Are the electron spin and magnetic moment parallel or antiparallel vectors?

Martín Rivas*
Dpto. de Física Teórica,
The University of the Basque Country,
Apdo. 644, 48080 Bilbao, Spain

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Abstract

A direct measurement of the relative orientation between the spin and magnetic moment of the electron seems to be never performed. The kinematical theory of elementary particles developed by the author and the analysis of the expectation value of Dirac's magnetic moment operator show that, contrary to the usual assumption, spin and magnetic moment of electrons and positrons might have the same relative orientation. Two plausible experiments for their relative orientation measurement are proposed.

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1 Introduction

The usual assumption concerning the magnetic dipole structure of the electron, states that if the electron is a spinning particle of negative charge which rotates along the spin direction then this motion will produce a magnetic moment opposite to the spin. In the case of the positron both magnitudes, spin and magnetic moment, will therefore have the same direction. But this interpretation is not supported by a classical analysis of spin, but rather by the guess that presumably spin and angular velocity are directly related.

Dirac's analysis [1] of the relativistic electron shows that the spin and magnetic moment operators are related by

$$\boldsymbol{\mu} = \frac{q\hbar}{2m}\boldsymbol{\sigma}, \tag{1}$$

where q is the electric charge and $\boldsymbol{\sigma}$ the spin matrix operator. For the electron $q = -e$, $e > 0$, and therefore the spin and magnetic moment are antiparallel vectors while they are

*wtpripem@lg.ehu.es

parallel for positrons since $q = +e$. However, if we make this discussion for the expectation values we obtain some indefiniteness in this relative orientation. Let us consider the particle analysed in the center of mass frame and in the Pauli-Dirac representation. Let

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

the usual Dirac's spinors in this frame. Spinors u_1 and u_2 are positive energy solutions and v_1 and v_2 the negative energy ones. The σ_z operator takes the form

$$\sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

so that u_1 and v_1 are spin up states and u_2 and v_2 spin down. If we take the expectation value of μ_z in all these states we obtain that for positive and negative energy states the magnetic moment has the opposite orientation to the corresponding expectation value of the spin. But if the negative energy states are considered to describe the antiparticle states then particle and antiparticle have the same relative orientation of spin and magnetic moment.

Now, let $C = i\gamma_2\gamma_0$ the charge conjugate operator. In the Pauli-Dirac representation it takes the form

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and the charge conjugate spinors $\tilde{u}_i = Cu_i$ and $\tilde{v}_i = Cv_i$ are given by

$$\tilde{u}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \tilde{u}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \tilde{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

so that the $\tilde{v}$ spinors represent positive energy states and $\tilde{u}$ negative energy states of a system which satisfies Dirac's equation for a particle of opposite charge. Now $\tilde{u}_2$ and $\tilde{v}_2$ are spin up states and the others spin down. But for this system we have to take for the magnetic moment operator the corresponding expression (1) with $q = +e$ and therefore the expectation values show that for positive and negative energy states of the positron spin and magnetic moment are parallel vectors. Then, although the expression of the magnetic moment operator obtained in Dirac's theory is unambiguous, the analysis of the expectation values leads to some contradiction and in any case it seems that particle and antiparticle may have the same relative orientation between both magnitudes. We shall see in section 2 that the analysis of a classical model of electron, which satisfies when quantized Dirac's equation, leads to the same indefiniteness in this relative orientation.

No explicit direct measurement of the relative orientation between spin and magnetic moment of the free electron, known to the author, can be found in the literature although very high precision experiments are performed to measure the magnitude of the magnetic moment and the absolute value of g, the gyromagnetic ratio. In the review article by

Rich and Wesley [2] the two main methods for measuring the anomaly factor of leptons $a = |g|/2 - 1$, are analysed: one kind involves precession methods which measure the difference between the spin precession frequency and its cyclotron frequency in a uniform magnetic field. The other are resonance experiments, like the ones developed by Dehmelt on a single electron in a Penning trap [3] where the cyclotron motion, magnetron motion and axial oscillation are monitored. All these measurements are in fact independent of whether spin and magnetic moment are indeed parallel or antiparallel, because they involve measurements of the spin precession frequencies in external magnetic fields.

