Ordinary differential equations
Topic 5

Laplace transforms

5.1 Definition, 5.2 Properties, 5.3 Inverse transform, 5.6 Linear equations with constant coefficients

5.1 Definition

- We will study a new concept, which will be useful in order to solve problems with initial conditions:

$f(t)$	$\longrightarrow$	$F(s)$
Function of the real variable t	Laplace transform	Function of the real variable s

- We will turn a differential equation into an algebraic equation
- It is very important from a theoretical point of view: it is widely used in the theory of circuits and in quantum mechanics

- The Laplace transform F of the function f will be defined by the following integral:

$$\mathcal{L}[f] = F(s) = \int_0^\infty e^{-st} f(t) dt$$

Exercise 5.1

- Show that the Laplace transform of $f = 1$ is the following:

$$\mathcal{L}[1] = \frac{1}{s} \text{ if } s > 0.$$

- Applying the definition

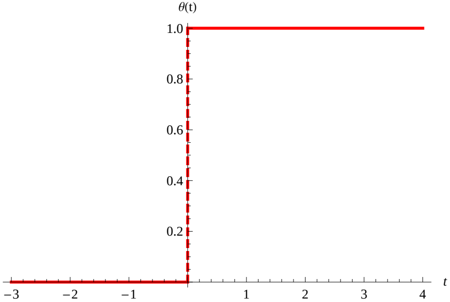
$$\mathcal{L}[1] = \int_0^\infty e^{-st} dt = \lim_{R\to\infty} \int_0^R e^{-st} dt =$$

$$\lim_{R\to\infty} \left[-\frac{e^{-st}}{s} \right]_0^R = \lim_{R\to\infty} \left[\frac{1}{s} - \frac{e^{-sR}}{s} \right] = \frac{1}{s}.$$

- Bear in mind that the expression is well defined because we have that $s > 0$.

- When calculating Laplace transforms, it is useful to use the following function: the Heaviside function. This is usually expressed as $\theta(t)$ and it is defined as:

$$\theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



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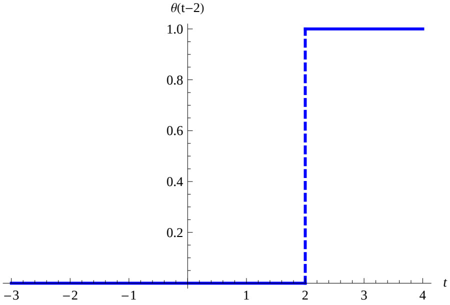
- Clearly

$$\theta(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$$

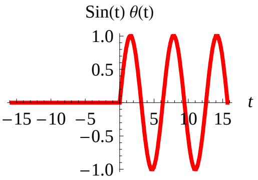
- For example, the function $\theta(t-2)$ will be:

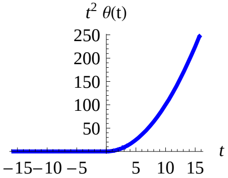


- Note that in order to calculate the Laplace transform of the function $f(t)$, the value of the function at $t < 0$ is not important, so we will always take:

$$f(t) = \theta(t) f(t).$$

- Here are two examples showing the effect of applying the Heaviside function:





- If $s > 0$ we have

$$\mathcal{L}[1] = \mathcal{L}[\theta(t)] = \frac{1}{s}.$$

- The following properties will be useful:
  - $\theta^2(t-a) = \theta(t-a)$,
  - $\theta(t-a)\theta(t-b) = \theta(t-\max(a,b))$.

The $\mathbf{F}(\alpha)$ space

- We need some condition in order for the integral in the Laplace transform to exist
- The function f will be or exponential order α if, given a constant α and the positive constants t_0, M, the following is satisfied

$$e^{-\alpha t}|f(t)| < M \qquad \forall t > t_0.$$

- The functions that satisfy the condition form the $\mathbf{F}(\alpha)$ space.

Exercise 5.2

- Prove that the functions 1, $\sin at$ and $\cos at$ belong to the $\mathbf{F}(0)$ space
- For $f = 1$

$$|1| \leq M \quad \forall M \geq 1, \quad \forall t > 0$$

For $f = \sin at$

$$|\sin at| \leq M \quad \forall M \geq 1, \quad \forall t > 0$$

For $f = \cos at$

$$|\sin at| \leq M \quad \forall M \geq 1, \quad \forall t > 0$$

Therefore, since $\alpha = 0$, then all the above functions belong to $\mathbf{F}(0)$..

