1. Consider the wave equation $u_{tt} = c^2 u_{xx}$ in an infinite one-dimensional system, with initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$ over $-\infty < x < \infty$ for all $t > 0$, where

$$f(x) = \begin{cases} 2 & -1 < x < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Plot $f(x)$. Check that

$$f(x - ct) = \begin{cases} 2 & -1 + ct < x < 1 + ct, \\ 0 & \text{elsewhere,} \end{cases}$$

write down $f(x+ct)$ and plot the solution $u(x,t)$ of the Initial Value Problem at times $t = 0$, $t = 1/2c$, $t = 1/c$ and $t = 2/c$. Verify that the initial displacement produces two waves moving in opposing directions (each of which carrying half the initial displacement).

2. Compute the characteristics of the equation $\phi_{xy} + xy\phi_{yy} - \phi_y = 0$.

3. Write down the equation of the characteristics of the PDE

$$u_{xx} + 3u_{xy} - 4u_{yy} - u_x + u_y = 0\,,$$

and, by means of an adequate change of variable, show that its general solution is

$$u(x,y) = F(y - 4x)e^{\frac{x+y}{5}} + g(x+y)\,,$$

where F and g are arbitrary functions. Next show that the solution that satisfies the boundary conditions

$$u|_{y=4x} = 5x + e^x\,, \qquad \text{y} \quad u|_{y=-x} = 1$$

is $u(x,y) = x + y + e^{(x+y)/5}$. (These boundary conditions are given over characteristic lines; nonetheless this problem has a unique solution. How come?)

4. Show that the general solution $u(x,y)$ of the PDE

$$y\frac{\partial^2 u}{\partial y^2} - x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = x^2$$

is $u(x,y) = \phi(xy) - x^2 y + f(x)$, where ϕ and f are arbitrary functions.

5. Classify the following equations and systems as hyperbolic, elliptic or parabolic. If they are mixed, give the domains for each case.

a) $u_{xx} = \rho(x)u_{tt} + h(x)u_t + f(x)e^{i\omega t}\,, \qquad \rho(x) > 0\,;$

b) $u_{xx} + (1-x)^2 u_{yy} = 6\,;$

c) $\phi_{xy} + xy\phi_{yy} - \phi_y = 0\,;$

d) $\dfrac{\partial^2 F}{\partial x^2} = \rho(x)\dfrac{\partial F}{\partial t} + h(x)F + f(x,t)\,;$

e) $u_x = (1-u)^2 u_{yy}\,;$

f) $\begin{cases} e_x + Ri &= 0, \\ i_x + Ce_t &= 0. \end{cases}$

6. Solve the following equations, in which u is a function of x and y ($p = u_x$, $q = u_y$):

a) $(x-y)\,p + (x+y)\,q = 0\,,$

b) $x/u\;p + u/y\;q = 0\,,$

c) $xy(p-q) = (x-y)u\,,$

d) $x(y-u)p + y(u-x)q = u(x-y)\,.$

7. Compute the general solutions of the following equations:
a) $u_x + u_y + u_z = 0$;
b) $xu_x + yu_y + zu_z = u$;
c) $u(u_x + u_y + u_z) = 1$;
d) $yzu_x + zxu_y + xyu_z = xyz$;
e) $(y-z)u_x + (z-x)u_y + (x-y)u_x = 0$.

8.* Find the solution of the following equation that goes through the curve $x = 0$, $y = u$:

$$(x + u_x)\, u_x = u_y \, .$$

9.* Find the solution of the following equation that goes through the curve $x^2 + y^2 = 1$, $z = 1$:

$$z_x^2 + z_y^2 = 1 \, .$$

10. Consider the vector field $\vec{V} = X(x,y,z)\mathbf{u}_x + Y(x,y,z)\mathbf{u}_y + Z(x,y,z)\mathbf{u}_z$. The curves tangent to the field at every point are called integral or vector curves, and they are the solution of the following system of equations:

$$\frac{dx}{X(x,y,z)} = \frac{dy}{Y(x,y,z)} = \frac{dz}{Z(x,y,z)} \, (= dt) \; .$$

Compute the vector lines and the orthogonal surfaces for the field $\vec{F} = x\mathbf{u}_x + y\mathbf{u}_y - z\mathbf{u}_z$.

