Ordinary differential equations
3rd topic

Higher-order equations

3.1 Geometric interpretation, 3.2 Existence-uniqueness theorem, 3.3 Equivalence between equation and systems, 3.4 Lowering the order, 3.5 Linear dependency of functions, 3.6 Linear differential equations, 3.7 Linear homogeneous equations, 3.8 Complete linear equations

3.1 Geometric meaning

- The geometric meaning of higher-order equations is a generalization of the first-order case
- Let us suppose that the equation of a family of flat curves depends on the parameters $C_1, C_2, \ldots, C_n$
  - The finite equation and its n derivatives will be the following:

$$
\begin{aligned}
\varphi(x, y, C_1, C_2, \ldots, C_n) &= 0, \\
\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y} y' &= 0, \\
\frac{\partial^2\varphi}{\partial x^2} + 2\frac{\partial^2\varphi}{\partial x \partial y} y' + \frac{\partial^2\varphi}{\partial y^2} y'^2 + \frac{\partial\varphi}{\partial y} y'' &= 0, \\
&\vdots \\
\frac{\partial^n\varphi^n}{\partial x^n} + \cdots + \frac{\partial\varphi}{\partial y} y^{(n)} &= 0,
\end{aligned}
$$

- It is possible to eliminate the n parameters using all those equations, to obtain the differential equation for the family

$$F(x, y, y', y'', \dots, y^{(n)}) = 0.$$

- The equation for the family of curves $\varphi(x, y, C_1, C_2, \dots, C_n) = 0$ is the solution to the differential equation, and each of the curves is an integral curve

Exercise 3.1

- Which is the differential equation of the unit circles?
- Denoting $C_1 = a$ and $C_2 = b$, we get $\varphi(x, y, C_1, C_2) = (x-a)^2 + (y-b)^2 - 1 = 0$. We need two more equations

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} y' = 2(x-a) + 2(y-b)y' = 0$$

and

$$\frac{\partial^2 \varphi}{\partial x^2} + 2\frac{\partial^2 \varphi}{\partial x \partial y} y' + \frac{\partial^2 \varphi}{\partial y^2} y'^2 + \frac{\partial \varphi}{\partial y} y'' =$$

$$2 + 0 + 2y'^2 + 2(y-b)y'' = 2(1 + y'^2) + 2(y-b)y'' = 0$$

The first derivative gives $(x-a)^2 = (y-b)^2(y')^2$ and combining this result with the finite equation we get $(y-b)^2(1+y'^2) = 1$

The finite equation and its first derivative give

$$\frac{1}{1+y'^2} = (y-b)^2.$$

The second derivative gives

$$(1+y'^2)^2 = (y-b)^2 y''^2.$$

Therefore, the equation we are after is the following

$$(1+y'^2)^3 = y''^2.$$

3.2 Existence-uniqueness theorem

- Let us write a differential equation in its normal form:

  $$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

  If the function f and $\partial f/\partial y$, $\partial f/\partial y'$, $\partial f/\partial y^{(n-1)}$ are continuous, and the equation has n initial conditions

  $$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y'_0 \\ &\vdots \\ y^{(n-1)}(x_0) &= y_0^{(n-1)} \end{aligned}$$

  then there exists a single solution to the equation satisfying the initial conditions.

3.3 Equivalence between equation and systems

- By adding variables and equations, we can always lower the order of an equation
- In fact, if we define new variables such as

$$y_1 \equiv y,\ y_2 \equiv y', \ldots, y_n \equiv y^{(n-1)},$$

the n-th order equation $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$ is equivalent to a system of n equations

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ &\vdots \\ y_n' &= f(x, y_1, y_2, \ldots, y_n). \end{aligned}$$

Exercise 3.2

- Write the equation of a forced oscillator as a system
- Denoting the dependent variable as x, and the independent variable as t, the equation for the oscillator is:

  $$\ddot{x} + \omega^2 x = \frac{F(x)}{m}.$$

  In order to write it as a system, we will introduce $x_1 = x$ and $x_2 = \dot{x}$.
  Thus, our system will be

  $$\dot{x}_1 = x_2,$$

  $$\dot{x}_2 = \frac{F(x_1)}{m} - \omega^2 x_1,$$

  .

3.4 Lowering the order

- There are not many procedures to solve higher-order equations
  - One possibility might be to lower the order
  - In some specific cases that can be done in a systematic way

Equations with the dependent absent

- Let us suppose that the equation can be written as $F(x, y', y'', \ldots, y^{(n)}) = 0$
  - It is then desirable to define $u \equiv y', u' \equiv y'', \ldots, u^{(n-1)} \equiv y^n$.
  - The differential equation obtained is of one order lower

$$F(x, u, u', \ldots, u^{(n-1)}) = 0.$$

- If the solution of the new equation is $\tilde{\varphi}(x, u, C_1, \ldots, C_{n-1}) = 0$
  - Then, undoing the change of variable $u = y'$ we obtain a family of first order differential equations $\tilde{\varphi}(x, y', C_1, \ldots, C_{n-1}) = 0$,
  - By solving this last equation, we get the general solution: $\varphi(x, y, C_1, \ldots, C_n) = 0$.

- Moreover, every singular solution of $F(x, u, u', \ldots, u^{(n-1)}) = 0$ will give us another first order differential equation. Its solutions will be singular solutions for the original equation.

