1. Show that for $t \neq 0$ the unit step or Heaviside step function $\theta(t)$ can be written as a complex integral:

$$\frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} \mathrm{d}\sigma \, \frac{e^{\sigma t}}{\sigma} \, ,$$

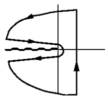
where s is a real positive number. What value would the integral take if $t = 0$?

2. Compute the inverse Laplace transform of the following functions directly using the inversion formula:

$$\text{a)} \, \frac{1}{s^2+a^2} \, ; \quad \text{b)} \, \frac{s}{s^2+a^2} \, ; \quad \text{c)} \, \frac{1}{s^3} \, ; \quad \text{d)} \, \frac{e^{-as}}{s} \quad (a \geq 0) \, .$$

3. Compute the Fourier transforms of

$$\text{a)} \, \frac{1}{x^2+a^2} \, ; \quad \text{b)} \, e^{-\alpha|x|} \, .$$

4.* Use the contour  to compute the inverse Laplace transform of

$$\frac{1}{\sqrt{s}} \, .$$

5.* Solve this integral equation:

$$f(x) = x + \int_0^x \mathrm{d}y \, f(y) \, .$$

(Hint: write the integral out as a convolution with Heaviside's step function, and compute its Laplace transform; as usual, make sure your result is correct. Notice that integral transforms are also useful for integral equations)

6. As usual, we shall re-examine the damped oscillator system. in yet another way. We want to obtain a particular solution of the family of differential equations

$$\ddot{f} + 2\gamma \dot{f} + \omega_0^2 f = g$$

in the form

$$f_p(t) = \int_{-\infty}^{+\infty} ds \, G(t-s) g(s) \, .$$

Which is the algebraic equation of which the Fourier transform of G must be a solution? That is to say, the condition that must be satisfied by the function

$$\hat{G}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt\, e^{i\omega t} G(t)\,.$$

Where are the singularities of the function $\hat{G}$? Classify these singularities in the all three distinct cases, i.e. a) subcritical damping; b) critical damping; c) overdamping. Try to understand resonance in this light. Is function G unique? Or else a set of functions? In which way would G change if the generalized boundary conditions were to change?

7. We compute the (cross-) correlation of two functions as

$$C(t) = \int_{-\infty}^{+\infty} \mathrm{d}\tau\, f^*(\tau) g(t+\tau)\,.$$

If both elements of the the correlation are equal, we call it autocorrelation. Show that the Fourier transform of the correlation is, up to a constant factor, the product of Fourier transforms. Compute the autocorrelation of $\theta(t)\exp(-\gamma t)\sin(\Omega t)$. What is underlying physical object we describe?

8. In some anisotropic systems in solid state and for low temperatures it is possible to observe a "charge-density wave" state, in which the charge density presents a modulation due to the change of structure in the crystal. In this problem we shall look into a very simplified model. Consider a system infinite in variables x and z, but of limited extent (from $-h$ to h) in the y direction. Assume the charge density inside the solid to be $A\cos(qx)$ (there is no z dependence, and regarding y, assume the form $\theta(h-|y|)$, with θ Heaviside's unit step function). Use the Fourier transform with respect to variable y to compute the electrostatic potential. Make sure that all sensible physical conditions are met by your solution (no charge outside the solid, symmetry, etc.)

9. Describe the propagator of the following problem in terms of a d-dimensional integral:

$$u_t = \mathbf{a}\cdot\nabla u + c^2\nabla^2 u\,, \qquad u(\mathbf{r},0) = f(\mathbf{r})\,.$$

[Explanation: write the solution to the problem in the form $u(\mathbf{r},t) = \int_{\mathbf{R}} \mathrm{d}\mathbf{x}\, G(\mathbf{r}-\mathbf{x},t) f(\mathbf{x})$, and use Laplace and Fourier transforms] Compute the explicit form of the propagator in the case $d=1$.

Also compute the Green's function, i.e. the means to the solution of the problem

$$u_t = \mathbf{a}\cdot\nabla u + c^2\nabla^2 u + f(\mathbf{r},t)\,, \qquad u(\mathbf{r},0) = 0\,.$$

(Hint: compute the Green's function in the case $\mathbf{a}=0$, and evaluate it at $\mathbf{x}+\mathbf{a}t$)