11. Liouville's theorem guarantees the conservation of volume in phase space under hamiltonian evolution. Thus we can write the evolution equation for the probability density on the one particle phase space, namely

$$\frac{\partial f}{\partial t} = \{H, f\} \; ,$$

where $H(q,p,t)$ is the hamiltonian of the system, and Poisson's bracket is defined for each couple of functions $g(q,p,t$ and $f(q,p,t)$ on phase space as

$$\{g, h\} = \frac{\partial g}{\partial q}\frac{\partial h}{\partial p} - \frac{\partial g}{\partial p}\frac{\partial h}{\partial q}$$

(bear in mind that the temporal variable is a mere spectator with regard to Poisson's bracket). Use the method of characteristics to solve the evolution of the probability density on phase space under the initial condition $f(q,p,0) = f_0(q,p)$, both for the free particle and for the harmonic oscillator (N.B.: since f is a function of three variables, the general solution of its evolution equation will be given in terms of an arbitrary two-variable function; additionally, if you do not want to overtax yourself, you might as well just compute the harmonic oscillator case and take the limit of the frequency going to zero to recover the free particle). Not using the method of characteristics, would you be able to infer the solution for a particle moving in a uniform gravitational field? (Hint: compare your results with the solution of the equations of motion for the particle, or, even more to the point, the equations for the characteristics with the equations of motion for the particle.)

Additional material:
Some problems from the latest exams

12. Let $\phi(t,x)$ be a function that satisfies the equation $x^2\phi_{xx} + x\phi_x = \phi_{tt}$ and the boundary conditions $\phi(t,1) = 0$, $\phi(t,2) = 0$, $\phi(0,x) = 1$, $\phi_t(0,x) = 0$, for $t > 0$ and $1 < x < 2$.
a) Classify the equation as hyperbolic, parabolic or elliptic.
b) Using the method of separation of variables obtain an expression for $\phi(x,t)$ in the form

$$\phi(t,x) = \sum_{n=1}^{\infty} c_n u_n(t,x) \ .$$

(Write clearly how the coefficients c_n are to be computed in terms of integrals. **You do not have to compute the integrals**.) (June 94, MMF course)

13. Find the integral surface $z(x,y)$ of the equation

$$x\ (x+y)\ \frac{\partial z}{\partial x}\ -\ y\ (x+y)\ \frac{\partial z}{\partial y}\ =\ (x-y)\ z$$

that includes the line $y = x+1,\ z = 1$. (Second partial exam 94, MMF course)

14. (September - 2002) Consider the curve given in parametric form as

$$x(t) = t\,, \quad y(t) = t\,, \quad u(t) = t^2\,.$$

Compute the integral surface of the following equation that goes through that line:

$$2xy\frac{\partial u}{\partial x} - 2xy\frac{\partial u}{\partial y} = (x-y)u\,.$$

What would happen if we were to look for the solution going through the curve $x(t) = t$, $y(t) = -t$, $u(t) = t^2$?

15. (September - 2003) Compute the solution of the equation

$$2y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 2xy(2y^2 - x^2)$$

that fulfills the condition $u(x,1) = x^2$ for $0 \le x \le 3$. If at all possible, compute the value of $u(2,2)$. If not possible, explain the reason.

16. (February - 2004) Solve, if feasible, the following problem. Which surface, solution of the equation

$$2xt\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = u$$

includes the curve given in parametric form by $x = \sinh(\tau)$, $t = 0$, and $u = \sinh(\tau)$, where τ goes from $-\infty$ to $+\infty$?