- Of course, if on top of a missing y variable, also higher derivatives $y', \ldots, y^{(m-1)}$ are missing, the change $u \equiv y^m$ will lower the equation to an equation of order $n - m$

- Let us make an example: $y''^2 = 240x^2y' = 0$

  As there is no y in the equation, let us define $u = y'$ to obtain $u' = \pm\sqrt{240}x\sqrt{u}$. Thus, we need to solve the integral $du/\sqrt{u} = \pm\sqrt{240}xdx$

  Its solution is $4\sqrt{u} = \sqrt{240}(x^2 + C_1) = \sqrt{15 \times 16}(x^2 + C_1)$, and undoing the change, we obtain the following equation: $y' = 15(x^2 + C_1)^2$.

  Expanding and integrating, we obtain the general solution $y = 3x^5 + 10C_1x^3 + 15C_1x + C_2$

  On the other hand, we cannot forget the singular solution $y' = 0$, which gives also the family of curves $y = C_3$

Exercise 3.3

- A point-like particle is falling down a vertical straight line due to gravity. If friction is proportional to the velocity, which is the velocity at every time step? Prove that there is a limiting velocity.
- The equation describing the velocity is $m\ddot{z} = -mg - k\dot{z}$. This, it is convenient to use $v = \dot{z}$ to rewrite the equation as $\dot{v} = -g - kv/m$.
  By direct integration
  $$v = -\frac{gm}{k} + Ce^{-\frac{k}{m}t}$$
  and clearly there is a limiting velocity
  $$\lim_{t\to\infty} v = -gm/k.$$
  Finally, undoing he change $\dot{z} = v$ and integrating, we obtain $z = -\frac{m}{k}\left(gt + Ce^{-\frac{k}{m}}\right)$

Autonomous Equations

- If the dependent variable is absent in the differential equation, it is called autonomous: $F(y, y', y'', \dots, y^{(n)}) = 0$.
  - Then, if $\varphi(x, y)$ is a solution, $\varphi(x - x_0, y)$ is also a solution for all x_0.
  - Therefore, one of the constants corresponds to where to set the origin, and the general solution is the following:

$$\varphi(x - x_0, y, C_1, \dots, C_{n-1}) = 0.$$

- one can use $u \equiv y'$ to lower the order, and thus

$$\begin{aligned} y'' &= \frac{du}{dy} u \\ y''' &= \frac{d^2u}{dy^2} u^2 + \left(\frac{du}{dy}\right)^2 u \\ &\vdots \\ y^{(n)} &= \frac{d^{n-1}u}{dy^{n-1}} u^{n-1} + \cdots + \left(\frac{du}{dy}\right)^{n-1} u \end{aligned} \tag{1}$$

- Substituting these in the original equation, we obtain a new equation of order $n-1$:

$$F(y, u, du/dy, d^2u/dy^2, \ldots, d^{n-1}u/dy^{n-1}) = 0.$$

- If the general solution to this equation is given by $\tilde{\varphi}(y, u, C_1, \ldots, C_{n-1}) = 0$, then, the change of variable $u = y'$ we obtain a new equation
- By solving this last equation, we get the general solution $\varphi(x - x_0, y, C_1, \ldots, C_{n-1}) = 0$.

Exercise 3.4

- Solve $y'' = (2y+1)y'$
- As it is autonomous, we will take $u \equiv y'$ (in order to simplify notation, we will take $\dot{u} = du/dy, \ddot{u}d^2u/dy^2, \dots$)
  As seen before, $y'' = udu/dy = u\dot{u}$, so the equation to solve reads
  $$\dot{u}u = (2y+1)u = 0.$$
  Clearly $u = \int(2y+1)dy = y^2 + y + C_1$ (Note that we have lost the solution $u = 0$).
  Undoing the change of variables we get $y' = y^2 + y + C$ and integrating [Spiegel, pg. 263, 1.12.1]:
  $$\frac{2}{\sqrt{4C_1 - 1}} \arctan\left(\frac{1+2y}{\sqrt{4C_1 - 1}}\right) = x + C_2.$$

Equidimensional-in-x differential equations

- These equations are invariant under $x \longrightarrow ax$

$$F(ax, y, a^{-1}y', a^{-2}y'', \ldots, a^{-n}y^{(n)}) =$$

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0.$$

  - This can be transformed into autonomous equations by $x \longrightarrow t \equiv \ln x$. Thus

$$\begin{aligned} x &= e^t \\ y' &= \frac{1}{x}\dot{y} \\ y'' &= \frac{1}{x^2}(\ddot{y} - \dot{y}) \\ &\vdots \\ y^{(n)} &= \frac{1}{x^n}\left[\frac{d^n y}{dt^n} + \cdots + (-1)^{n-1}(n-1)!\frac{dy}{dt}\right]. \end{aligned}$$

- It can be seen that the original equation $F(x, y, y', y'', \dots) = 0$ is equivalent to $F(1, y, \dot{y}, \ddot{y} - \dot{y}, \dots) = 0$.
- Since the last equation is autonomous, it is convenient to use $u \equiv \dot{y}$ to find its solution. This will give us a new first order equation that needs to be solved.

Exercise 3.5

- Solve $xy'' = yy'$
- Bearing in mind $y' = \dot{y}/x$ and $y'' = (\ddot{y} - \dot{y})/x^2$, the equation reads:

$$x\left[\frac{1}{x^2}(\ddot{y} - \dot{y})\right] = \frac{y\dot{y}}{x}.$$

This is equivalent to $\ddot{y} = (1 + y)\dot{y}$ (careful! Remember to check $y = 0$).
But this is a known result, since the changes $y \to 2y$ and $x \to t$ take us to the equation solved in exercise 3.4
.
By using that solution, and undoing $2y \to y$ and $t \to x$
, the solution to the new equation is

$$\frac{2}{\sqrt{4C_1 - 1}} \arctan\left(\frac{1 + y}{\sqrt{4C_1 - 1}}\right) = t + C_2.$$

There is also the singular solution $y = C_3$.