Another exercise

- Which is the exponential order of the function $e^{\sin t}$?
- Since $|\sin t| \leq 1$,

$$\frac{1}{e} \leq e^{\sin t} \leq e \quad \forall t > 0,$$

and

$$|e^{\sin t}| \leq e \quad \forall t > 0.$$

Thus, if we choose $M = e$, we find that the exponential order is 0.

And yet another one

- Which is the exponential order of $e^{(1+\cos t)t}$?
- Since $(1+\cos t)t < (1+|\cos t|)t < 2t$, we have that at least for $\forall\, t > 1$ then

$$e^{(1+\cos t)t} \leq e^{2t} \quad \forall t > 1.$$

The exponential order is 2.
(actually, more careful analysis shows that it holds for $\forall\, t > 0.739$).

Exercise 5.4

- Prove that if $f \in \mathbf{F}(\alpha)$ and $g \in \mathbf{F}(\beta)$, then $fg \in \mathbf{F}(\alpha + \beta)$
- We know for f that for some constants M_1 and t_1

$$|f(t)| \leq M_1 e^{\alpha t} \, \forall t > t_1.$$

And for g, given the constants M_2 and t_2

$$|g(t)| \leq M_2 e^{\beta t} \, \forall t > t_2.$$

Then, we will have that

$$|f(t)g(t)| \leq M_1 M_2 e^{\alpha + \beta t} \, \forall t > max(t_1, t_0)$$

so this means that $fg \in \mathbf{F}(\alpha + \beta)$.

- If $f(t) \in \mathbf{F}(\alpha)$ then the following properties will be satisfied:
  - The transform $F(s)$ will be defined for $s > \alpha$
  - The function $sF(s)$ will be bounded at $s \to \infty$, so $\lim_{s\to\infty} F(s) = 0$.
- For example, since $1 \in \mathbf{F}(0)$, the transform $\mathcal{L}[1]$ will be defined for $s > 0$. Moreover, $\lim_{s\to\infty} \mathcal{L}[1] = 0$
  - In this case this is obvious, since we have seen previously that

$$\lim_{s\to\infty} \mathcal{L}[1] = \frac{1}{s}$$

- From now on, we will assume that the function $f(t)$ will belong to the appropriate $\mathbf{F}(\alpha)$.

Exercise 5.5

- Prove the following

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a} \quad \text{if} \quad s > a$$

-

$$\mathcal{L}\left[e^{at}\right] = \int_0^\infty e^{-st} e^{at} dt = \lim_{R\to\infty} \int_0^R e^{(a-s)t} dt$$

$$\lim_{R\to\infty} \left[\frac{e^{(a-s)t}}{a-s}\right]_0^R =$$

$$\left[\frac{e^{(a-s)R}}{a-s} - \frac{1}{a-s}\right] = \frac{1}{s-a} \quad \forall s > a.$$

5.2 Properties
Linearity

- From the linearity of the integral, we get:

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)],$$

  where a and b are constants.
- Therefore, if our differential problem is linear, we will not lose the linearity when performing the transform.

Exercise 5.6

- Use linearity and the transform of the exponential to prove

$$\mathcal{L}[\cosh at] = \frac{s}{s^2 - a^2}, \quad \mathcal{L}[\sinh at] = \frac{a}{s^2 - a^2},$$

$$\text{for } s > |a|$$

without calculating any integral.

- On the one hand

$$\cosh at = \frac{e^{at} + e^{-at}}{2}, \qquad \sinh at = \frac{e^{at} - e^{-at}}{2},$$

and on the other

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \text{ if } s > a,$$

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a} \text{ if } s > -a.$$

- Then,

$$\mathcal{L}[\cosh at] = \frac{\mathcal{L}[e^{at}]}{2} + \frac{\mathcal{L}[e^{-at}]}{2}$$

$$= \frac{1}{2(s-a)} + \frac{1}{2(s+a)} = \frac{s}{s^2 - a^2} \quad \text{for } s > |a|.$$

- In a similar way we have,

$$\mathcal{L}[\sinh at] = \frac{\mathcal{L}[e^{at}]}{2} - \frac{\mathcal{L}[e^{-at}]}{2}$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{a}{s^2 - a^2} \quad \text{for } s > |a|.$$

Theorem of displacement

- If $F(s) = \mathcal{L}[f(t)]$ for $s > \alpha$, then, using the definition of the Laplace transform,

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{-st}e^{at}f(t)dt =$$

$$\int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a) \quad \text{for } s-a > \alpha.$$

- The transform of the product between $f(t)$ and e^{at}, is the translation of the transform of the function $f(t)$.

