1. Investigate whether the method of separation of variables can be applied to problems in which the following partial differential equations turn up, ignoring the possible difficulties derived of boundary conditions:

$$\begin{aligned} &a) \qquad xu_{xx} + u_t = 0\,, \\ &b) \qquad tu_{xx} + xu_t = 0\,, \\ &c) \qquad u_{xx} + u_{xt} + u_t = 0\,, \\ &d) \qquad [p(x)u_x]_x - r(x)u_{tt} = 0\,, \\ &e) \qquad u_{xx} + (x+y)u_{yy} = 0\,. \end{aligned}$$

2. A sphere of radius R behaves according to the heat equation; that is, temperature $T(r,t)$ obeys the equation

$$\frac{\partial T}{\partial t} = c^2 \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} \right)\,,$$

where c is a constant. Compute the function T for all points of the sphere for all instants $t > 0$, under the following set of conditions:

a) temperature on the surface of the sphere is 0 at all times;
b) temperature is finite at all points of the sphere, including its center $r = 0$;
c) the initial value of temperature is

$$T(r,0) = \frac{T_0 R}{r} \sin\left(\frac{\pi r}{R}\right)\,.$$

(*Hint:* Check that the function $S(r,t) = rT(r,t)$ must obey $S_t = c^2 S_{rr}$. What should the value of S be at the center of the sphere?

3. Use the results of a previous example sheet to check that

$$\frac{d}{dx}\left[x^2\left(J_m^2(x) - J_{m+1}(x)J_{m-1}(x)\right)\right] = 2xJ_m^2(x)\,.$$

Using this obtain an expression for the scalar product $\int_0^1 dx\, x J_m(k_i x) J_m(k_j x)$, with $k_i = \sqrt{\lambda_i^{(m)}}$ and $k_j = \sqrt{\lambda_j^{(m)}}$ being roots of $J_m(x)$.

What are the numbers $\lambda_j^{(m)}$? (*Hint:* Sturm-Liouville operators ring a bell? Why is there an x in the scalar product.)

4. The transversal vibrations of an elastic membrane are well described by the wave equation $u_{tt} = a^2\nabla^2 u$, where $u(t,r,\theta)$ stands for the transversal displacement. Consider a circular membrane of fixed rim of unit radius. Assume that at instant $t = 0$ the displacement does not depend on the polar angle θ, and the membrane is let go with no initial velocity, i.e.

$$u(t,1,\theta) = 0, \quad t > 0 \qquad\qquad u(0,r,\theta) = f(r), \quad u_t(0,r,\theta) = 0 \qquad 0 \le r \le 1\,,$$

where $f(r)$ is the initial configuration , and satisfies the condition $f(1) = 0$. Additionally, u must always be bounded. Show that the function u is correctly written as

$$u(t,r) = \sum_{n=1}^{\infty} c_n J_0(\sqrt{\lambda_n} r)\cos(a\sqrt{\lambda_n} t)\,.$$

Obtain closed expressions for the coefficients c_n. Compute them fully in the case $f(r) = (1-r^2)^2$.

5. Compute the function $u(t,r,\theta)$ that describes the transversal displacement of a circular membrane of unit radius under the initial conditions $u(0,r,\theta) = J_1(\sqrt{\lambda_1^{(1)}}r)\cos\theta$ and $u_t(0,r,\theta) = 0$. Draw an approximate depiction of the initial conditions.

6. Compute the solution of the equation $u_{tt} - c^2 u_{xx} = F(x)\cos\omega t$ in the interval $0 < x < \pi$ for $t > 0$, subject to the initial and boundary conditions $u(0,t) = u_x(\pi,t) = 0$ and $u(x,0) = u_t(x,0) = 0$, in the following situations:
a) ω differs from all natural frequencies, $\{\omega_n\}_{n=1}^{\infty}$.
b) ω is very close to the natural frequency ω_m of a normal mode.
c) ω and ω_m are the same (how do we call this phenomenon?)

7. Further extending the previous problem, consider the equation $u_{tt} - c^2 u_{xx} = F(x,t)$ in the interval $0 < x < \pi$ for $t > 0$, subject to the initial and boundary conditions $u(0,t) = u_x(\pi,t) = 0$ and $u(x,0) = u_t(x,0) = 0$. Write the solution in terms of a series.