Equidimensional-in-y differential equations

- This are invariant under the scaling $y \longrightarrow ay$

$$F(x, ay, ay', ay'', \dots, ay^{(n)}) =$$

$$F(x, y, y', y'', \dots, y^{(n)}) = 0.$$

- they can be turned into autonomous equations by $u = y'/y$
  Then

$$\begin{aligned} y' &= yu, \\ y'' &= y(u' + u^2), \\ &\vdots \\ y^{(n)} &= y(u^{(n-1)} + \cdots + u^n). \end{aligned}$$

- The original equation $F(x, y, y', y'', \dots) = 0$ is equivalent to $F(x, 1, u, u', u' + u^2 \dots) = 0$
- Solving this last one and undoing $u = y'/y$ we get a first order equation. Solving this one we obtain the solution.

Exercise 3.6

- Solve $yy'' = y'^2$
- Since it is a second order equation, we have to use $y' = yu$ and $y'' = y(u' + u^2)$
  We then obtain $y(y(u' + u^2)) = (yu)^2$, and simplifying the factors of y, we are losing the solution $y = 0$
  The new equation reads $u' = 0$, which is directly integrable to $u = C_1$
  Undoing the change of variables, we get $y' = C_1 y$, and the general solution is then

$$\ln y = C_1 x + C_2, \quad y = e^{C_1 x + C_2}.$$

  Note that the solution $y = 0$ is not really lost, since it is recovered in the limit $C_2 \to -\infty$. So $y = 0$ is not a singular solution, it is a particular solution.

- Let us supposed that the equation is an exact derivative

$$F(x, y, y', \dots, y^{(n)}) = \frac{d}{dx} G(x, y, y', \dots, y^{(n-1)}) = 0.$$

- Then, the quadrature

$$G(x, y, y', \dots, y^{(n-1)}) = C$$

will give as a first integral.

- The quadrature is a differential equation of order $n - 1$, and this will be the new equation to be solved.

Exercise 3.7

- Solve $yy'' + y'^2 = 0$
- It can be seen "by eye" that

$$yy'' + y'^2 = \frac{d}{dx}(yy') = 0.$$

therefore, $yy' = C_1$ is a first integral.
Lastly, we can get the solution

$$y^2 = 2C_1 x + C_2$$

- Sometimes, an equation that is not exact can be made exact by an integrating factor or by suitable transformations.

Exercise 3.7

- Show that the equation $yy'' - y'^2$ can be made exact by dividing it by y^2. Is there any singular solution?
- One could guess that

$$\frac{yy'' - y'^2}{y^2} = \frac{d}{dx}\left(\frac{y'}{y}\right) = 0.$$

Then, $y'/y = C_1$ is a first integral. Solving it we get the general solution:

$$\ln y = C_1 x + C_2,$$

$$y = e^{C_1 x + C_2}$$

In principle the solution $y = 0$ could have been lost, but it is part of the general solutions in the limit $C_2 \to -\infty$

3.5 Linear dependency of functions

- The group of solutions of any homogeneous linear differential equation is a linear vector space.
  - The usual addition and multiplication operators of functions induce that structure
  - The space is the subspace of the infinite dimensional space of regular functions
- Let us use $y_k(x)$ to denote any solution
- As we know, if all functions $y_k(x)$ are solutions to our linear equations then $y = \sum_{k=1}^{\infty} c_k y_k(x)$ is also a solution, but as we will see later, it can be written in an easier way

- If the domain of definition for the functions is I

  $$\{y_k(x) : k = 1, \ldots, n; x \in I\}$$

  the regular functions will be linearly independent if and only if

  $$\sum_{k=1}^{\infty} c_k y_k(x) = 0 \quad (\forall x \in I)$$

  holds only for the case $c_1 = c_2 = c_3 = \cdots = c_n = 0$.
- For example, the set of powers $1, x, x^2, \ldots, x^n$ is linearly independent in any domain
  - Let us see this: If the coefficients of the polynomial $c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^n = 0$ are not all zero, then the polynomial will only be zero in its roots.
  - But since there are at most n roots, the roots cannot feel all the domain

- In order to study the linear dependency, it is useful to use the Wronskian:
- For the functions $\{y_k(x) : k = 1, \ldots, n; x \in I\}$ the Wronskian is defined as

$$W[y_1, \ldots, y_n] = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \ldots & \ldots & \ddots & \ldots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}$$

- On the other hand, if the set $y_k(x) : k = 1, \ldots, n; x \in I$ is linearly dependent, it is possible to find a set of constants $c_1, c_2, \ldots, c_n$ not all zero for which $\sum_{k=1}^{\infty} c_k y_k(x) = 0$ for all points x in its definition domain.
- But the first $n-1$ derivatives of that equation will also be zero in all the domain. Therefore:

$$
\begin{array}{cccc}
c_1 y_1(x) n+ & c_2 y_2(x)+ & \cdots + & c_n y_n(x) = 0, \\
c_1 y_1'(x)+ & c_2 y_2'(x)+ & \cdots + & c_n y_n'(x) = 0, \\
\vdots & \vdots & \vdots & \vdots, \\
c_1 y_1^{(n-1)}(x)+ & c_2 y_2^{(n-1)}(x)+ & \cdots + & c_n y_n^{(n-1)}(x) = 0.
\end{array}
$$

