_n = \frac{1}{n!} f^{(n)}(x_0), \qquad f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x - x_0)^n.$$

- Two series are the same if the coefficients corresponding to each order are the same

$$f(x) = g(x) \Leftrightarrow a_n = b_n.$$

- On the other hand, if $\rho > 0$, the function $f(x)$ is analytic around the point $x = x_0$.
  - If f and g are analytic, then $\alpha f + \beta g$, fg and f/g are analytic.
  - For example, polinomials and $\sin x$, $\cos x$, $\exp x$, $\sinh x$ and $\cosh x$ are analytic functions around any point
  - But, for example, $\ln(1 + x)$ has $\rho(0) = 1$.

Exercise 6.2

- Find the convergence radius and the sum of the following series:

$$f_1(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$f_2(x) = 1 - \frac{x}{2} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots$$

$$f_3(x) = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$f_4(x) = 1 + x^2 + \frac{3}{4}x^4 + \frac{1}{2}x^6 + \frac{5}{16}x^8 + \frac{3}{16}x^{10} + \dots$$

- For the first case, it is clear that for $x_0 = 0$ one has:

$$a_n = \frac{(-1)^{n+1}}{n} \quad \forall n > 1.$$

On the other hand,

$$\rho(0) = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| =$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}\frac{1}{n}}{(-1)^{n+2}\frac{1}{n+1}} = \right| = \left| \frac{n+1}{n} \right| = 1.$$

Finally,

$$f_1(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = \ln(1+x) \qquad (|x| < 1).$$

- In the second case, for $x_0 = 0$ we have:

$$a_n = \frac{(-1)^n}{(2n)!} \quad \forall n > 0.$$

Therefore,

$$\rho(0) = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| =$$

$$\lim_{n\to\infty} \left| \frac{(-1)^n \frac{1}{(2n)!}}{(-1)^{n+1} \frac{1}{(2(n+1))!}} \right| = \Big| 2(n+1) \Big| = +\infty.$$

Finally

$$f_2(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{(2n)!} = \cos\sqrt{x} \qquad (|x| < +\infty).$$

- In the third case we have for $x_0 = 0$:

$$a_n = (n+1) \quad \forall n > 0.$$

Therefore,

$$\rho(0) = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| =$$

$$\lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| = 1.$$

Finally,

$$f_3(x) = \sum_{n=1}^{\infty} (n+1)x^n = (1-x)^{-2}. \qquad (|x| < 1).$$

- For the fourth case, as all the powers are even, we can define $y = x^2$ and then write $f_4(x) = g_4(y(x)) = \sum_{n=0}^{\infty} b_n y^n$. Then, we have

$$b_n = \frac{n+1}{2^n} \quad \forall n > 0.$$

Therefore

$$\rho(0) = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right| =$$

$$\lim_{n\to\infty}\left|\frac{(n+1)2^{n+1}}{(n+2)2^n}\right| = 2.$$

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$$f_4(x) = \sum_{n=1}^{\infty}\frac{n+1}{2^n}(x^2)^n = \sum_{n=1}^{\infty}(n+1)\left(\frac{x^2}{2}\right)^n =$$

$$\left(1-\frac{x^2}{2}\right)^{-2}. \qquad (|x^2| < 2).$$

6.2 Series solutions

- From now on, we will find solutions for the equation $y'' + P(x)y' + Q(x)y = 0$ using two types of series:
  - ordinary series

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

  - and Frobenius series

$$y = (x - x_0)^{\lambda} \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

  where the power λ is the index of the series

- In order to find the solution around a point, we have to choose a convenient series, and for that we need to study the nature of the point.
- If the functions $P(x)$ and $Q(x)$ are analytic around the point x_0, then the point is ordinary. Otherwise, the point is singular.
- But even if x_0 is a singular point, if the functions

$$p(x) \equiv (x - x_0)P(x), \qquad q(x) \equiv (x - x_0)^2 Q(x)$$

are analytic around that point, it will be a regular singular point (in other words, when the functions $P(x)$ and $Q(x)$ have a first order and second order pole respectively). In other case, the point will be an irregular singular point.