- Using the result of the previous exercise:

$$\mathcal{L}[e^{at}\cosh bt] = \frac{s-a}{(s-a)^2-b^2}, \quad \text{for } s > |b| + a,$$

$$\mathcal{L}[e^{at}\sinh bt] = \frac{b}{(s-a)^2-b^2}, \quad \text{for } s > |b| + a.$$

- Remember that we are working with the hypothesis that the functions $f(t)$ are zero for $t > 0$.

Exercise 5.8

- Use a change of variables to prove the following: if $F(s) = \mathcal{L}[f(t)]$ for $s > \alpha$, and $\alpha > 0$, then

$$\mathcal{L}[\theta(t-a)f(t-a)] = e^{-as}F(s), \text{ if } s > \alpha.$$

- From the definition:

$$\mathcal{L}[\theta(t-a)f(t-a)] = \int_0^\infty e^{-st}\theta(t-a)f(t-a)dt =$$

$$\int_a^\infty e^{-st}f(t-a)dt.$$

Now, changing variables to $\tau = t - a$

$$\mathcal{L}[\theta(t-a)f(t-a)] = \int_0^\infty e^{-s(\tau+a)}f(\tau)d\tau =$$

$$e^{-sa}\int_0^\infty e^{-s\tau}f(\tau)d\tau = e^{-sa}F(s), \text{ for } s > \alpha.$$

Exercise 5.9

- Prove, without calculating any integral that for $a > 0$:

$$\mathcal{L}[\theta(t-a)] = \frac{e^{-as}}{s}, \text{ for } s > 0.$$

- We know that the transform of $f(t) = 1$ for $s > 0$ is

$$\mathcal{L}[1] = \frac{1}{s} \text{ for } s > 0.$$

  Now, using the result of 5.8,

$$\mathcal{L}[\theta(t-a)] = e^{-sa}/s, \text{ for } s > 0,$$

  But in order for the previous result to be acceptable we need $a > 0$, because for $a < 0$ we have

$$\lim_{s\to\infty} e^{-sa}/s \neq 0.$$

Change of scale
Exercise 5.10

- Let us suppose that when $a > 0$ and $s > \alpha$, then $F(s) = \mathcal{L}[f(t)]$. Prove that

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right), \text{ if } s > \alpha a.$$

- Applying the definition

$$\mathcal{L}[f(at)] = \int_0^\infty e^{-st} f(at)dt.$$

Now, changing variables to $\tau = at$:

$$\mathcal{L}[f(at)] = \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}\tau} f(\tau)d\tau = \frac{1}{a}F\left(\frac{s}{a}\right),$$

if $s > \alpha a$.

Derivatives

- Let us suppose that $f, f', \dots, f^{(n)} \in F(\alpha)$ and $F(s) = \mathcal{L}[f(t)]$ for all $s > \alpha$.
- Supposing that the derivative of f is continuous in $[0, \infty)$, let us calculate the transform of the derivative f':

$$\int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s \int_0^\infty e^{-st} f(t)dt.$$

- Bearing in mind that the exponential order of the function $f(t)$ is α, then $\lim_{s\to\infty} e^{-st} f(t) = 0$ for all $s > \alpha$.
- Thus

$$\mathcal{L}[f'(t)] = sF(s) - f(0), \text{ for } s > \alpha.$$

- By induction, we can get, for $s > \alpha$:

$$\mathcal{L}[f'(t)] = sF(s) - f(0).$$

$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0),$$

$$\vdots$$

$$\mathcal{L}[f^n(t)] = s^nF(s) - s^{n-1}f(0) -$$

$$s^{n-2}f'(0) - \cdots - sf^{n-2}(0) - f^{(n-1)}(0).$$

- We will suppose that $f(0)$, $f'(0)$, $f''(0) \cdots$ exist as right limits

Exercise 5.11

- Use $\mathcal{L}[f'(t)] = sF(s) - f(0)$ to get the transform of e^{at}.
- $$\mathcal{L}[ae^{at}] = \mathcal{L}[\frac{d}{dt}\left(e^{at}\right)] = s\mathcal{L}[e^{at}] - e^{at}|_{t=0} = s\mathcal{L}[e^{at}] - 1.$$