- The previous system of equations defines a linear homogeneous system with unknowns c_k at every point x
  - since we are assuming that there is linear dependency, the solutions to this system are not zero
  - Therefore, in all points of the domain, the determinant of the system (the Wronskian) is not zero
- This is the main conclusion: the Wronskian of a linearly dependent set of functions is zero for all points in its definition domain..
- This is way if the Wronskian is not identically zero, the functions will be linearly independent

Exercise 3.10

- Show that $1, x, x^2, \dots, x^n$ are independent using the Wrosnkian
- It is clear that:

$$W = \begin{vmatrix} 1 & x & x^2 & x^3 & \dots & x^n \\ 0 & 1 & 2x & 3x^2 & \dots & nx^{n-1} \\ 0 & 0 & 2 & 6x & \dots & n(n-1)x^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & n! \end{vmatrix},$$

but $W = 1 \times 2 \times 6 \cdots \times n! \neq 0$ so they are linearly independent

Exercise from Feb-03 exam

- Discuss whether the functions $x - 2$, $x^3 - x$ and $6x^3 - 3x - 6$ are linearly independent in the real line.
- We can answer this by studying the linear combination

$$c_1(x-2) + c_2(x^3 - x) + c_3(6x^3 - 3x - 6) = 0$$

in three different points
For example, the points $x = 0$, $x = 2$, $x = -1$ give the following system:

$$-2c_1 - 6c_3 = 0,$$

$$6c_2 + 36c_3 = 0,$$

$$-3c_1 - 4c_2 - 6c_3 = 0.$$

- From the second equation $c_2 = -6c_3$, and substituting in the third we get $-3c_1 - 3c_2 = 0$, so $c_1 = -c_2$.
  The first, in turn, gives $2c_1 = -6c_3$, but using the previous result $2c_1 = c_2$
  Combining all relations we get $c_1 = 0 = c_2 = c_3$, and there is no linear dependency.

3.6 Linear differential equations

- These equations can be written as:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n y = b(x).$$

  - Dividing the equation by a_0 the only thing that changes is the definition domain
  - In general, we will take $a_0 = 1$
  - Moreover, we will take $a_1, a_2, \ldots, a_n$ and b to be continuous in the domain I (when $b = 0$ the equation will be homogeneous, and not-homogeneous otherwise).

- We may also use the following to operators to make the notation easier

$$D \equiv \frac{d}{dx},$$

$$L \equiv D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x).$$

- Thus, the operator L will act upon the functions $f(x)$ which are defined over the domain I:

$$\begin{aligned}(Lf)(x) &= f^{(n)}(x) + a_1(x)f^{(n-1)}(x) + \cdots + \\ &\quad a_{n-1}(x)f'(x) + a_n f(x).\end{aligned}$$

- Linear non-homogeneous equations can be written as: $Ly = b$.
- Besides, the operator is a linear operator

$$L(c_1 f_1 + c_2 f_2) = c_1 L f_1 + c_2 L f_2$$

for any constants c_1 and c_2

Exercise

- Write the equation for a harmonic oscillator with frequency ω using the operator D
- With the usual notation

$$\ddot{x} + \omega^2 x = 0,$$

Using the operator D

$$D^2 x + \omega^2 x = 0,$$

Therefore, we will have $L = D^2 + \omega^2$, and using L, the original equation can be written as $Lx = 0$

3.7 Homogeneous linear equations

- For homogeneous linear equations we have

$$Ly = 0.$$

- Besides, the principle of superposition and the linearity of the operator L are equivalent. If we use y_k to represent the solutions to the homogeneous equation we get

$$Ly_k = 0 \Rightarrow L\sum_{k=1}^{\infty} c_k y_k = \sum_{k=1}^{\infty} c_k L y_k = 0.$$

  - The previous results proves that the set of solutions of a homogeneous linear equation forms a vector space
  - The dimension of the vector space is related to the Wronskian

3.1 Theorem

Let us consider n solutions for an n dimensional linear homogeneous equation defined in the domain I: $Ly_k = 0$. The following three sentences are equivalent:

1. The functions y_k are linearly dependent in I.
2. The Wronskian for y_k is identically zero in I.
3. The Wrosnkian for y_k is zero in one point $x_0 \in I$.

- On the other hand, the dimension for the solution-space for a linear homogeneous equation of order n cannot be less than n.
- To see this, we need to use the existence&uniqueness theorem, which says that for initial conditions given by

$$\begin{aligned} y_1(x_0) &= 1 \\ y_1'(x_0) &= 0 \\ &\vdots \\ y_1^{(n-1)}(x_0) &= 0 \end{aligned}$$

  there is a unique solution.
- We can construct similar n initial value problems:

$$\begin{array}{cccc} y_1(x_0) = 1 & y_2(x_0) = 0 & \dots & y_n(x_0) = 0, \\ y_1'(x_0) = 0 & y_2'(x_0) = 1 & \dots & y_n'(x_0) = 0, \\ \vdots & \vdots & \vdots & \\ y_1^{(n-1)}(x_0) = 0 & y_2^{(n-1)}(x_0) = 0 & \dots & y_n^{(n-1)}(x_0) = 1. \end{array}$$

- Due to the uniqueness&existence, the solutions to each of those initial conditions are different
- Therefore, their Wrosnkian is not zero
- We have thus constructed n linearly independent solutions to our linear homogeneous equation. But the number of such constructions is infinite (for example, choosing a constant $C \neq 0$ instead of 1 in each one).
- The set of n solutions to an n order linear homogeneous equation is called the fundamental system of solutions.

3.2 theorem

If we choose n linearly dependent solutions (y_k) for a homogeneous linear equation of order n, then any other solution can be written in a unique way as a linear combination of constant coefficients of the solutions (y_k).