- For convenience, we will always consider that the solution will be calculated around $x_0 = 0$
  - We can always do that by translation
  - or changing a change of variables such as $x = 1/t$ in case we need the solution at infinity

Exercise 6.3

- Classify the singular points of the following equation:

$$x^2(x^2-1)^2y'' - 2x(x+1)y' - y = 0.$$

- We clearly have

$$P(x) = -\frac{2x(x+1)}{(x^2(x^2-1)^2)} = -\frac{2}{(x(x+1)(x-1)^2)},$$

$$Q(x) = -\frac{1}{(x^2(x-1)^2(x+1)^2)}.$$

The singular points are $x=0$, $x=-1$ and $x=1$. Taking into account the definitions, $x=0$ and $x=-1$ are regular, but $x=1$is irregular.

6.3 Ordinary points

- Let us suppose that the coefficients $P(x)$ and $Q(x)$ of the equation $y'' + P(x)y' + Q(x)y = 0$ are analytic around $x = 0$, then the series

$$P(x) = \sum_{n=0}^{\infty} P_n x^n, \quad Q(x) = \sum_{n=0}^{\infty} Q_n x^n,$$

are convergent in $|x| < \rho$ for some $\rho > 0$

- Let us now consider the solution and its derivatives
  - $$y = \sum_{n=0}^{\infty} c_n x^n,$$
    $$y' = \sum_{n=1}^{\infty} n c_n x^{n-1},$$
    $$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

- In order to compare the series, we want all of them to have order n:
  - In the series for y' we will make the change $n \to n+1$

$$y' = \sum_{n=1}^{\infty} nc_n x^{n-1} = \sum_{n+1=1}^{\infty} (n+1)c_{n+1}x^{(n+1)-1} =$$

$$\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n,$$

  - and for the series y'', we will make the change $n \to n+2$

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} =$$

$$\sum_{n+2=2}^{\infty} (n+2)((n+2)-1)c_{n+2}x^{(n+2)-2} =$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n.$$

- Using the properties of the series revised earlier, we have
  - $$Qy = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} Q_{n-k}c_k\right]x^n,$$
  - $$Py' = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)P_{n-k}c_{k+1}\right]x^n,$$
  - $$y'' = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n.$$
- And thus

$$y'' + Py' + Qy = \sum_{n=0}^{\infty}\left\{(n+2)(n+1)c_{n+2}+\right.$$

$$\left.\sum_{k=0}^{n}\left[Q_{n-k}c_k + (k+1)P_{n-k}c_{k+1}\right]\right\}x^n.$$

- Thus, for the series that we have proposed to be the solution to the equation, the following relation has to be satisfied order by order for all n, in other words, for $n = 0, 1, 2, \ldots$ we need:

$$(n+2)(n+1)c_{n+2}+\sum_{k=0}^{n}[Q_{n-k}c_k + (k+1)P_{n-k}c_{k+1}] = 0$$

- It can be seen that c_0 and c_1 are free parameters, we can choose them as we wish
- Then, if $c_0, c_1, c_2, \ldots, c_{n+1}$ are known, then we can calculate c_{n+2} by means of

$$c_{n+2} = -\frac{1}{(n+2)(n+1)}\sum_{k=0}^{n}[Q_{n-k}c_k + (k+1)P_{n-k}c_{k+1}]$$

Exercise 6.4

- Use the method of series to solve the following equation

$$(x^2 - 1)y'' + 4xy' + 2y = 0$$

- The following will be helpful

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n = \dots$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n.$$

- Then we have

$$(x^2-1)y''+4xy'+2y = x^2\left(\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}\right) -$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + 4x\left(\sum_{n=1}^{\infty} nc_n x^{n-1}\right) +$$

$$2\left(\sum_{n=0}^{\infty} c_n x^n\right) = \left(\sum_{n=2}^{\infty} n(n-1)c_n x^n\right) +$$

$$4\left(\sum_{n=1}^{\infty} nc_n x^n\right) + \sum_{n=0}^{\infty}\left(2c_n-(n+2)(n+1)c_{n+2}\right)x^n$$

- Term by term

$$c_0 = c_2 \qquad c_1 = c_3$$

$$c_m = c_{m+2}$$

$$y(x) = c_0 \sum_{n\, odd} x^n + c_1 \sum_{n\, even} x^n =$$

$$y(x) = \frac{c_0\, x}{1-x^2} + \frac{c_1}{1-x^2}$$

6.5 Method of Frobenius

- We will now obtain the general solution corresponding to an ordinary point or to a regular singular point
- For convenience, we will write our equation as:

$$x^2y'' + xp(x)y' + q(x)y = 0.$$

- Since the origin is by hypothesis ordinary or regular singular, we have that the series

$$p(x) = xP(x) = \sum_{n=0}^{\infty} p_n x^n, \quad q(x) = x^2Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

  are convergent for $|x| < \rho$ for a given $\rho > 0$
- It can be seen that one sufficient and necessary condition for the origin to be ordinary is $p_0 = q_0 = q_1 = 0$.

