1. Newton's method We want to solve the algebraic equation $g(x) = 0$ by means of an iterative numerical method. Let us write the sequence of approximations in terms of a recurrence relation: $x_{n+1} = x_n + \Delta_n$. If we indeed desire the successive approximations to be improving, we need the values of $g(x_{n+1})$ to be small. On the other hand, we know that $g(x_{n+1}) = g(x_n) + g'(x_n)\Delta_n + O(\Delta_n^2)$. Let us then define $\Delta_n = -g(x_n)/g'(x_n)$, and, therefore,

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}\,.$$

Using the previous method to compute the solutions of the equation $x^2 - 3x + 2 = 0$
Compute that solution of equation $x - 2\sin x = 0$ which is close to $x_0 = 2$.

2. Solve the equation $x^3 + (4+\epsilon)x^2 + 5x + 2 = 0$ using a perturbation method close to the point $x_0 = -2$ (ϵ is a small number). What would happen if we tried a perturbation expansion close to the point $x_0 = -1$? (*Hint:* look into the equation $(x+1)^2 + \epsilon x^2 = 0$ in order to have a better grasp of what is going on here.)

3. Would you be able to solve equation $z^2 - 2\epsilon z - 2\epsilon = 0$ using an ordinary perturbation expansion in powers of ϵ? Why or why not? Try using a change of variable of the form $z = \epsilon^p w$.

4. Use a perturbation expansion to solve the following problem:

$$y'' + (1+\epsilon x)y = 0\,, \qquad y(0) = 0\,, \quad y'(0) = 1\,.$$

Is the expansion valid if $x > 1/\epsilon$?

5. Use a perturbation expansion to solve the following problem: to order ϵ^2:

$$y'' - y' + \epsilon y^2 = 0\,, \qquad y(0) = 1\,, \quad y'(0) = 0\,.$$

6. Use Lindstedt-Poincaré's method to solve the following problem:

$$y'' + y + \epsilon y|y| = 0\,, \qquad |\epsilon| \ll 1\,, \quad x > 0\,, \quad y(0) = 1\,, \quad y'(0) = 0\,.$$

7. Let us compute the precession of the perihelion of Mercury using Binet's generalized equation,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{m}{h^2} + 3mu^2\,.$$

The last term is the perturbation term. m is the solar mass, and h is related to the angular momentum of the particle. Using Lindtstedt's method, compute the precession.

8.* (**Tough problem**) Consider an otherwise homogeneous string at one point of which a small mass m has been attached. The ends of the string are fixed, and the joint string-small mass system is subject to tension τ.

1.- Show that in the small oscillation approximation the vertical displacement of the string, $u(x,t)$, obeys the equation

$$\rho(x)u_{tt} = \tau u_{xx}\,,$$

where

$$\rho(x) = \rho_0 + m\delta(x-a)$$

is the linear density of the joint system. ρ_0 is the linear density of the homogeneous string.

2.- Let us now investigate normal modes. That is, let us write the function $u(x,t)$ as $v(x)e^{-i\omega t}$. v will then be subject to the boundary conditions $v(0) = v(b) = 0$. What is the differential equation that it must obey?

3.- Let us define the function $G_0(x,\xi;k)$ as the solution of the following problem, whenever it exists:

$$\left(\frac{d^2}{dx^2} + k^2\right) G_0(x,\xi;k) = -\delta(x-\xi)\,;$$
$$G_0(0,\xi;k) = G_0(b,\xi;k) = 0\,.$$

Show that the normal frequencies of the joint string-mass system obey the relation

$$\frac{m}{\tau} G_0(a,a;\omega\sqrt{\rho_0/\tau})\omega^2 = 1\,.$$

4.- Show that the n-th normal frequency will be given by the following perturbation expansion, if indeed it is the case that $m << \rho_0 b$):

$$\omega_n = \sqrt{\frac{\tau}{\rho_0}}\frac{n\pi}{b}\left(1 - \frac{m}{\rho_0 b}\sin^2\left(\frac{n\pi a}{b}\right) + O\left(\left(\frac{m}{\rho_0 b}\right)^2\right)\right)\,.$$

9. Compute the asymptotic expansion of the function

$$f(x) = \int_0^\infty dt\,\frac{e^{-xt}}{1+t^2}$$

around $x = \infty$.

10. Consider the function

$$f(x) = xe^{x^2}\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} xe^{x^2} \int_x^\infty dt\, e^{-t^2}\,.$$

Compute an asymptotic expansion around $x = \infty$ by integration by parts (*Hint:* it is advisable to carry out the change of variables $\tau = t^2$ in the integral). Show that the expansion is divergent for all x.