- For example, for the equation $y'' + \omega^2 y = 0$ we have the following as fundamental system of solutions: $\{\cos \omega x, \sin \omega x\}$. Their Wronskian is ω and the general solution is $y = A \cos \omega x + B \sin \omega x$. This can be written in many other ways.

Exercise 3.13

- Show that the set $\{1, e^x, e^{-x}\}$ is a fundamental system for the equation $y''' - y' = 0$. Find another fundamental system. Write the general solution using both systems and check that they are equivalent.
- The Wronskian of the fundamental system is:

$$W = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2.$$

Since it is not zero, the system is independent.
now we have to show that any given linear combination is a solution of the differential equation:

$$y = A + Be^x + Ce^{-x}, \; y' = Be^x - Ce^{-x}$$

$$y'' = Be^x + Ce^{-x} = y, \; y''' = Be^x - Ce^{-x} = y'.$$

Thus, since it is a solution, we have shown that $\{1, e^x, e^{-x}\}$ is a fundamental system.

- We can guess that the set $\{1, \sinh x, \cosh x\}$ is a good candidate. The Wronskian is

$$W = \begin{vmatrix} 1 & \sinh x & \cosh x \\ 0 & \cosh x & \sinh x \\ 0 & \sinh x & \cosh x \end{vmatrix} = \cosh^2 x - \sinh^2 x = 1.$$

The equivalence between both systems is clear

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

- Let us study more closely the link between fundamental systems and linear equations:
- Each fundamental systems corresponds to a single linear homogeneous equation (at least if $a_0 = 1$ in the equation)
  - Let us imagine that a set of n functions is the fundamental system of two operators L_1 and L_2:

$$L_1 y_k = y_k^n + a_1(x) y^{(n-1)} + \cdots + a_{n-1}(x) y'(x) + a_n(x) y = 0,$$

$$L_2 y_k = y_k^n + \tilde{a}_1(x) y^{(n-1)} + \cdots + \tilde{a}_{n-1}(x) y'(x) + \tilde{a}_n y = 0.$$

  - Then, the set is also a fundamental system for the operator $L_1 - L_2$: $L_1 y_k - L_2 y_k = (L_1 - L_2) y_k = 0$.
  - But, the order of the operator $L_1 - L_2$ is $n - 1$:

$$(L_1 - L_2) y_k = \cancel{(y_k^n - y_k^n)} + (a_1(x) - \tilde{a}_1(x)) y^{(n-1)} + \cdots +$$
$$(a_{n-1}(x) - \tilde{a}_{n-1}(x)) y'(x) + (a_n(x) - \tilde{a}_n(x)) y = 0.$$

  - Thus,the operator $L_1 - L_2$ of order $n - 1$ admits a fundamental system of order n. Since that is impossible, $L_1 - L_2$ has to be the null-operator, so $L_1 = L_2$

- It is easy to construct the equation that corresponds to a fundamental system.
  - If the system is $\{y_1, \ldots, y_n\}$, any other solution to the equation will be written as a linear combination of these.
  - Thus, the system $y, y_1, \ldots, y_n$ and thus $W[y_1, \ldots, y_n, y] = 0$.
  - The equation defined by $W[y_1, \ldots, y_n, y] = 0$ will be a linear homogeneous equation for y, and it will have y_k as independent solutions
  - In that equation, $y^{(n)}$ will be the highest derivative and a_0 its coefficient.
  - It can be seen that $W[y_1, \ldots, y_n] = a_0 \neq 0$
  - Dividing the whole equation by a_0 we will get the only normalized linear homogeneous equation that has the initial system as a fundamental solution

- For example, x and x^{-1} are linearly independent in any domain that does not contain the origin

  The linear homogeneous equation corresponding to them is

$$W[x, x^{-1}, y] = \begin{vmatrix} x & x^{-1} & y \\ 1 & -x^{-2} & y' \\ 0 & 2x^{-3} & y'' \end{vmatrix} =$$

$$-\frac{2}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = -\frac{2}{x}\left(y'' + \frac{y'}{x}y' + \frac{y}{x^2}\right) = 0.$$

- In order to write the equation in normal form we have to divide it by $W[x, x^{-1}] = -2x^{-1}$:

$$y'' + \frac{y'}{x} + \frac{y}{x^2} = 0$$

.

Exercise 3.14

- Find the linear homogeneous equation that has the system $\{x, e^x\}$ as its fundamental system.
- First we need the Wronskian:

$$W[x, e^x, y] = \begin{vmatrix} x & e^x & y \\ 1 & e^x & y' \\ 0 & e^x & y'' \end{vmatrix} = xe^x y'' + e^x y - xe^x y' - e^x y'' =$$

$$e^x\left((x-1)y'' - xy' + y\right) = 0.$$

Then, the equation is:

$$y'' - \frac{xy'}{x-1} + \frac{y}{x-1}y,$$

and it is defined in all domains that do not contain $x = 1$.

- Liouville (and also independently Abel and Ostrogradski) found the formula that describes how the Wronskian evolves from point to point:

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a_1(u)du} \quad \forall x \in I.$$

  This formula assumes that $a_0 = 1$.
- Besides, since the exponential is non-zero, it is clear that in order for the W to be zero in all its domain, it is enough for it to be zero in one point.

- In general, there is no general way of solving linear equations, but we can ease the process if we know one particular solution,
- Let us suppose that one know one particular solution y_1. According to the method of D'Alembert we can lower the order of the equation by performing a change of variables
$$y = y_1 \int u dx$$
.