8.* *Paradox.* In the previous problem the series can be decomposed as a sum of a finite number of series of the form $g(\chi) = \sum_{k=0}^{\infty} \sin\left((2k+1)\chi/2\right)/(2k+1)$. Define the function $f(z) = \sum_{k=0}^{\infty} z^{k+1/2}/(2k+1)$; differentiate it, sum the derivative, and, using the fact that $f(0) = 0$, obtain a compact expression for $f(z)$ in the region $|z| < 1$. Use this result to compute $g(\chi)$ (hint: $\sin(\zeta) = \Im[\exp(i\zeta)]$): ¡it is a constant! As a consequence, the series in the previous problem seems to be identically zero. What is going on here?

9.* Solve the problem of heat conduction in an isolated sphere (the normal derivative of the temperature - normal to the surface of the sphere, that is - must be zero), with initial condition $T(r,\theta,\phi,t=0) = f(r,\theta,\phi)$ (*Hint:* It might be advisable at some point to make a change of dependent variable $R(r) = r^{-1/2}S(r)$, and hence use the properties of Bessel functions of semientire order.)

10. Compute the function $u(x,t)$ that is a solution of the equation $u_{xx} = u_{tt} + \sin x$ under the boundary conditions $u(0,t) = 7(1-t)$, $u_x(\pi,t) = 0$, $u(x,0) = 7$, $u_t(x,0) = 0$.

11. Obtain the solution $u(x,y)$ of Laplace's equation on the rectangle $0 \le x \le a$, $0 \le y \le b$, under the following boundary conditions:

$$\begin{aligned} u(0,y) &= 0\,, \qquad u(a,y) = f(y)\,, \qquad 0 \le y \le b\,, \\ u(x,0) &= 0\,, \qquad u(x,b) = g(x)\,, \qquad 0 \le x \le a\,. \end{aligned}$$

12. The temperature at the ends of a thin rod is fixed, i.e. $u(0,t) = 0$, $u(l,t) = 0$. Let us assume that at all instants $t < 0$ the temperature at all points of the rot is zero, and that at instant $t = 0$ the system is perturbed in such a way that the initial temperature (for instant $t = 0$) is $u(x,0) = u_0\delta(x - x')$. The point x' is not one of the ends of the rod. Compute the temperature distribution for the instants $t > 0$. How would you use the previous result to compute the evolution of more general temperature distributions, keeping the boundary conditions fixed?

13. The Laplacian in spherical coordinates (r,θ,ϕ) is written as follows:

$$\nabla^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\mathrm{sen}\theta}\partial_\theta(\mathrm{sen}\theta\partial_\theta) + \frac{1}{r^2\mathrm{sen}^2\theta}\partial_{\phi\phi}$$

True or false? $\nabla^2(1/r) = 0$. (Reflect on a corresponding physical situation)

14. The two-dimensional Laplacian in polar coordinates (r, φ) is written as follows:

$$\nabla^2 = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\varphi^2 .$$

True or false? $\nabla^2 \ln r = 0$.

15. Consider a spherical planet with a surface distribution of temperatures that is only latitude dependent, and such that its internal temperature distribution does not depend on the angle *phi* either. Compute the temperature $u(r, \theta)$ in the stationary state under the condition that the surface temperature be $u(R, \theta) = \sin^2\theta$.

16.* (**The method of images**) In this problem we shall solve the problem of heat conduction on the half-line $x \geq 0$:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 ; \qquad x \geq 0 ; \quad t \geq 0 ;$$
$$u(x,0) = f(x) ; \quad u(0,t) = 0 , \qquad \text{y } u \text{ acotada} .$$

Here the Fourier transform cannot be directly used, and we shall go a different route: write the solution in the form

$$u(x,t) = \int_0^\infty ds\ [G(s-x,t) - G(s+x,t)]\, f(s) ,$$

and compute G.

An alternative would be solve the following problem:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 ; \qquad x \in \mathbf{R} ; \quad t \geq 0 ;$$
$$u(x,0) = \begin{cases} f(x) , & x > 0; \\ -f(-x) , & x < 0; \end{cases} \qquad \text{eta } u \text{ acotada} .$$

Are this two methods in fact different?

17.* Consider the following family of problems:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} - \frac{c^2}{l} u_x + \frac{c^2 \mu}{l^2} u ; \\ u(L,t) = e^{\gamma L/l} u(0,t) , &\qquad u_x(L,t) = e^{\gamma L/l} u_x(0,t) ; \\ u(x,0) = f(x) , &\qquad u_t(x,0) = g(x) . \end{aligned}$$

L is the length of the interval, and l, on the other hand, a characteristic diffusion length. Both parameters μ and γ are adimensional. When γ takes the value $1/2$ the method of separation of variables is directly applicable; for other values of γ, however, it becomes rather involved. Why? Compute the solution in the case $\gamma = 1/2$. Propose a physical interpretation.

