1. Does the following inhomogeneous problem have a Green's function?

$$\frac{d}{dx}\left[\frac{1}{x^2}\frac{dy}{dx}\right] = 2(x+1)\ , \qquad\qquad y(1) = y(2) = 0 \ \ ?$$

If the answer is positive, compute the Green's function. If not, explain why.

2. One of the following two boundary problems can be solved through Green's function methods; not so the other one:

$$\begin{aligned}
&i) \qquad \left(\frac{y'}{x+1}\right)' = \cos x \qquad\qquad y(0) = 0, \qquad y(\sqrt{2}) + y'(\sqrt{2}) \ = \ 0\\
&ii) \qquad \left(\frac{y'}{x+1}\right)' = \frac{x^2}{\sqrt{2}} + \sqrt{2}x \qquad\qquad y(0) = 0, \qquad y(\sqrt{2}) - y'(\sqrt{2}) \ = \ 0
\end{aligned}$$

a) Which one cannot be solved by the method? Why?
b) Compute the Green's function for the other problem.

3.* Consider the following Sturm-Liouville operator:

$$L[y](x) = -x^2 y''(x) - xy'(x)\,, \qquad D_L = \{y \in C^2(1, e^\pi)\,|\,y(1) = 0\,\&\,y'\,(e^\pi) = 0\}\,.$$

Compute the eigenfunctions and eigenvalues of the operator (that is to say, those that are non-trivial solutions to the equation $L[y] = \lambda y$). What form should the inner product take for the eigenfunctions to be orthogonal? Solve the following Sturm-Liouville problem:

$$L[y](x) = \lambda y + \ln(x)\,, \qquad y(1) = 0 \quad \& \quad y'\,(e^\pi) = 0\,.$$

Compute the inverse of the operator $L_\lambda = L - \lambda$ (i.e., the Green's function G_λ such that $L_\lambda G_\lambda(x, z) = \delta(x - z)$).

4. Consider the following Sturm-Liouville operator:

$$L[y](x) = -y'' - y' - \frac{1}{4}y\,, \qquad D_L = \{y \in C^2(0, 1)\,|\,y(0) = 0\,\&\,y'(1) + \frac{1}{2}y(1) = 0\}\,.$$

Compute the eigenfunctions and eigenvalues of the operator. Solve the following Sturm-Liouville problem:

$$y'' + y' + \frac{1}{4}y = e^{-x/2}\,, \qquad y(0) = 0\,, \quad y'(1) + \frac{1}{2}y(1) = 0\,.$$

Compute the inverse of the operator $L_\lambda = L - \lambda$ (i.e., the Green's function G_λ such that $L_\lambda G_\lambda(x, z) = \delta(x - z)$).

5. Compute the Green's function for the problem $y''(x) = f(x)$, $y(0) = e^\mu y(\pi)$, $y'(0) = e^{-\mu}y'(\pi)$. What happens when μ tends to zero? (Follow-on: keeping the boundary conditions fixed, what would be the solution of the problem defined by the equation $y''(x) + \lambda y(x) = \delta(x - z)$?)

6. Let us examine the family of problems defined by

$$x^2 y'' + 2xy' + \left(\lambda + \frac{1}{4}\right) y = f(x)\,, \qquad y(1) = y(e) = 0\,.$$

In which cases does the problem have a single solution? Why? Compute the eigenfunctions and eigenvalues of the associated operator.

Expand the function $1/\sqrt{x}$ in that basis.

If it exists, write the solution of the problem for the value $\lambda = 0$ as a linear integral transform of the function $f(x)$ (i.e., use the compact form of the Green's function).

Use the previous results to compute the solution of

$$x^2 y'' + 2xy' + \frac{1}{4} y = \frac{1}{\sqrt{x}}\,, \qquad y(1) = y(e) = 0\,.$$

7. (September 2006) Consider the family of inhomogeneous problems parameterised by λ

$$y'' + \frac{2}{x} y' - \frac{\lambda}{x^2} y = f(x)\,, \qquad y(1) = 0 = y'(2)\,.$$

For which values of λ could we write the solution in the form $\int_1^2 \mathrm{d}\xi\, H(x,\xi) f(\xi)$? Compute the function H for $\lambda = -1/4$, if it makes sense.

8. (February 2007) Compute the Green's function $G(t, t_0)$ that solves

$$\frac{\partial^2}{\partial t^2} G + \alpha \frac{\partial}{\partial t} G = \delta(t - t_0)$$

under the initial conditions $G(0, t_0) = \frac{\partial}{\partial t} G(0, t_0) = 0$ for $t_0 > 0$. Hence solve

$$\ddot{x} + \alpha \dot{x} = A e^{-\beta t} \theta(t)\,, \quad x(0) = \dot{x}(0) = 0\,,$$

where θ is Heaviside's step function.

9. Compute the solution to the family of problems

$$y^{(\mathrm{IV})} = f(x)\,, \qquad y(0) = y'(0) = y(1) = y'(1) = 0\,,$$

written as an integral (that is to say in the format $y(x) = \int_0^1 \mathrm{d}\xi\, G(x,\xi) f(\xi)$).

10. (February 2008) A particle is moving in a parabolic potential. At a particular instant in time, when the particle is passing through the location of the potential minimum, an external sinusoidal time dependent force is suddenly switched on. After an interval, which is taken as data, the particle is back at the location of the minimum for the first time. Use the method of Green's functions to study the problem. Discuss the existence and uniqueness of the solution you propose. Try to give a physical explanation for the exceptional cases. Which is the speed of the particle at the onset of the external force?