- Let us try to understand that. First we construct:

$$
\begin{aligned}
a_n \{y &= y_1 \int u dx\} \\
a_{n-1} \{y' &= y_1' \int u dx + y_1 u\} \\
a_{n-2} \{y'' &= y_1'' \int u dx + 2y_1' u + y_1 u'\} \\
&\vdots \\
1 \{y^{(n)} &= y_1^{(n)} \int u dx + n y_1^{(n-1)} u + \cdots + y_1 u^{(n-1)}\}
\end{aligned}
$$

- Adding all the equations we get:

$$
Ly = (Ly_1) \int u dx + (a_{n-1} y_1 + a_{n-2} 2y_1' + \cdots + n y_1^{(n-1)}) u +
$$

$$
(a_{n-2} y_1 + \dots) u' + \cdots + (\cdots + y_1) u^{(n-1)} = 0.
$$

- Since y_1 is a solution, we have $Ly_1 = 0$. Then,

$$Ly = (a_{n-1}y_1 + a_{n-2}2y_1' + \cdots + ny_1^{(n-1)})u+$$

$$(a_{n-2}y_1 + \dots)u' + \cdots + y_1 u^{(n-1)} =$$

$$\tilde{a}_n(x)u + \tilde{a}_{n-1}(x)u' + \cdots + y_1 u^{(n-1)} = 0.$$

- Therefore, the change of variables has enable us to get an equation with a lower order.

- For general second order homogeneous linear equations: $y'' + a_1(x)y' + a_0(x)y = 0$ The useful formula from the method of d'Alembert is the following:

$$(a_1(x)y_1(x) + 2y_1'(x))u + y_1(x)u' = 0.$$

- The last equation is separable and easy to solve:

$$\int \frac{du}{u} = -\int \frac{(a_1(x)y_1(x) + 2y_1'(x))}{y_1(x)} =$$

$$-\int (a_1(x) + \frac{2y_1'(x)}{y_1(x)})dx =$$

$$\ln u - \ln C_2 = -\int a_1(x)dx - \ln y_1^2.$$

- This is,

$$u = C_2 \frac{\exp(-\int a_1(x)dx)}{y_1^2}.$$

- But since $y = y_1 \int u dx$, our solution is:

$$y = C_1 y_1 + C_2 y_1 \int \frac{\exp(-\int a_1(x) dx)}{y_1^2} dx.$$

Exercise 3.17

- Solve $(x^2+1)y'' - 2xy' + 2y = 0$.
- We can find "by eye" that a solution is $y = x$.
  On the other hand, writting the solution in normal form we get:
  $$y'' - \frac{2x}{x^2+1}y' + \frac{2}{x^2+1}y,$$
  and so, $a_1 = -2x/(x^2+1)$.
  Applying the formula we have obtained before
  $$y = C_1 x + C_2 x \int \frac{\exp(\int \frac{2x}{x^2+1} dx)}{x^2} dx =$$
  $$C_1 x + C_2 x \int \frac{\exp(\ln((x^2+1))}{x^2} = C_1 x + C_2 x \int \frac{x^2+1}{x^2} =$$
  $$= C_1 x + C_2(x^2 - 1)$$
  This is the general solution.

- Use the method of d'Alembert and $y_1 = e^{kx}$ to prove the following result:

$$y'' - 2ky' + k^2 y = 0 \Leftrightarrow y = C_1 e^{kx} + C_2 x e^{kx}.$$

- In that equation $a_1 = -2k$, therefore,

$$y = C_1 e^{kx} + C_2 e^{kx} \int \frac{\exp(\int 2k dx)}{e^{2kx}} dx$$

$$y = C_1 e^{kx} + C_2 e^{kx} \int dx = C_1 e^{kx} + C_2 e^{kx} \int dx.$$

Then, we get,

$$y'' - 2ky' + k^2 y = 0 \Rightarrow y = C_1 e^{kx} + C_2 x e^{kx}$$

proving the result.

- On the other hand, this is fundamental system e^{kx}, xe^{kx}:

$$W = \begin{vmatrix} e^{kx} & xe^{kx} \\ ke^{kx} & e^{kx} + kxe^{kx} \end{vmatrix} =$$

$$e^{kx}(e^{kx} + kxe^{kx}) - (ke^{kx})(xe^{kx}) = e^{2kx} \neq 0.$$

What equation does the system correspond to?

$$W = \begin{vmatrix} e^{kx} & xe^{kx} & y \\ ke^{kx} & e^{kx} + kxe^{kx} & y' \\ k^2e^{kx} & 2ke^{kx} + k^2xe^{kx} & y'' \end{vmatrix} =$$

$$e^{2kx}(y'' - 2y'k + k^2y) = 0.$$

Dividing by the Wrosnkian we get the equation in normal form

$$y'' - 2y'k + k^2y = 0$$

Thus, we have proved that

$$y'' - 2ky' + k^2y = 0 \Leftarrow y = C_1e^{kx} + C_2xe^{kx}.$$

Usual particular solutions

- What conditions do the coefficients of a linear homogeneous equation of order n have to satisfy in order to accept the following as particular solutions?
  a) $y_1 = x$, b) $y_1 = x^2$, c) $y_1 = e^x$, d) $y_1 = e^{-x}$.
- a) For $y_1 = x$, $y^{(n)} = 0 \, \forall n > 1$, then
  $Ly = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-2}y'' + a_{n-1}y' + a_n y = a_{n-1} + a_n x = 0$.
  The condition reads $a_{n-1} = -a_n x$, but the other a_m coefficients are unconstrained $\forall n-1 > m > 0$.
- b) For $y_1 = x^2$, $y^{(n)} = 0 \, \forall n > 2$, then
  $Ly = 2a_{n-2} + 2a_{n-1}x + a_n x^2 = 0$.
  The condition reads $2(a_{n-2} + a_{n-1}x) = -a_n x$, but the other a_m coefficients are unconstrained
  $\forall \, n-2 > m > 0$.