18.* In the study of the time evolution of a qubit (spin) under the influence of a classical white noise driving term, the probability density of finding a given pure state characterized by the Bloch angles obeys the following Fokker-Planck equation:

$$\partial_t P(\theta, \phi, t) = \left[\eta^2 \partial_\theta^2 + \frac{\eta^2}{4\tan^2\theta}\partial_\phi^2 + \frac{\omega}{2}\partial_\phi\right] P(\theta, \phi, t) .$$

η is a parameter denoting the strength of the coupling of the external field and the qubit (it is zero when the evolution of the qubit is free); ω is the frequency for the free evolution (the free qubit evolves according to the Hamiltonian $\hbar\omega\sigma_z/2$); ϕ and θ are the angles parameterising the Bloch sphere (the state of a qubit can be written as $|\psi\rangle = \cos\theta|+\rangle + e^{i\phi}\sin\theta|-\rangle$, and ϕ and θ range from 0 to 2π and π respectively). Describe the evolution of $P(\theta,\phi,t)$ for a generic initial distribution and the for the initial density $P(\theta,\phi,0) = \delta(\theta-\theta_0)\delta(\phi-\phi_0)$. What is the limit towards which the density function P tends as time goes to infinity? (In other words, is there something akin to a stationary density?) [ref: M. Schulz and S. Trimper, *Persistence of Quantum Information*, `quant-ph/0609221`]

Additional material:

Some problems from the latest exams

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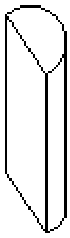
19. (February 2004) A homogeneous cylinder, of height h and radius a, conducts heat. The surface of the cylinder is kept at a fixed temperature, except for its lower end, which has temperature $T_0 + 100$ K. Obtain the stationary temperature distribution inside the cylinder.

20. (February 2004) Which is the smallest normal frequency of a drum shaped as a semicircle? (Hint: consider the modes of a circular drum.)

21. (September 2004) Which is the smallest normal frequency of the cavity shown in the figure?



22. (September 2002) Solve the following initial and boundary value problem (Hint: do not forget about orthogonality):

$$\begin{aligned} &\phi_t = \phi_{xx} + 2\phi_x\,, \qquad 0 \le x \le \pi\,, \quad t \ge 0\,;\\ &\phi(0,t) = 2t\,, \quad \phi(\pi,t) = 2t+2\,;\\ &\phi(x,0) = 2x/\pi\,. \end{aligned}$$

23. (September 2003) The temperature of a cylindrical body of infinite height and radius R with an inner heat source which is homogeneous and constant follows the equation

$$\frac{\partial T}{\partial t} = a^2\nabla^2 T + b^2\,,$$

with b and a real numbers. The initial temperature is zero everywhere inside the cylinder. Additionally, the cylinder radiates through its surface, according to the relation

$$T + R\frac{\partial T}{\partial r} = 0 \quad \text{en} \quad r = R$$

for all instant and angle. Compute the temperature distribution for all times after the initial one.

24. (February 2005) Solve the following problem:

$$u_t = c^2 u_{xx}\,, \qquad x \in [0, L]\,, \quad t > 0\,,$$
$$u(0,t) = 1\,, \quad u(L,t) = 2e^{-\mu t}\,, \quad u(x,0) = 3x + 1\,.$$

25. (September 2005) Solve the following boundary value problem,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{x}\frac{\partial \phi}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \phi}{\partial y^2} = 0\,,$$

with ϕ a function of x and y, over the region $x \geq a$, which obeys the conditions $\partial\phi/\partial x = 0$ at $x = a$, and

$$\frac{\partial \phi}{\partial x} \to A\cos y\,, \qquad \frac{1}{x}\frac{\partial \phi}{\partial y} \to -A\sin y$$

as $x \to +\infty$.

26. (February 2006) A gymnast jumps on a jump mat of a 5 m radius. As the mat again goes through the initial position (which was the stationary one), the upward speed is given by the formula $(5\text{ m} - r) \times 10/\text{s}$, where r is the distance to the centre. Compute the shape of the mat for all later instants.