11. Compute the asymptotic expansion of the following integral for $\lambda \to \infty$:

$$\int_0^\infty dt\, e^{\lambda(t(1+i)-t^3)} \,.$$

12. Compute the asymptotic expansion of the following integral for $\lambda \to \infty$:

$$\int_0^\infty dt\, e^{-\lambda t} \frac{\sqrt{t} e^t}{1+t} \,.$$

13. Compute the asymptotic expansion of the following integral for large ω:

$$\int_3^5 dt\, \frac{\cos \omega t}{1+t^2} \,.$$

14.* Show that as $\lambda \to \infty$ it holds that

$$\int_{-\infty}^\infty dx\, \frac{e^{i\lambda x}}{(1+x^2)^\lambda} \sim \frac{\sqrt{(2-\sqrt{2})\pi}}{\lambda^{1/2}} e^{(1-\sqrt{2})\lambda} \left(2\sqrt{2}-2\right)^{-\lambda} \,.$$

15.* (Very long, if the correct notation is not adequately chosen from the outset!) Apply the method of Lindstet-Poincaré to the following system of equations and compare with the exact solution:

$$\begin{aligned} \dot{x} =& -4y + 2z + \epsilon z \,, \\ \dot{y} =& 4x - 4z \,, \\ \dot{z} =& -2x + 4y - \epsilon x \,. \end{aligned}$$

You should check the applicability of the method before plunging into the computation proper.

Additional material:

Some problems from the latest exams

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16. (February 2006) Compute an approximation to the eigenfunctions and eigenvalues of the operator $Ly = -y'' - \epsilon x^2 y$, $y(0) = y(1) = 0$, that is, those values of λ for which $Ly = \lambda y$ has nontrivial solutions and the solutions themselves. x is an adimensional quantity, and ϵ a very small number ($|\epsilon| \ll 1$). Could any function defined on the interval $(0,1)$ be expanded in terms of the eigenfunctions? Why? If possible, how would you compute the coefficients?

17. (February, 2005) Compute an approximate solution of the following problem. Discuss its range of validity.

$$y'' = (1+x)^2 y\,, \qquad y(0)=1\,, \quad y'(\infty)=0\,.$$

18. (September, 2005) Compute the first terms of the asymptotic expansion of the following function as $t \to \infty$:

$$f(t) = \int_1^\infty \mathrm{d}y \frac{y^{-t-1}}{1+(\ln y)^2}\,.$$

19. (February, 2004) Compute an approximate solution, valid for positive x, of equation

$$y'' - \frac{1}{x}y' + (1 - \frac{5}{4x^2} + \frac{1}{x^4})y = 0\,.$$

(Hint: obtain the equivalent equation $v'' + \phi(x)v = 0$ by a change of variables $y(x) = \mu(x)v(x)$)

20. (September, 2004) Compute an approximate solution, valid for positive x, of equation

$$y'' - 2y' + \left(2 + \frac{2}{x^2} + \frac{1}{x^4}\right) y = 0\,.$$

21. (September, 2003) Obtain an approximate even solution of the following equation, valid for all x:

$$y'' + (1+\epsilon x^2)^2 y = 0\,.$$

22. (February, 2002) Compute the asymptotic expansion as $x \to \infty$ of the following integral. Is the expansion convergent? Explain your answer.

$$\int_0^\infty dx\, e^{-xt}\log(1+x)\,.$$

23. (February, 2002) Consider the equation $y'' - g^2(x)y = 0$, with a "slow" g. If g were constant the solutions would be exponentials.

Let $\epsilon \ll 1$ be a positive real number. Obtain an approximate solution valid for all x, of the following problem:

$$\epsilon^2 y'' = (1+x^2)^2 y\,,$$
$$y(0)=0\,, \quad y'(0)=1\,.$$

24. (September, 2007) Compute at least the first three terms of a series expansion for large λ of the following integral (Hint: a change of variable might prove advantageous)

$$\int_1^\infty \mathrm{d}\tau \, \frac{\tau^{-\lambda-1}}{1+\tau} \, .$$

25. (February, 2008) In the expanding Universe the equation of wave propagation is

$$\frac{1}{c^2} u_{tt} = a^2(t) \nabla^2 u \, ,$$

where ∇^2 is the Laplacian, c the speed of light and $a(t)$ a function of time, called "scale factor".

Assume a physical situation in which the distance from a plane does not impinge on wave propagation, and the plane is given by (say) a spiral galaxy. Let it be the case that the waves are zero at the edge of the galaxy. Use separation of variables, as well as the WKBJ method for the temporal part, to obtain the mathematical description of such a situation. Under which conditions is the approximation a good one? Among all the different modes, point out those for which the approximation is best, given the following data: the radius of the galaxy is 10^5 light-years, Hubble's constant ($H = \dot{a}/a$) has the value $(1.3 \times 10^{10}$ year$)^{-1}$, and the scale factor is of the form $Ct^{2/3}$.

26. (September, 2008) Compute at least the first two terms of a series expansion for large s of the following integral

$$\int_{-\pi/4}^{5\pi/8} \mathrm{d}t \, e^{-s\cos^2 t} \sinh t \, .$$