1. Compute the Fourier series for each of the following functions. Assume that the functions are periodically extended out of the stated intervals.

$$
\begin{array}{lll}
a)\ f(x) = \sin^2 x, & -\pi \le x \le \pi & T = 2\pi \\
b)\ f(x) = \begin{cases} 0 \\ x \end{cases} & \begin{array}{r} -\pi \le x < 0 \\ 0 \le x < \pi \end{array} & T = 2\pi \\
c)\ f(x) = (1 - x^2) & -1 \le x \le 1 & T = 2
\end{array}
$$

2. Consider an even function $f(t)$, piecewise continuous and periodic, its period being T. Which of the following expansions is (or are) correct expressions for the Fourier series of $f(t)$?

$$a)\quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t), \quad \text{donde} \quad \omega = 2\pi/T,\ a_n = \frac{2}{T}\int_0^T f(x)\cos(n\omega x)dx$$

$$b)\quad \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos(n\omega t), \quad \text{donde} \quad \omega = 2\pi/T,\ b_n = \frac{2}{T}\int_{-T/4}^{3T/4} f(x)\cos(n\omega x)dx$$

$$c)\quad \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\Omega t), \quad \text{donde} \quad \Omega = \pi/T,\ c_n = \frac{1}{T}\int_0^{2T} f(x)\cos(n\Omega x)dx$$

$$d)\quad \sum_{n=-\infty}^{\infty} d_n e^{in\omega t}, \quad \text{donde} \quad \omega = 2\pi/T,\ d_n = \frac{1}{T}\int_0^T f(x)e^{-in\omega x}dx$$

Compute in each case the four first terms of the expansion of the constant function $f(t) = 1$

3. Consider functions $f(x)$ whose values in the interval $(0, \pi)$ match those of the cosine function. a) Plot one such function $f(x)$, periodic of period π, and compute its Fourier series expansion. b) Find a Fourier series only with sines and another with cosines for functions $f(x)$, and plot them. Will the partial sums give rise to Gibbs' phenomenon?

4. From a function $f(x)$ defined for $0 \le x \le L$, we can obtain both a sine or cosine series by using the odd and even extensions of F, respectively. These are not the only possibilities; here we shall show how to compute other Fourier series that agree with $f(x)$ on its domain

a) Extend f to $(L, 2L]$ in some way. Next extend the new function to $(-2L, 0)$ as an odd function, and, hence, to the whole real line as a periodic function of period $4L$ Show that to this final function there corresponds a Fourier series in sines of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/2L)$$

where

$$b_n = \frac{1}{L}\int_0^{2L} \mathrm{d}x\, f(x)\sin(n\pi x/2L)\,.$$

b) Let us now extend f to $(L, 2L]$ so that it be symmetric with respect to $x = L$, that is, $f(2L - x) = f(x)$ for $0 \leq x \leq L$. Now extend the function to $(-2L, 0)$ as an odd function, and hence to the whole real line as periodic function of period $4L$. Show that to this final function there corresponds a Fourier series in sines of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin[(2n-1)\pi x/2L]$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin[(2n-1)\pi x/2L] dx$$

Both series converge to the original function in the interval $(0, L)$. As a matter of fact, the second one converges to the original function in the interval $(0, L]$ (Why?).

5. Compute the Fourier series expansion of the function $f(x) = x$ in the interval $(-\pi, \pi)$. Use this expansion to prove

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}\,.$$

6. Compute the Fourier series expansion of the function $f(x) = \exp(x)$ in the interval $(-1, 1)$. What is the value of the expansion at the point $x = 2$?

7. Integrate term by term the expansion of the previous example. Now, by expanding in the same interval an appropriate function, show that $\int \mathrm{d}x \exp(x) = \exp(x) + c$!! Prove the following identity:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2\pi^2} = \frac{e + 1/2 - e^2/2}{e^2 - 1}\,.$$

What would happen if we were to differentiate term by term the expansion of the previous example? Show that one would be led to the following nonsensical statement: $\sum_{m=1}^{\infty}(-1)^m = -1/2$.

8. Consider a square periodic wave, of period T. Show that low-pass filter need not be very good in order to allow most of the power through (that is, compute how many harmonics are required for the transmission of 90% of the power).

9. Check that the Fourier expansion of the function $f(x) = |x|$ in the interval $(-\pi, \pi)$ is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2}\,.$$

Integrate term by term, and compute the following series

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3}\,.$$