All attempts of Stern-Gerlach type on unpolarised beams to separate electrons in inhomogeneous magnetic fields have failed and Bohr and Pauli claimed that this failure was a consequence of the Lorentz force on charged particles which blurred the splitting. Nevertheless Seattle experiments on a single electron [4] show a "continuous Stern-Gerlach type" of interaction producing an "axial" oscillation of the particle in the direction orthogonal to its cyclotron motion. But these experiments are not able to determine the relative orientation between these magnitudes. Batelaan et al. [5], propose an alternative device which according to Dehmelt's suggestions minimize the Lorentz force by using an external magnetic field along the electron velocity but maximize the spin force by using large magnetic field gradients in that direction. They obtain numerically a polarization of the electron beam along the direction of motion. Perhaps an experimental setup in these terms will be able to clarify the relative orientation between these two vector magnitudes.

After giving in the next section a short review to the kinematical formalism of spinning particles developed by the author, in section 3 a direct experiment and in 4 an indirect experiment will be suggested to check the relative orientation of spin and magnetic moment.

2 Clasical spinning particles

The kinematical theory of elementary spinning particles [6] produces a classical description of spin and an elementary particle in this formalism is a pointlike object. The charge of the particle is located in that point and its motion can also be interpreted as the combination of a translational motion of its center of mass and a harmonic motion of the center of charge around the center of mass. Once the spin direction is fixed, the motion of the point charge is completely determined. If we consider as *the particle* the positive energy solution and of negative electric charge, then the spin and magnetic moment for both the electron and positron are described by parallel vectors. If we consider that the particle has positive charge we get the opposite orientation. We thus obtain the same indefiniteness as in the previous analysis of the expectation value of Dirac's magnetic moment operator.

Let us review the main highlights of the mentioned approach:

- The classical variables that characterise the initial and final state of a classical elementary spinning particle in a Lagrangian approach are precisely the variables used as parameters of the kinematical group of space-time symmetries or of any of its homogeneous spaces. Any element of the Poincaré group can be parametrised in terms of the time and space translation and the relative velocity and orientation among inertial observers. Therefore, a relativistic spinning particle is described by the variables time t, position $\boldsymbol{r}$, velocity $\boldsymbol{v}$ and orientation $\boldsymbol{\alpha}$. We shall call to these variables the kinematical variables and the manifold they span the kinematical space of the system.

- A classical spinning particle is thus described as a point with orientation. The particle moves and rotates in space. Point $\boldsymbol{r}$ describes its position in space while $\boldsymbol{\alpha}$ describes its spatial orientation. But what point $\boldsymbol{r}$ describes is the position of the charge, which is in general a different point than its center of mass $\boldsymbol{q}$, and in general $\boldsymbol{r}$ describes a harmonic motion around $\boldsymbol{q}$, usually called this motion the *zitterbewegung*.

- When expressed the Lagrangian in terms of the kinematical variables it becomes a homogeneous function of first degree of the derivatives of the kinematical variables and consequently it also depends on $\dot{\boldsymbol{v}}$, the acceleration of point $\boldsymbol{r}$, and on $\dot{\boldsymbol{\alpha}}$ or equivalently on the angular velocity $\boldsymbol{\omega}$. It turns out that it can be written as

$$L = T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}} + \boldsymbol{V}\cdot\dot{\boldsymbol{v}} + \boldsymbol{W}\cdot\boldsymbol{\omega}, \tag{2}$$

where $T = \partial L/\partial\dot{t}$, $\boldsymbol{R} = \partial L/\partial\dot{\boldsymbol{r}}$, $\boldsymbol{V} = \partial L/\partial\dot{\boldsymbol{v}}$ and $\boldsymbol{W} = \partial L/\partial\boldsymbol{\omega}$.

- For a free relativistic particle, when analyzing the invariance under the different one-parameter subgroups of the Poincaré group, Noether's theorem determines the usual constants of the motion which take the following form in terms of the above magnitudes: Energy,

$$H = -T - \boldsymbol{v}\cdot\frac{d\boldsymbol{V}}{dt},$$

linear momentum,

$$\boldsymbol{P} = \boldsymbol{R} - \frac{d\boldsymbol{V}}{dt}, \tag{3}$$

kinematical momentum

$$\boldsymbol{K} = \frac{H}{c^2}\boldsymbol{r} - \boldsymbol{P}t - \frac{1}{c^2}\boldsymbol{S}\times\boldsymbol{v}, \tag{4}$$

and angular momentum

$$\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{S}, \tag{5}$$

where the observable $\boldsymbol{S}$, takes the form

$$\boldsymbol{S} = \boldsymbol{v}\times\boldsymbol{V} + \boldsymbol{W}. \tag{6}$$