  On the other hand,

  $$\mathcal{L}[ae^{at}] = a\mathcal{L}[e^{at}].$$

  Then,

  $$a\mathcal{L}[e^{at}] = s\mathcal{L}[e^{at}] - 1,$$

  and so

  $$\left(\mathcal{L}[e^{at}](a - s)\right) = -1, \qquad \mathcal{L}[e^{at}] = \frac{1}{s - a} \text{ if } s > a.$$

Exercise 5.12

- Use the definition of the transform, and then induction, to prove:

$$\mathcal{L}[tf(t)] = -F'(s) \quad \text{if } s > \alpha,$$

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s) \quad \text{if } s > \alpha$$

- Bearing in mind that

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

we can take a derivative with respect to t to get

$$F'(s) = -1 \int_0^\infty e^{-st} t f(t) dt$$

- Taking further derivatives

$$F''(s) = (-1)^2 \int_0^\infty e^{-st} t^2 f(t) dt,$$

$$F'''(s) = (-1)^3 \int_0^\infty e^{-st} t^3 f(t) dt,$$

$$\vdots$$

$$F^{(n)}(s) = (-1)^n \int_0^\infty e^{-st} t^n f(t) dt.$$

And then we get:

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s) \quad \text{if } s > \alpha.$$

Exercise 5.13

- Prove (without performing any integral)

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \quad \text{if } s > 0,$$

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}} \quad \text{if } s > \alpha$$

- Bearing in mind that

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s) \quad \text{if } s > \alpha,$$

and in this case $f(t) = 1$, and $\mathcal{L}[f(t)] = 1/s$, then

$$\mathcal{L}[t^n] = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) = \frac{n!}{s^{n+1}} \quad \text{if } s > 0.$$

- Let us study the second case now.

  First we denote $\mathcal{L}[t^n] = F(s)$.
  Using the displacement theorem

$$\mathcal{L}[t^n e^{at}] = F(s-a)$$

  In this last expression, we do the change $s \longrightarrow s-a$ to get:

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}.$$

- Remember that

$$\mathcal{L}\left[\theta(t)f(t)\right] = \mathcal{L}[f(t)] = F(s)$$

then, it is convenient to write this in the most general way as

$$\mathcal{L}[\theta(t)tf(t)] = -F'(s)$$

$$\mathcal{L}[\theta(t)t^n f(t)] = (-1)^n F^{(n)}(s)$$

$$\mathcal{L}[\theta(t)t^n] = \frac{n!}{s^{n+1}} \quad \text{if } n = 1, 2, \cdots$$

$$\mathcal{L}[\theta(t)t^n e^{at}] = \frac{n!}{(s-a)^{n+1}} \quad \text{if } n = 1, 2, \cdots$$

- Some other useful transforms (Use the book!)

$$\mathcal{L}[\theta(t)\sin(at)] = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}[\theta(t)t\sin(at)] = \frac{2as}{(s^2 + a^2)^2}$$

$$\mathcal{L}[\theta(t)\cos(at)] = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}[\theta(t)t\cos(at)] = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

5.3 Inverse transform

- There is a general formula to get the inverse transform, but we will mostly do it by inspection
- The inverse transform is denoted as:

$$\mathcal{L}^{-1}[F(s)] = f(t).$$

- We will also use that it is linear:

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)].$$

- Let us try one example:

$$F(s) = \frac{1}{s(s+1)^2}.$$

We can decompose it in simple fractions

$$\frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}.$$

Using tables and the theorem of displacement we can get:

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = \theta(t), \quad \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = \theta(t)e^{-t}.$$

$$\mathcal{L}^{-1}\left[-\left(\frac{1}{s+1}\right)^2\right] = -\theta(t)te^{-t}.$$

