Quantization of generalized spinning particles: New derivation of Dirac's equation

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Quantization of generalized Lagrangian systems suggests that wave functions for elementary particles must be defined on the kinematical space rather than on configuration space. For spinning particles the center of mass and center of charge are different points. Their separation is of the order of the Compton wavelength. Spin-1/2 particles arise if the classical model rotates but no half integer spins are obtained for systems with spin of orbital nature. Dirac's equation is obtained when quantizing the classical relativistic spinning particles whose center of charge is circling around its center of mass at the speed c. Internal orientation of the electron completely characterizes its Dirac's algebra.

I. INTRODUCTION

In previous works[1,2] we have found a Lagrangian formulation of classical spinning particles where the spin is produced by the *Zitterbewegung* and rotational motion of the particle around its center of mass. The novelty with respect to other approaches is the definition of a classical particle. The usual canonical formulation defines a classical particle as a system whose phase space is a homogeneous space of the Poincaré group. In our approach it is the kinematical space of the system that is required to be a homogeneous space of the kinematical group of space–time transformations. The kinematical space of a system is the manifold spanned by the initial (or final) boundary variables of the corresponding variational problem in terms of a Lagrangian function. This definition of a classical particle implies that a general Lagrangian must depend on the acceleration of the particle and thus necessarily we must work within a generalized Lagrangian formalism. Section II is devoted to a concise summary of the mentioned references to show the spin structure for both relativistic and nonrelativistic particles. One of the salient features for a spinning particle is the existence of two position vectors. One defines the center of mass and the other is interpreted as the center of charge position.

Feynman's quantization of the above generalized Lagrangian systems implies that wave functions must be squared-integrable functions defined on the kinematical space. This is shown in Sec. III. Thus a general wave function will be a function of ten variables: time t, position $\mathbf{r}$, velocity $\mathbf{u}$, and orientation α.

Some authors[3–5] have claimed to consider the dependence of the wave function on time and position as well as on orientation variables. To our knowledge is in the work of Bopp and Hagg[3] where for the first time the wave function is allowed to depend on additional angular variables, in particular Euler's angles and they succeeded in obtaining spin-1/2 wave functions. Finkelstein[4] classified the internal structures of a rigid body according to its invariance under the Lorentz group and where internal variables bear the generic name of "orientation." Bacry and Kihlberg[5] made use of an alternative parameterization to discuss some properties of a certain class of homogeneous spaces of the Poincaré group to describe spinning particle wave functions. In the relativistic case, Dahl[6] obtained Dirac's equation by assuming dependence of the wave function on the orientation.

Sections IV and V analyze different quantized examples for nonrelativistic and relativistic particles, respectively. In Sec. V when quantizing the relativistic system of a particle circling around its center of mass with velocity c we obtain Dirac's equation. The Dirac algebra of internal

observables is completely determined by the spin components on the spatial and body axes and by the orientation of the particle such that any other internal observable like velocity and acceleration can be expressed in terms of them. The way we relate these kinematical variables to the essential internal observables leads to different representations of Dirac's equation.

II. GENERALIZED LAGRANGIAN SYSTEMS

Let us consider a classical system for which its Lagrangian is a function of time and of the generalized coordinates and their time derivatives up to order k, i.e., $L(t, q_i, ..., q_i^{(k)})$, where $q_i^{(k)} = d^k q_i / dt^k$. We can alternatively write the Lagrangian in such a way that it becomes a real function $L(x^j, \dot{x}^l)$ defined on a manifold X and its tangent space. The manifold X, called the kinematical space of the system, is the manifold spanned by the time t, the generalized coordinates q_i, and their derivatives $q_i^{(r)}$ up to order $k-1$, such that the Lagrangian written in this way is a first order homogeneous function of the derivatives of the kinematical variables.

Then it has the general expression

$$L(x, \dot{x}) = F_i(x, \dot{x}) \dot{x}^i, \tag{1}$$

where the $F_i(x, \dot{x}) = \partial L / \partial \dot{x}^i$, are homogeneous functions of zero degree of the derivatives $\dot{x}^i$, and the dot means differentiation with respect to some invariant evolution parameter τ. Because of this homogeneity condition, the Lagrangian formalism in terms of the kinematical variables is parameter independent. To obtain from Eq. (1) dynamical equations, the corresponding constraints among the kinematical variables have to be considered.