- c) For $y_1 = e^x$, $y^{(n)} = y \, \forall n > 0$, then $Ly = (1 + a_1(x) + \cdots + a_{n-2} + a_{n-1} + a_n)e^x = 0$. The condition reads $(1 + a_1(x) + \cdots + a_{n-2} + a_{n-1} + a_n) = 0$.

- d) For $y_1 = e^{-x}$, $y^{(n)} = (-1)^n y \, \forall n > 0$, then $Ly = (1 - a_1(-1)^{(n-1)} + \cdots + a_{n-3}(-1)^3 + a_{n-2}(-1)^2 - a_{n-1} + a_n)e^{-x} = 0$. The condition reads $(1 - a_1(-1)^{(n-1)} + \cdots + a_{n-3}(-1)^3 + a_{n-2}(-1)^2 - a_{n-1} + a_n) = 0$.

- what happens if we do not know a particular solution?
- We can try a couple of other changes of variables.

Exercise 3.21

- Perform the change $x \to t \equiv \int \sqrt{Q} dx$ in the equation $y'' + P(x)y' + Q(x)y = 0$, and prove that when $2PQ' + Q' = 0$ we can find solutions. Solve the following equation:

$$xy'' - y' + 4x^3 y = 0.$$

For what other cases can this change of variables by useful?

- Let us calculate derivatives using the chain-rule:

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \dot{y}\sqrt{Q},$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\dot{y}\sqrt{Q}\right)\frac{dt}{dx} = \ddot{y}Q + \frac{\dot{y}}{2\sqrt{Q}}Q'.$$

The equation now reads:

$$\ddot{y}Q + \frac{\dot{y}}{2\sqrt{Q}}Q' + P\dot{y}\sqrt{Q} + Qy = 0.$$

- If we have $Q' + 2PQ = 0$, then we get:

$$\ddot{y} + y = 0.$$

On the other hand, when

$$\frac{Q'}{2\sqrt{Q}} + P\dot{y}\sqrt{Q} = CQ$$

the equation turns into

$$\ddot{y} + C\dot{y} + y = 0$$

and its solution can be given by exponentials.

- Let us solve $xy'' - y' + 4x^3y = 0$. In this case $P = -Q/x = 4x^2$, $2PQ = -8x$, and $Q' = 8x$. Using the change of variables we have just seen, we get $\ddot{y} + y = 0$.
The general solution is thus $y = A\cos t + B\sin t$, but using $t = \int\sqrt{4x^2}dx = \int 2xdx = x^2$,the final result can be written as:

$$y = A\cos(x^2) + B\sin(x^2).$$

Exercise 3.22

- The following change of variables is called Liouville's transform:
$$y = ue^{-\frac{1}{2}\int P(x)dx}.$$
Use it to prove that the equation $y'' + P(x)y' + Q(x) = 0$ can be written in the following way:
$$u'' + f(x)u = 0.$$
Show that when the coefficient $f(x) \equiv (4Q - P^2 - 2P')/4$ is constant, it helps us get solutions. Find the general solution for $xy'' + 2y' + xy = 0$.
- Taking derivatives we get:
$$y' = e^{-\frac{1}{2}\int P(x)dx}\left(u' - u\,(1/2)\,P\right).$$

Taking derivatives again:

$$y'' = e^{-\frac{1}{2}\int P(x)dx}(u'' - u'P + (1/4)uP^2 - (1/2)uP')$$

Our equation now reads:

$$u'' - u\left(\frac{P^2}{4} + \frac{P'}{2} - Q\right) = 0.$$

- Let us solve $xy'' + 2y' + xy = 0$ now.
  In this case $P = 2/x$, $P' = -2/x^2$ and $Q = 1$. Thus, $P^2/4 + P'/2 - Q = 1/x^2 - 1/x^2 - 1 = -1$, therefore $f(x) = 1$ and performing the change, the equation is now $u'' + u = 0$.
  The general solution is $u = A\cos x + B\sin x$; therefore, the general solution is

$$y = ue^{-\frac{1}{2}\int P(x)dx} = ue^{-\frac{1}{2}\int (2/x)dx} =$$

$$\frac{u}{x} = \frac{1}{x}\left(A\cos x + B\sin x\right).$$

3.8 Complete linear equations

- From linearity, we get
  - $Ly_1 = b_1, \quad Ly_2 = b_2 \Rightarrow L(a_1 y_1 + a_2 y_2) = a_1 b_1 + a_2 b_2,$
  - $Ly_1 = 0, \quad Ly_2 = b \Rightarrow L(y_1 + y_2) = Ly_1 + Ly_2 = b,$
  - $Ly_1 = Ly_2 = b \Rightarrow L(y_1 - y_2) = Ly_1 - Ly_2 = 0.$
- Thus, the solution for the complete linear equation is the sum of the general solution for the homogeneous equation and a particular solution.