The final answer being

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)^2}\right] = \theta(t)\left(1-(1+t)e^{-t}\right).$$

Exercise 5.15

- Find the inverse transform of the following functions:

$$F(s) = \frac{s}{s^3 - s^2 - s + 1}$$

$$F(s) = \frac{2s}{(s^2 + 1)^2}$$

- For the first case we can decompose it as

$$\frac{s}{s^3 - s^2 - s + 1} = \frac{1}{2(s-1)^2} + \frac{1}{4(s-1)} - \frac{1}{4(s+1)}$$

We know that:

$$\mathcal{L}[e^{at}t^n] = \mathcal{L}[\theta(t)e^{at}t^n] = \frac{n!}{(s-a)^{n+1}}.$$

Then

$$\frac{1}{(s-1)^2} = \mathcal{L}[\theta(t)e^t t], \ \frac{1}{(s-1)} = \mathcal{L}[\theta(t)e^t],$$

$$\frac{1}{(s+1)} = \mathcal{L}[e^{-t}].$$

and,

$$F(s) = \frac{1}{2(s-1)^2} + \frac{1}{4(s-1)} - \frac{1}{4(s+1)} =$$

$$\frac{1}{2}\mathcal{L}[\theta(t)e^t t] + \frac{1}{4}\mathcal{L}[\theta(t)e^t] - \frac{1}{4}\mathcal{L}[\theta(t)e^{-t}],$$

The solution is then

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{4}\theta(t)(e^t(2t+1) - e^{-t}).$$

- The second case is easier:

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\frac{d}{ds}(-\frac{1}{s^2+1})] =$$

$$\mathcal{L}^{-1}[\frac{d}{ds}\mathcal{L}[-\sin t]] = t \sin t.$$

5.6 Linear equations with constants coefficients

- Due to the properties of the Laplace transform, the derivatives become multiplications
- We can use that property to solve the initial value problem of a differential equation (or system of equations) with constant coefficients using the following method:
  1. Calculate the Laplace transform of the differential problem
  2. Solve the algebraic problem
  3. Find the inverse transform
- Laplace transforms are specially useful when the inhomogeneous term is defined in parts

Exercise 5.23

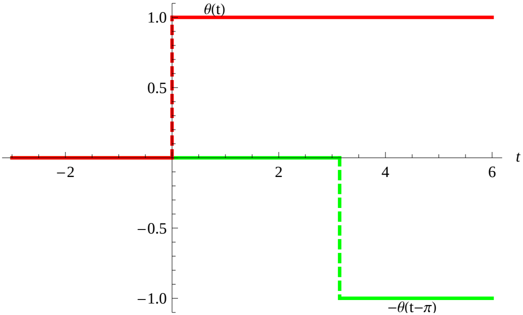
- LEt us consider the following differential equation

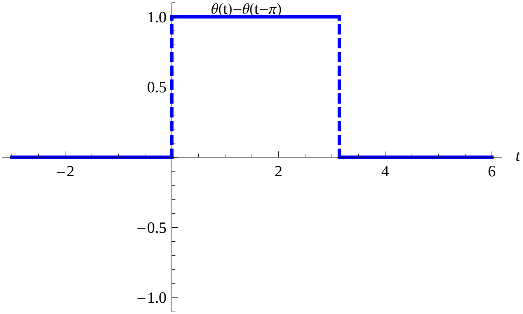
$$\ddot{x} + x = \begin{cases} 1, & \text{if } 0 < t < \pi, \\ 0, & \text{if } t > \pi, \end{cases} \qquad x(0) = \dot{x}(0) = 0.$$

  Show that the inhomogeneous term is $\theta(t) - \theta(t - \pi)$ and solve the problem. Is the solution continuous?