The relativity principle states that dynamical equations must be independent of the inertial observer and this implies that Lagrangians must transform as[7]

$$L(gx, g\dot{x}) = L(x, \dot{x}) + \frac{d\alpha(g; x)}{d\tau} = F_i(x, \dot{x}) \dot{x}^i + \frac{\partial \alpha(g; x)}{\partial x^i} \dot{x}^i, \tag{2}$$

where g is an element of the kinematical group G. Group G defines the class of equivalent inertial observers, and $\alpha(g; x)$ is a gauge function[7] defined on the manifold $G \times X$.

Gauge functions satisfy the identity

$$\alpha(g'; gx) + \alpha(g; x) - \alpha(g'g; x) = \xi(g', g), \tag{3}$$

where $\xi(g', g)$ is an exponent of the group G,[8] and g and g' two arbitrary group elements.

The mechanical action of the system from x_1 to x_2 is defined as the integral of the Lagrangian along the classical path $\bar{x}(\tau)$. It becomes a function A of the end points of the trajectory in X space

$$A(x_2, x_1) = \int_{\tau_1}^{\tau_2} L(\bar{x}(\tau), \dot{\bar{x}}(\tau)) d\tau \tag{4}$$

such that it transforms among inertial observers according to

$$A(gx_2, gx_1) = A(x_2, x_1) + \alpha(g; x_2) - \alpha(g; x_1), \tag{5}$$

with $gx(\tau)$ being the path followed by the particle as measured by observer O'.

End points x_1 and x_2, are precisely the end points that are held fixed in the corresponding variational problem. They are independent as initial and final conditions although there should exist differential constraints among the kinematical variables when considered as generalized coordinates.

In Refs. 1 and 2 a classical elementary particle is defined as a classical system for which the kinematical space X is a homogeneous space of the kinematical group G. This corresponds to

relating the possible inertial free motions with the one-parameter subgroups of G. Since a particle is left at x_1 and reaches x_2 freely, there exists a finite group element $g \in G$ such that $x_2 = g x_1$ and thus X is a homogeneous space of G.

All the kinematical groups considered, in particular the Galilei and Poincaré groups, are ten parameter groups. They are parameterized in an equivalent way in terms of the following parameters $(b, \mathbf{a}, \mathbf{v}, \boldsymbol{\alpha})$, where b and $\mathbf{a}$ with dimensions of time and length, represent the time and space translation, respectively, $\mathbf{v}$ is the relative velocity among observers, and the dimensionless magnitude $\boldsymbol{\alpha}$ represents their relative orientation, expressed in terms of a suitable parameterization of the rotation group. It turns out that the kinematical space of a particle with the highest structure is a ten-dimensional manifold whose variables share the same dimensionality as the above group parameterization. This manifold represents a spinning particle whose kinematical variables are identified with time t, position $\mathbf{r}$, velocity $\mathbf{u} = d\mathbf{r}/dt$, and orientation $\boldsymbol{\alpha}$. Since the Lagrangian also depends on their derivatives, it shows dependence on the angular velocity $\boldsymbol{\omega}$ as well as on the particle acceleration $d\mathbf{u}/dt = d\mathbf{r}^2/dt^2$ and this dependence on the second order derivatives of the position vector $\mathbf{r}$ is why we must necessarily work within a generalized Lagrangian formalism. Among the kinematical variables we find that $\mathbf{u}$ and $\boldsymbol{\alpha}$ are translationally invariant and thus characterize the internal structure of the system. We shall call them internal variables, to distinguish from t and $\mathbf{r}$, which describe the space–time evolution.

In this general case, the position of the particle and its center of mass do not coincide, and spin is related to the rotation and internal motion (*Zitterbewegung*) around the center of mass of the particle. If the Lagrangian shows no dependence on the acceleration, the spin is only of a rotational nature, and the position and center of mass position define the same point. Finally if in addition to this, the Lagrangian does not depend on the orientation we just have a spinless point particle for which the kinematical variables reduce to time and position.