17. (September - 2004) Compute the general solution of the equation

$$y\,(2u+1)\,\frac{\partial u}{\partial x} + x\left(4ux^2 + 2x^2\right)\frac{\partial u}{\partial y} = 2xy(2x^2 - 1)\,.$$

Is there any solution that includes the curve

$$\left\{y = x^2\,, \quad u = x\,, \quad x > 0\right\}\ ?$$

And the curve

$$\left\{y=\sqrt{x^4+x}\,,\quad u=x^2\,,\quad x>0\right\}?$$

Why?

18. (February - 2005) Classify the equation

$$y\frac{\partial^2 f}{\partial x^2}+(y-x)\frac{\partial^2 f}{\partial x\partial y}-x\frac{\partial^2 f}{\partial y^2}+a\frac{x-y}{x+y}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)=(x^2+y^2)(x+y)^2\,.$$

Solve the equation in the case $a=1$ (compute the general solution). What would happen if $a\neq 1$? Again for $a=1$, compute, if it exists, the solution such that $f(x,0)=0$, $\partial_y f(x,0)=0$. Compute as well, if it exists, the solution such that $f(x,x)=0$, $\partial_y f(x,x)=0$. Give full explanation for your replies.

19. (February - 2005) An infinite string on which waves propagate with speed c has an initial displacement given by

$$y(x)=\begin{cases}\sin(\pi x/a) & -a\le x\le a,\\ 0 & |x|>a.\end{cases}$$

The string is released with 0 velocity at the instant $t=0$, and its later displacement is described by $y(x,t)$. use Heaviside step functions to obtain a general expression of the displacement as a function of time for all x. In particular for i) $x=0$; ii) $x=a$; iii) $x=a/2$.

20. (September - 2005) Solve the following problems, if the solution exists, and clearly describe the region of validity of the solution. If no solution exists, give the reason for that event.

$$(1)\qquad \tan x u_x+u_y=1\,,\qquad u(\pi/2,y)=0\,.$$

$$(2)\qquad 3xu_x+2yu_y=1\,,\qquad u(\sqrt{81-t^3},t)=0\,,\quad t<3\,.$$

21. (February - 2006) Find those solutions of the equation

$$x\frac{\partial u}{\partial x}+(x+y+1)\frac{\partial u}{\partial y}=axu$$

that include one of the lines of the family $u=7$, $y=-1+\alpha x\ln x$, with α a parameter. Is there any restriction on the possible values of α? Why?

22. (September - 2006) Find those solutions of the equation

$$2x\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=u$$

that include the line $u=y$, $x=1$.

23. (February - 2007) Small transverse displacements of thin flexible tube filled with an incompressible fluid are governed by

$$\frac{\partial^2 u}{\partial t^2}+2v\frac{\partial^2 u}{\partial x\partial t}+\left(v^2-\frac{T}{\rho A}\right)\frac{\partial^2 u}{\partial x^2}=0\,.$$

Here u is the transverse displacement, v is the velocity of the fluid flow, T the tension of the tube, ρ the fluid density, and A the cross section of the tube.

(1) Show that the general solution of the equation entails two waveforms travelling with different speeds. In which case is it possible that both travel to the left? Give a physical argument for your result. Check your result by considering the limit $v\to 0$.

(2) Assume that initially the tube has a small displacement $a \cos kx$ and is released from rest. Find its subsequent motion.

24. (September - 2007) For which values of α is there a unique solution of the equation

$$x(x+2y)z_x - (2x+y)yz_y = (x-y)z$$

that includes the line $z = x^2y^2$, $x + y = \alpha$? Provide the solution, whenever it exists, and give a reason for the non-existence/non-uniqueness, when appropriate.

25. (February - 2008) If possible, solve the following differential equation. What happens if $v = c$?

$$u_{tt} + 2vu_{xt} + \left(v^2 - c^2\right) u_{xx} = \frac{\beta}{u} \left[\left(v^2 - c^2\right) u_x^2 + u_t^2 + 2vu_t u_x\right] .$$

26. (September - 2008) Compute the general solution of

$$ux \cosh y \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u \cosh y \, .$$

Particularize for the solution that includes $y = 0$, $x = e^{-u}$.