- The linear momentum (3) is not lying along the velocity $\boldsymbol{v}$ of point $\boldsymbol{r}$. Point $\boldsymbol{r}$ does not represent the center of mass position. If in terms of the last term in (4) we define the position vector

$$\boldsymbol{k} = \frac{1}{H}\boldsymbol{S}\times\boldsymbol{v},$$

then the center of mass position can be defined as $\boldsymbol{q} = \boldsymbol{r}-\boldsymbol{k}$, such that the kinematical momentum (4) can be written as

$$\boldsymbol{K} = \frac{H}{c^2}\boldsymbol{q} - \boldsymbol{P}t.$$

The time derivative of this expression leads for the linear momentum to the form

$$\boldsymbol{P} = \frac{H}{c^2}\frac{d\boldsymbol{q}}{dt},$$

which is the usual relativistic expression of the linear momentum in terms of the center of mass velocity. Observable $\boldsymbol{k}$ is the relative position of point $\boldsymbol{r}$ with respect to the center of mass $\boldsymbol{q}$.

- The observable $\boldsymbol{S}$ is the classical equivalent of Dirac's spin operator, because it satisfies the free dynamical equation

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{P} \times \boldsymbol{v},$$

which does not vanish because $\boldsymbol{P}$ and $\boldsymbol{v}$ are not parallel vectors and where the velocity operator in Dirac's theory becomes $\boldsymbol{v} = c\boldsymbol{\alpha}$ in terms of Dirac's $\boldsymbol{\alpha}$ matrices.

- The structure of the spin observable is twofold. One, $\boldsymbol{v} \times \boldsymbol{V}$, is related to the zitterbewegung or motion of the charge around its center of mass and another $\boldsymbol{W}$ which comes from the rotation of the particle.

- The magnetic moment is produced by the charge motion and is thus related only to the zitterbewegung part of the spin. It is because the spin has another contribution coming from the rotation that a pure kinematical interpretation of the gyromagnetic ratio [7] has been given.

- The classical system that when quantized satisfies Dirac's equation [8] corresponds to a particle whose charge is moving at the speed of light, and therefore if $v = c$ is constant, the acceleration is always orthogonal to the velocity.

- If we take in (4) the time derivative of this expression and afterwards its scalar product with vector $\boldsymbol{v}$, since $v = c$ we get the relationship

$$H - \boldsymbol{P} \cdot \boldsymbol{v} - \frac{1}{c^2}\left(\boldsymbol{S} \times \frac{d\boldsymbol{v}}{dt}\right) \cdot \boldsymbol{v} = 0$$

which is the classical equivalent of Dirac's equation.

- The center of mass observer is defined by the conditions $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$. For this observer we see from (5) that the spin $\boldsymbol{S}$ is a constant of the motion. If the system has positive energy $H = +mc^2$, from (4) we get

$$m\boldsymbol{r} = \frac{1}{c^2}\boldsymbol{S} \times \boldsymbol{v},$$

so that the charge of the particle is describing circles in a plane orthogonal to $\boldsymbol{S}$ as depicted in part a) of figure 1. Part b) is the time reversed motion of this particle which corresponds to its antiparticle or to a particle that in the center of mass frame has energy $H = -mc^2$. If the particle is negatively charged then particle and antiparticle have their magnetic moment along the spin direction. If we consider as the particle the positively charged one we obtain the opposite orientation for the magnetic moment. In any case the magnetic moment of the particle and antiparticle have the same relative orientation with respect to the spin.

The radius of this motion is

$$R_o = \frac{S}{mc} = \frac{\hbar}{2mc} = \frac{1}{2}\lambda_C = 3.8 \times 10^{-11} \text{ cm}.$$

The frequency of this motion is

$$\omega_o = \frac{mc^2}{S} = \frac{2mc^2}{\hbar} = 1.55 \times 10^{21}\,\text{s}^{-1}.$$

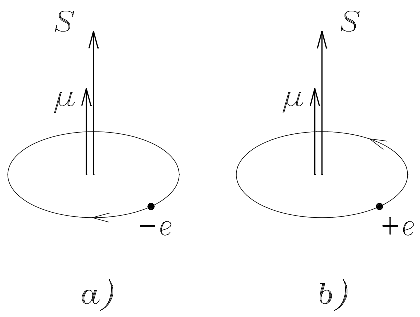

Figure 1: Charge motion of the electron a) and positron b) in the center of mass frame.