- Let us study the following relation with a graphic:

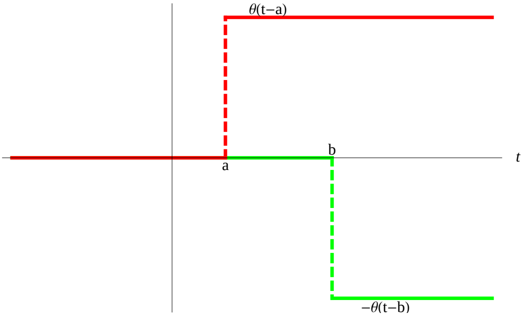
$$\theta(t) - \theta(t - \pi) = \begin{cases} 1, & \text{if } 0 < t < \pi, \\ 0, & \text{if } t > \pi. \end{cases}$$

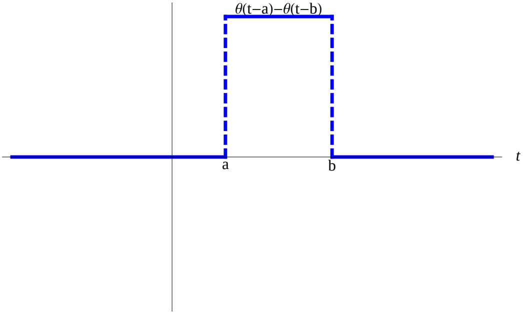

θ(t)
1.0
0.5
−0.5
−1.0
−2
2
4
6
t
−θ(t−π)


θ(t)−θ(t−π)
1.0
0.5
−0.5
−1.0
−2
2
4
6
t

- In general we will have:

$$\theta(t-a)-\theta(t-b)=\begin{cases}1, & \text{if } a<t<b,\\ 0, & \text{otherwise.}\end{cases}$$


θ(t−a)
a
b
t
−θ(t−b)


θ(t−a)−θ(t−b)
a
b
t

- Let us solve the problem now

  If we call $X(s) = \mathcal{L}[x[t]]$.
  Then

$$\mathcal{L}[\ddot{x}[t]] = s^2 X(s) - sx(0) - \dot{x}(0) = s^2 X(s),$$

  where we have used $x(0) = \dot{x}(0) = 0$.

- The equation to solve is

$$\ddot{x} + x = \theta(t) - \theta(t - \pi).$$

  Then,

$$\mathcal{L}[\ddot{x} + x] = \mathcal{L}[\theta(t) - \theta(t - \pi)].$$

  Using previous results we get:

$$\mathcal{L}[\ddot{x} + x] = s^2 X(s) + X(s) = (s^2 + 1)X(s)$$

- On the other hand

$$\mathcal{L}[\theta(t)] = \frac{1}{s}, \quad \mathcal{L}[\theta(t-\pi)] = \frac{e^{-\pi s}}{s}.$$

The transform then reads:

$$(s^2+1)X(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s} = \frac{1-e^{-\pi s}}{s}$$

$$X(s) = \frac{1-e^{-\pi s}}{s(s^2+1)} = \left(1-e^{-\pi s}\right)\left(\frac{1}{s} - \frac{s}{s^2+1}\right).$$

- We need to calculate the inverse transform now.
  - Remembering the following relation:

$$\mathcal{L}[\theta(t-a)f(t-a)] = e^{-as}F(s)$$

defining $F(s) = \mathcal{L}[f(t-a)]$.
It can be seen that:

$$X(s) = \left(1 - e^{-\pi s}\right)\left(\mathcal{L}[\theta(t)] - \mathcal{L}[\theta(t)\cos t]\right) =$$

$$\left(\mathcal{L}[\theta(t)] - \mathcal{L}[\theta(t)\cos t]\right) - e^{-\pi s}\left(\mathcal{L}[\theta(t)] - \mathcal{L}[\cos t]\right) =$$

$$\mathcal{L}[\theta(t)] - \mathcal{L}[\theta(t)\cos t] -$$

$$\mathcal{L}[\theta(t-\pi)] - \mathcal{L}[\theta(t-\pi)\cos(t-\pi)] =$$

$$\mathcal{L}[\theta(t)(1-\cos t)] - \mathcal{L}[\theta(t-\pi)(1-\cos(t-\pi))].$$

- Then

$$X(s) = \mathcal{L}[x(t)] = \mathcal{L}[\theta(t)(1 - \cos t) -$$

$$\theta(t - \pi)(1 - \cos(t - \pi))],$$

and the solution is

$$x(t) = \theta(t)(1 - \cos t) - \theta(t - \pi)(1 - \cos(t - \pi)).$$

- In order to check the the continuity, we can write it as

$$x(t) = \begin{cases} 1 - \cos t & \pi > t > 0 \\ -2\cos t & t > \pi \end{cases}.$$