- Thus, the complete linear equation is solved in two steps:
  - First find n linearly independent solution of the homogeneous to compute the general solution:

$$Ly = 0 \Leftrightarrow y = \sum_{k=1}^{n} C_k y_k.$$

  - Find one particular solution of the complete equation

$$Ly_p = b.$$

- The general solution for the complete equation is then $y = \sum_{k=1}^{n} C_k y_k + y_p$

$$Ly_p = b \Leftrightarrow y = \sum_{k=1}^{n} C_k y_k + y_p.$$

- For example, let us consider the following linear equation $y''' - y' = 1$
  - In exercise 3.13 we found out that the general solution of the homogeneous equation is $y = A + Be^{x} + Ce^{-x}$
  - In this case, it is easy to see that one particular solution is $y = -x$
  - Then, we reach the general solution:

$$y = A + Be^{x} + Ce^{-x} - x.$$

Exercise 3.23

- Find the general solution of $y'' + y = x$.
- The general solution for the homogeneous is clearly $y = A\cos x + B\sin x$
- On the other hand, we can see that a particular solution is $y_p = x$
- Therefore, the complete solution is $y = A\cos x + B\sin x + x$

- The most difficult part of finding the general solution for the complete equation is to find the particular solution
- There are some systematic methods to find the particular solution, and we will study one of them

Variation of parameters

- Let us suppose that we know the general solution to a linear homogeneous equation $\sum_{k=1}^{n} C_k y_k$
- We will suppose that a particular solution to the complete equation will be given by $y_p = \sum_{k=1}^{n} g_k(x) y_k$. We will obtain $g_k(x)$ by the following method:
- First, we will impose that the following relations are satisfied

$$
\begin{array}{ccccccc}
g_1' y_1 & + & g_2' y_2 & + \dots + & g_n' y_n & = & \sum_{k=1}^{n} g_k' y_k = 0 \\
g_1' y_1' & + & g_2' y_2' & + \dots + & g_n' y_n' & = & \sum_{k=1}^{n} g_k' y_k' = 0 \\
\vdots & & \vdots & \ddots + & \vdots & & \vdots \\
g_1' y_1^{(n-2)} & + & g_2' y_2^{(n-2)} & + \dots + & g_n' y_n^{(n-2)} & = & \sum_{k=1}^{n} g_k' y_k^{(n-2)} = 0 \\
g_1' y_1^{(n-1)} & + & g_2' y_2^{(n-1)} & + \dots + & g_n' y_n^{(n-1)} & = & \sum_{k=1}^{n} g_k' y_k^{(n-1)} = b
\end{array}
$$

- Using the relation, we can construct the following

$$\begin{array}{lllll} a_n & \{y_p & = & \sum_{k=1}^n g_k y_k & & \} \\ a_{n-1} & \{y_p' & = & \sum_{k=1}^n g_k y_k' & + \quad [\sum_{k=1}^n g_k' y_k = 0] & \} \\ a_{n-2} & \{y_p'' & = & \sum_{k=1}^n g_k y_k'' & + \quad [\sum_{k=1}^n g_k' y_k' = 0] & \} \\ \vdots & & \vdots & & & \\ a_1 & \{y_p^{(n-1)} & = & \sum_{k=1}^n g_k y_k^{(n-1)} & + \left[\sum_{k=1}^n g_k' y_k^{(n-2)} = 0\right] & \} \\ 1 & \{y_p^{(n)} & = & \sum_{k=1}^n g_k' y_k(n) & + \left[\sum_{k=1}^n g_k' y_k^{(n-1)} = b\right] & \} \end{array}$$

- Adding all terms:

$$Ly_p = \sum_{k=1}^{n} g_k L y_k + b,$$

and since the functions y_k are a solution, we end up with $Ly_p = b$.

- On the other hand, the conditions imposed over $g_k(x)$ form a linear system
  - The determinant, is the Wronskian of the y_k solutions of the homogeneous equation
  - Since the Wronskian is not zero, the solution is not trivial and is moreover unique:

$$g_k'(x) = f(x) \Rightarrow g_k(x) = \int f_k(x)dx + C_k.$$

- Thus, we obtain

$$y_p = \sum_{k=1}^{n} \left(\int f_k(x)dx \right) y_k + \sum_{k=1}^{n} C_k y_k,$$

and since it has n free constants, it is really a general solution of the complete equation

- As an example, let us analyse $y'' - y = x^2$
  - We know that the solution to the homogeneous equation is $y = C_1 e^x + C_2 e^{-x}$
  - Let us check then a particular of the form $y_p = g(x)e^x + h(x)e^{-x}$
  - We have to study the following relations

$$g' y_1 + h' y_2 = 0, \qquad g' y_1' + h' y_2' = b.$$

  - Therefore $\quad g' e^x + h' e^x = 0, \qquad g' e^x - h' e^x = x^2.$
- It is easily seen that $g' = x^2 e^{-x}/2$ and $h' = -x^2 e^x/2$, therefore we have

$$g = -\frac{1}{2}(x^2 + 2x + 2)e^{-x}/2 + C_1,$$

$$h = -\frac{1}{2}(x^2 - 2x + 2)e^x + C_2,$$

and the general solution is

$$y = C_1 e^x + C_2 e^{-x} - x^2 - 2.$$

Exercise 3.24

- Find the general solution for $y'' + y = 1/\cos x$
- The general solution for the homogeneous is $y = C_1 \cos x + C_2 \sin x$, so then $g' \cos x + h' \sin x = 0, \quad -g' \sin x + h' \cos x = 1/\cos x$. which can be rewritten as

$$g' \cos x \sin x + h' \sin^2 x = 0,$$

$$-g' \cos x \sin x + h' \cos x^2 = 1.$$

Adding both equations, we get $h' = 1$, so $g' = -\tan x$ and $h = x + C_1$, $g = \log(\cos x) + C_2$.
The general solution thus reads

$$y = (\log(\cos x) + C_2) \cos x + (x + C_1) \sin x.$$