- The method of Frobenius consists of trying a solution of the type

$$y = \sum_{n=0}^{\infty} c_n x^{n+\lambda} \quad (c_0 \neq 0).$$

  This series will be convergent at least for $0 < |x| < \rho$
- We can obtain then:

$$xy' = \sum_{n=0}^{\infty} (n+\lambda) c_n x^{n+\lambda}$$

$$x^2 y'' = \sum_{n=0}^{\infty} (n+\lambda-1)(n+\lambda) c_n x^{n+\lambda}$$

- Using the series expansions of $q(x)$ and $p(x)$ we get:
  - $$qy = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} q_{n-k} c_k \right] x^{n+\lambda},$$
  - $$xpy' = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} (k+\lambda) p_{n-k} c_k \right] x^{n+\lambda}.$$
  - All in all, this is the equality we have to solve

$$(n+\lambda)(n+\lambda-1)c_n + \sum_{k=0}^{n} \left[(k+\lambda)p_{n-k} + q_{n-k} \right] c_k = 0$$

for $n = 0, 1, 2, \ldots$

- It is convenient to define the index function:

$$\mathcal{I}(u) \equiv u(u-1) + p_0 u + q_0.$$

- Using this definition, our main equation reads

$$\mathcal{I}(n+\lambda)c_n + \sum_{k=0}^{n-1} \left[(k+\lambda)p_{n-k} + q_{n-k}\right] c_k = 0$$

- Now, taking $n = 0$ we get

$$\mathcal{I}(\lambda)c_0 = (\lambda(\lambda-1) + p_0 u + q_0)c_0 = 0$$

and using $c_0 \neq 0$, the previous equation gives the possible values for λ.

- From Frobenius' theorem, the largest of these index (λ_1) will always gives as a bounded solution:

$$y = \sum_{n=0}^{\infty} c_n x^{n+\lambda_1} \quad (c_0 \neq 0).$$

- Once we know one solution, we can use the method of d'Alembert to get the second solution

- If the equation to solve is given to us in the following form

$$h(x)y'' + xp(x) + q(x)y = 0$$

where $h(x)$ is a polynomial, it is convenient to multiply the equation with some power of x, such the smallest power in the term corresponding to y'' is x^2
- Doing this, the method can be applied in the same way, but the index equation will change

Example: September 2005

- Solve the following equation

$$x(x-1)y'' + 3y' - 2y = 0$$

Example: September 2002

- Solve the following equation

$$x(x-3)y'' - (x^2-6)y' + 3(x-2)y = 0.$$

Example: September 2006

- Solve the following equation

$$xy'' + xy' + y = 0.$$

Bessel's equation

- Let us study the following equation:

$$x^2y'' + xy' + (x^2 - \mu^2)y = 0$$

  - The origin is a regular singular point
  - We will then use a Frobenius series

- By direct calculation we get

$$(\lambda^2 - \mu^2)c_0 = 0,$$

$$[(\lambda + 1)^2 - \mu^2)c_1 = 0,$$

$$[(\lambda + n)^2 - \mu^2)]c_n + c_{n-2} = 0,\ n = 2, 3, \ldots$$

- Therefore, the indices of the equation are $\lambda = \pm\mu$
- It can be seen that the equation has two solutions given by:

$$y_1 = J_\nu(x) \;\; \text{eta} \;\; y_2 = J_{-\nu}(x),$$

where

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\lambda + k + 1)} \left(\frac{x}{2}\right)^{\mu+2k} .$$

- This is know as Bessel Function of the first kind
- One would think that the general solution to Bessel's equation is the following:

$$y = AJ_\nu(x) + BJ_{-\nu(x)},$$

but since

$$W[J_\nu(x), J_{-\nu}(x)] = -\frac{2\sin(\nu\pi)}{\pi x}$$

the solutions are not linearly independent when ν is an integer

- In that case we have to define Bessel functions of the second kind

$$Y_\nu(x) = \frac{\cos(\nu x) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu x)}.$$

- It can be seen that Y_ν is a solution of the equation and $W[J_\nu(x), Y_\nu(x)] \neq 0 \, \forall \nu$
- The general solution is then

$$y = AJ_\nu(x) + BY_\nu(x).$$