- It turns out that although the particle is pointlike, because of the zitterbewegung the charge has a localized region of influence of size $2R_o$, which is Compton's wave length. The latest LEP experiments at CERN establish an upper bound of 10^{-17}cm for the radius of the charge of the electron, which is consistent with this pointlike interpretation, while its quantum mechanical behaviour is produced for distances of its Compton wave length, six orders of magnitude larger.

- Properly speaking what this formalism shows is that the magnetic moment $\boldsymbol{\mu}$ of the electron is not an intrinsic property like the charge. It is produced by the motion of the charge and therefore is orthogonal to the zitterbewegung plane. But at the same time, in the center of mass frame, the electron has an oscillating electric dipole of magnitude eR_o lying on the zitterbewegung plane. Its time average value vanishes and in low energy interactions the effect of this electric dipole is negligible but in high energy physics we have to take into account the detailed position of the charge and thus the electric dipole contribution is not negligible. This electric dipole is not related to a loss of spherical symmetry of some charge distribution. The charge distribution is spherically symmetric because it is just a point. This dipole is just the instantaneous electric dipole moment of the charge with respect to the center of mass.

As an approximation we can consider the classical electron as a point, its center of mass, where we also locate the charge. But at the same time we have to assign to this point two electromagnetic properties, a magnetic moment lying along or opposite to the spin direction and an oscillating electric dipole, of frequency ω_o, on a plane orthogonal to the spin.

A more detailed analysis of the dynamics gives rise to the dynamical equations for the center of mass $\boldsymbol{q}$ and center of charge $\boldsymbol{r}$ in an external electromagnetic field as given by [9]:

$$m\ddot{\boldsymbol{q}} = \frac{e}{\gamma(\dot{q})}\left[\boldsymbol{E} + \dot{\boldsymbol{r}} \times \boldsymbol{B} - \dot{\boldsymbol{q}}\left([\boldsymbol{E} + \dot{\boldsymbol{r}} \times \boldsymbol{B}] \cdot \dot{\boldsymbol{q}}\right)\right], \tag{7}$$

$$\ddot{\boldsymbol{r}} = \frac{1 - \dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{r}}}{(\boldsymbol{q} - \boldsymbol{r})^2}\,(\boldsymbol{q} - \boldsymbol{r}). \tag{8}$$

Here, an over dot means a time derivative and the external fields are defined at the charge position $\boldsymbol{r}$, and it is the velocity of the charge that produces the magnetic force term.

3 A direct measurement

Since the classical model depicted in figure 1 satisfies when quantized Dirac's equation [8] it is legitimate to use it for analysing the interaction with an external electromagnetic field from a classical viewpoint. We can alternatively use the Bargmann-Michel-Telegdi equation for the spin evolution[10], but this approach assumes a minimal coupling for the charge and also an anomalous magnetic moment coupling. However in our approach since the spinning electron is a point charge such that the magnetic moment is a consequence of the zitterbewegung, we only have to consider a minimal coupling prescription which is closer to quantum electrodynamics in which no anomalous magnetic moment coupling for the electron is present.

The proposed experiment is to send a beam of transversally polarised electrons or positrons and check the interaction of their magnetic moment with an external magnetic field. As suggested by Batelaan et al. [5] we shall consider a region of low or negligible magnetic field but with a non-negligible field gradient such that the deflection of the beam is mainly due to the magnetic dipole structure.

We consider a beam moving along the positive direction of the OY axis and with the transversal spin pointing along the positive OZ axis. The external magnetic field will be produced by two conducting wires parallel to the OY axis, separated by a distance $2b$ and contained in the YOZ plane of a cartesian frame (see figure 2). If they carry a current in the same direction, then the magnetic field vanishes along the OY axis and is very low in its neighborhood. The square depicted in the figure represents the region, using for computation, where the initial position of the center of mass of the electrons in the beam is contained. We have found no experimental evidence of one such a device which could be useful to analyse the magnetic moment of free charged particles as an alternative to the Stern-Gerlach magnets which do not work properly with charged particles.