27. (February 2006) Consider a circular sector with angle $\pi/6$. If you were to use it as a drum, what would be its smallest natural frequency? Let us now assume that the sector conducts heat and that the radius is a. What would be the stationary temperature distribution under the following conditions? (The temperature is described by a function u with variables radius r and angle ϕ.)

$$u(r,0) = 0\,, \quad u\left(r, \frac{\pi}{6}\right) = 0\,, \quad 0 \leq r \leq a\,; \qquad u(a,\phi) = \phi\,, \quad 0 \leq \phi < \frac{\pi}{6}\,.$$

Classify the equation you just solved.

28. (September 2006) Consider a thin rod of length 2 m and insulated laterally (heat only flows inside the rod). Initially the temperature u is

$$\left[(4/\pi^2)\sin(\pi x/2\text{ m}) + 500\right]\text{K}\,.$$

The left and the right ends are both attached to a thermostat, and the temperature at the left side is fixed to 500 K, while the right end is maintained at 100 K. There is also a heater attached to the rod adding a constant heat flow $q(x) = Q\sin(\pi x/2\text{ m})$. Find the temperature field $u(x,t)$ of the rod at any time. What happens when $t \longrightarrow \infty$?

29. (September 2006) Hannibal Chew, the eye designer for the Tyrell Corporation (Blade Runner), has a number of eyes submerged to three parts of four in a liquid with a temperature close to the human body, while the other fourth is in contact with the air at room temperature of 20 °C. Compute the temperature distribution inside one of those eyes. If your answer is a series, compute numerically the first two terms.

30. (February 2007) An infinite conducting cylinder presents constant temperature throughout when, suddenly, it is immersed in a fluid which acts as a thermal reservoir with a different temperature. Describe the time evolution of the temperature at each point of the cylinder. In which way would your result be different if the cylinder were of finite height? Give the temperature at each point of the cylinder for all times after the immersion in this latter case as well.

31. (February 2007) A thermally conducting solid bounded by two concentric spheres is such that the internal boundary is kept at a constant temperature identical over that surface while the outer boundary is fixed with a distribution proportional to $1 - \cos\theta$. Compute the steady state temperature of the solid.

32. (September 2007) Find the potential inside a infinite cylinder of dielectric material, for which one outer half-circumference is kept at potential V while the other outer half is kept at potential $-V$.

33. (September 2007) An infinite plate with a measured thickness is inserted in a heat bath. Before doing so, the temperature distribution inside the plate was only a function of the transversal coordinate, was symmetric with respect to the central plane of the plate, and had a single maximum. Under these conditions, describe the evolution of the temperature distribution. To complete the problem, choose a definite model for the initial temperature distribution and carry out fully all computations for it.

34. (February 2008) An uncharged infinite conducting cylinder is introduced in a constant electric field, perpendicular to its axis. What is the electric potential after the introduction of the cylinder?

35. (February 2008) In the expanding Universe the equation of wave propagation is

$$\frac{1}{c^2} u_{tt} = a^2(t) \nabla^2 u \,,$$

where ∇^2 is the Laplacian, c the speed of light and $a(t)$ a function of time, called “scale factor”.

Assume a physical situation in which the distance from a plane does not impinge on wave propagation, and the plane is given by (say) a spiral galaxy. Let it be the case that the waves are zero at the edge of the galaxy. Use separation of variables, as well as the WKBJ method for the temporal part, to obtain the mathematical description of such a situation. Under which conditions is the approximation a good one? Among all the different modes, point out those for which the approximation is best, given the following data: the radius of the galaxy is 10^5 light-years, Hubble’s constant ($H = \dot{a}/a$) has the value $(1.3 \times 10^{10}\ \text{year})^{-1}$, and the scale factor is of the form $Ct^{2/3}$.

36. (September 2008) Two concentric spheres, the outer one’s radius double the inner one’s, are each cut in two hemispheres by the same non-conducting plate. The upper inner hemisphere and the lower exterior one are both kept at the same potential $-V$, whereas the other two hemispheres (upper outer and lower inner) are found to be at V potential. Compute the electrostatic potential in the region between spheres.

37. (September 2008) Solve the equation

$$u_{\rho\rho} + \frac{1}{\rho} u_\rho - u_{tt} = 0$$

under the following conditions: 1) the solution is regular in the $\rho = 0$ axis; 2) the solution behaves asymptotically ($\rho \to \infty$) as $\sqrt{5/\pi\rho}(\cos 5\rho + \sin 5\rho) \cos 5t$.