A. Classical Galilei particles

The kinematical group is the Galilei group. The kinematical space spanned by variables $(t, \mathbf{r}, \mathbf{u}, \boldsymbol{\alpha})$ interpreted as time, position, velocity and orientation, respectively, gives rise to a system whose Lagrangian can be written by Eq. (1) as

$$L = -T\dot{t} + \mathbf{Q} \cdot \dot{\mathbf{r}} + \mathbf{U} \cdot \dot{\mathbf{u}} + \mathbf{Z} \cdot \boldsymbol{\omega},$$

with $T = -\partial L / \partial \dot{t}$, $Q_i = \partial L / \partial \dot{r}^i$, $U_i = \partial L / \partial \dot{u}^i$, and $Z_i = \partial L / \partial \omega^i$, the constraint $\mathbf{u} = d\mathbf{r}/dt = \dot{\mathbf{r}}(\tau)/\dot{t}(\tau)$, and where the angular velocity $\boldsymbol{\omega}$ is a linear function of the derivatives $\dot{\boldsymbol{\alpha}}$ of the corresponding parameterization of internal orientation.

Noether's theorem when applied to the Galilei group leads to the ten constants of the motion

$$H = T - \mathbf{u} \cdot \frac{d\mathbf{U}}{dt},$$

$$\mathbf{P} = \mathbf{Q} - \frac{d\mathbf{U}}{dt},$$

$$\mathbf{G} = m\mathbf{r} - \mathbf{P}t - \mathbf{U},$$

$$\mathbf{J} = \mathbf{r} \times \mathbf{P} + \mathbf{u} \times \mathbf{U} + \mathbf{Z} = \mathbf{L} + \mathbf{S}$$

interpreted as usual as the total energy, linear momentum, Galilei or static momentum, and angular momentum, respectively.

We see that total angular momentum is the sum of the orbital part $\mathbf{L}$ and the spin $\mathbf{S}$ that has two terms: One is $\mathbf{u} \times \mathbf{U}$, related to the dependence of the Lagrangian on the acceleration and the other is $\mathbf{Z}$, which comes from its dependence on the angular velocity. If we define $\mathbf{U} = m\mathbf{k}$ in terms

of the magnitude $\mathbf{k}$ with dimension of length, then $\dot{\mathbf{G}}=0$ leads to $\mathbf{P}=md(\mathbf{r}-\mathbf{k})/dt$ and the vector $\mathbf{q}=\mathbf{r}-\mathbf{k}$ defines the center of mass position, while $\mathbf{k}$ represents the relative position of point $\mathbf{r}$ with respect to the center of mass. For a free particle, the center of mass moves with constant velocity while point $\mathbf{r}$ is moving around it, and this internal motion is interpreted as the *Zitterbewegung*.

B. Classical relativistic particles

We shall consider three different homogeneous spaces of the Poincaré group. We have first a homogeneous space spanned by variables $(t,\mathbf{r},\mathbf{u},\boldsymbol{\alpha})$ with the same interpretation as above, but now with $u<c$.

The constants of the motion can be written as

$$H=T-\mathbf{u}\cdot\frac{d\mathbf{U}}{dt},$$

$$\mathbf{P}=\mathbf{Q}-\frac{d\mathbf{U}}{dt},$$

$$\boldsymbol{\pi}=-H\mathbf{r}/c+\mathbf{P}ct+\mathbf{D},$$

$$\mathbf{J}=\mathbf{r}\times\mathbf{P}+\mathbf{u}\times\mathbf{U}+\mathbf{Z}=\mathbf{L}+\mathbf{S},$$

where the different observables in these expressions are obtained from the Lagrangian with the same definitions as above but now the generalized angular velocity $\boldsymbol{\omega}$ is a linear function of both $\dot{\boldsymbol{\alpha}}$ and $\dot{\mathbf{u}}$, and the $\mathbf{D}$ function is expressed in a rather involved way in terms of $\mathbf{U}$ and $\mathbf{S}$. For explicit expressions the reader is referred to Ref. 2.

If in terms of $\mathbf{D}$ we define the position vector $\mathbf{k}$, $\mathbf{D}=H\mathbf{k}/c$, then time differentiation of the Poincaré momentum $\boldsymbol{\pi}$ leads to $\mathbf{P}=(H/c^2)d(\mathbf{r}-\mathbf{k})/dt$ so that $\mathbf{q}=\mathbf{r}-\mathbf{k}$ represents the center of mass position and $\mathbf{k}$ is again the relative position of point $\mathbf{r}$ with respect to the center of mass.