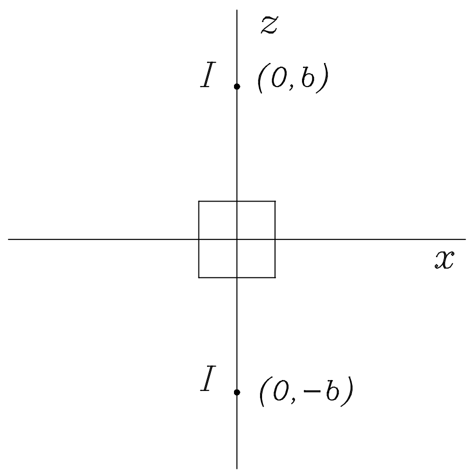

Figure 2: A transversally polarised electron beam of square cross section is sent along the OY axis into the magnetic field created by two conducting wires, separated a distance $2b$, perpendicular to the figure and carrying a current I in the same direction.

The magnetic field produced by the current takes the form:

$$B_x = \frac{k(z+b)}{x^2+(z+b)^2} + \frac{k(z-b)}{x^2+(z-b)^2}, \quad B_y = 0,$$
$$B_z = -\frac{kx}{x^2+(z+b)^2} - \frac{kx}{x^2+(z-b)^2},$$

where $k = I/2\pi\epsilon_0 c^2$, I is the intensity of the current and ϵ_0 the permittivity of the vacuum.

To compute numerically the motion of a polarised electron beam in an external magnetic field, we shall use the above dynamical equations (7) and (8) with initial conditions such that the zitterbewegung plane of each electron is the XOY plane and the charge motion produces a magnetic moment pointing along the positive OZ axis. For the center of mass position we shall consider the electrons uniformly distributed in the mentioned square region.

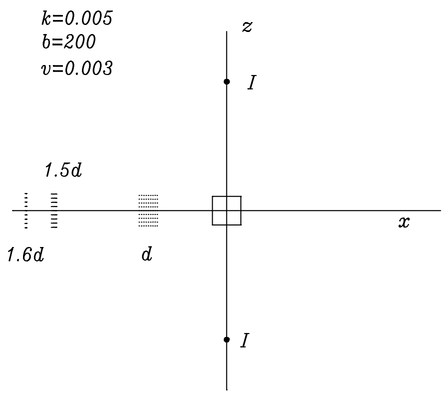

Figure 3: Position of the beam after travelling a distance d, $1.5d$ and $1.6d$ along the OY direction in the magnetic field created by the two parallel currents. The separation between the wires is $200\lambda_C$, the velocity of the center of mass of electrons is $0.003c$ and the current $I = 3$A.

In figure 3 we depict the situation of the polarised electron beam after travelling some distance inside the magnetic field region. The dots represent the center of mass position of a sample of particles which in the incoming beam are distributed uniformly in the shown square region. With the magnetic moment of the electrons pointing upwards we see a deflection (and also a focusing effect) to the left. The deflection is of the same amount to the right for electrons with the magnetic moment pointing down. It is checked that the deflection is independent of the initial position of the electron charge compatible with the initial center of mass position. We also obtain an equivalent deflection for many other values of the separation between wires, current and initial beam velocity.

We thus expect from this experiment that if the beam is deflected to the left then spin and magnetic moment are parallel vectors, while they are antiparallel for right deflection. The interaction does not modify the spin orientation so that we can check the polarization of the beam at the exit by some direct method of electron absorption like the one devised for measuring the spin of the photon in circularly polarised light beams [11].

Although this device is considered for analysing charged particles the deflection is produced in the low magnetic field region so that it is mainly due to the interaction with the magnetic moment. This is one of the reasons to consider this device as an alternative to the Stern-Gerlach magnets to separate unpolarised beams.

4 An indirect measurement

As an indirect experiment we shall measure the relative orientation of spin and magnetic moment of electrons bounded in atoms.

Let us consider a material system formed by atoms of some specific substance. Let us send a beam of circularly polarised light of such an energy to produce electron transitions on these atoms from an S-state ($l = 0$ orbital angular momentum) into another S-state. Let us assume that the photons of the circularly polarised beam have their spins pointing forward. In this case the transition only affects to the electrons with the spin pointing backwards such that after the transition the excited electrons have their spins in the forward direction. If we now introduce a magnetic field in the forward direction to observe the Zeeman splitting then the measurement of the additional interaction energy $-\boldsymbol{\mu} \cdot \boldsymbol{B}$ will give us only one of the two expected transition lines of the emission spectrum from which we determine the relative orientation between $\boldsymbol{\mu}$ and $\boldsymbol{B}$ and therefore between $\boldsymbol{\mu}$ and $\boldsymbol{S}$.

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