And as this is satisfied:

$$\lim_{t \to \pi^-} = 1 - \cos\pi = 2, \quad \lim_{t \to \pi^+} = -2\cos\pi = 2.$$

The function is continuous

February 2008

- Solve the following initial value problem:

$$\ddot{x} + 2\dot{x} - 3x = \begin{cases} t & 0 < t < 1\,, \\ 0 & t > 1\,, \end{cases} \qquad x(0) = \dot{x}(0) = 0.$$

- The problem can be rewritten as:

$$\ddot{x} + 2\dot{x} - 3x = t[\theta(t) - \theta(t-1)] =$$

$$t\theta(t) - (t-1)\theta(t-1) - \theta(t-1)$$

Taking its Laplace transform and bearing in mind $x(0) = \dot{x}(0) = 0$, one gets

$$(s^2 + 2s - 3)X(s) = (s+3)(s-1)X(s) =$$

$$\frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right).$$

- We can solve for $X(s)$ now:

$$X(s) = \frac{1-e^{-s}}{s^2(s+3)(s-1)} - \frac{e^{-s}}{s(s+3)(s-1)} =$$

$$(1-e^{-s})\left(\frac{1}{4(s-1)} - \frac{1}{36(s+3)} - \frac{2}{9s} - \frac{1}{3s^2}\right) -$$

$$e^{-s}\left(\frac{1}{4(s-1)} + \frac{1}{12(s+3)} - \frac{1}{3s}\right)$$

- To solve the problem, we need another "recipe". Remember the relation

$$\mathcal{L}[e^{at}f(t)] = F(s-a) \text{ if } \mathcal{L}[f(t)] = F(s)$$

Let us define

$$g(t) = e^{at}f(t) \text{ with } F(s-a) = G(s).$$

Using that

$$\mathcal{L}[\theta(t-b)g(t-b)] = e^{-bs}G(s) \text{ if } \mathcal{L}[g(t)] = G(s)$$

Then

$$\mathcal{L}[\theta(t-b)g(t-b)] = \mathcal{L}[\theta(t-b)e^{a(t-b)}f(t-b)] =$$

$$e^{-bs}F(s-a).$$

- So what is the value of the following ?

$$\mathcal{L}^{-1}[\frac{e^{-bs}}{s-a}].$$

In this case we have $f(t) = \theta(t)$ and then

$$\mathcal{L}^{-1}[\frac{e^{-bs}}{s-a}] = \theta(t-b)e^{a(t-b)}\theta(t-b) = \theta(t-b)e^{a(t-b)},$$

since $\theta^2(t-b) = \theta(t-b)\ \forall b.$

- So using the inverse transform one gets:

$$x(t) = \theta(t)\left(\frac{e^t}{4} - \frac{e^{-3t}}{36} - \frac{t}{3} - \frac{2}{9}\right) -$$

$$\theta(t-1)\left(\frac{e^{t-1}}{2} + \frac{e^{-3(t-1)}}{18} - \frac{t-1}{3} - \frac{5}{9}\right).$$

September 2008

- Solve the following initial value problem:

$$\ddot{x} + x = \begin{cases} \cos t & 0 < t < \pi\,, \\ 0 & t > \pi\,, \end{cases} \qquad x(0) = \dot{x}(0) = 0.$$

- Rewriting the problem as:

$$\ddot{x} + x = (\theta(t) - \theta(t-\pi))\cos t =$$

$$\theta(t)\cos t + \theta(t-\pi)\cos(t-\pi)$$

Taking its Laplace transform and using $x(0) = \dot{x}(0) = 0$:

$$(s^2+1)X(s) = \frac{s}{s^2+1}(1 + e^{-\pi s}).$$

$$X(s) = \frac{s(1+e^{-\pi s})}{(s^2+1)^2} = -\frac{(1+e^{-\pi s})}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right).$$

- The inverse transform gives:

$$x(t) = \frac{1}{2}\left\{\theta(t)(t\sin t) + \theta(t-\pi)[(t-\pi)\sin(t-\pi)]\right\} =$$

$$\frac{1}{2}[\theta(t)t - \theta(t-\pi)(t-\pi)]\sin t,$$