The nine-dimensional manifold spanned by variables $(t,\mathbf{r},\mathbf{u},\boldsymbol{\alpha})$ but now with $u=c$, is another homogeneous space of the Poincaré group $\mathscr{P}$. The constants of the motion have the same general expressions as above except that the Poincaré momentum in this case reads

$$\boldsymbol{\pi}=-H\mathbf{r}/c+\mathbf{P}ct-\mathbf{u}\times\mathbf{S}/c \tag{6}$$

and the angular velocity is only expressed in terms of the time derivatives of the orientation $\boldsymbol{\alpha}$. The relative position vector $\mathbf{k}$ comes from $-\mathbf{u}\times\mathbf{S}=H\mathbf{k}$ and again $\mathbf{P}=(H/c^2)d\mathbf{q}/dt$, where $\mathbf{q}=\mathbf{r}-\mathbf{k}$.

The time derivative of $\boldsymbol{\pi}$ and the scalar product with $\mathbf{u}$ give rise to the Poincaré invariant relation

$$H-\mathbf{P}\cdot\mathbf{u}-\frac{1}{c^2}\left(\frac{d\mathbf{u}}{dt}\times\mathbf{u}\right)\cdot\mathbf{S}=0. \tag{7}$$

Since $\mathbf{u}\cdot\dot{\mathbf{u}}=0$ the particle describes a circle with velocity c for the center of mass observer, in a plane orthogonal to $\mathbf{S}$ which is constant in this frame.

It is the quantization of this system that leads to Dirac's equation.

Finally, the homogeneous space $X=\mathscr{P}/SO(3)$ spanned by variables $(t,\mathbf{r},\mathbf{u})$ with $u>c$ describes particles with internal tachyonic motion and no rotation, since for $u>c$ no general homogeneous space of $\mathscr{P}$ with angular variables can be defined.

C. Two position vectors

Vector **r**, transforming as a position vector, is interpreted as the position of the particle. But if the Lagrangian depends on the acceleration, a new position vector **q** has been defined in terms in which the total linear momentum can be expressed as $\mathbf{P}=md\mathbf{q}/dt$ in the Galilei case and $\mathbf{P}=(H/c^2)d\mathbf{q}/dt$ in the relativistic formulation as seen above. This vector **q** clearly represents the center of mass position. Then, what position does the vector **r** represent?

Interaction with some external field suggests to interpret **r** in certain situations as the charge position. In fact, let $L_0(x,\dot{x})$ be the Poincaré invariant Lagrangian of a generalized classical spinning free particle. If there is some interaction with an external field, we have to add an interacting term $L_I(x,\dot{x},\phi_i)$ depending on the kinematical variables of the system x, their first derivatives $\dot{x}$, and on some external sources $\phi_i(x,\dot{x})$ which are in general functions of x and $\dot{x}$. However L_I must necessarily be a first order homogeneous function of $\dot{x}$. For instance, if we assume a minimal coupling we can write $L_I = -\phi(t,\mathbf{r})\dot{t}+\mathbf{A}(t,\mathbf{r})\cdot\dot{\mathbf{r}}$, linear in the derivatives and where the source functions are only time and position dependent. The interaction Lagrangian L_I is Poincaré invariant if $(\phi,\mathbf{A})$ transform like a Minkowski four-vector as happens if the source is an external electromagnetic field. But the potentials and thus the electric and magnetic fields are evaluated not at the center of mass position **q** but at the position **r**. In this minimal coupling interaction, **r** seems to be the electric charge position. For arbitrary interactions, there will be terms depending on $\dot{\mathbf{u}}$ and ω and the source functions could also be derivative dependent, and position **r** will be related to the position of some other attribute or generalized charge of the particle.

The kind of particle this formalism describes is a rigid rotator. Let us assume a rigid body of any shape and mass m. If charged, with total charge e arbitrarily distributed, the center of mass and center of charge are not in general coincident points. Rotation can be described by the evolution of the corresponding orthogonal frame linked to the body. The free motion of this model is a straight line trajectory for the center of mass at constant speed and a rotation of the body fixed axis. This rotation can be decomposed into the rotation of the center of charge around the center of mass and finally a possible rotation around the direction determined by these two points. When interacting with an external electromagnetic field, dynamical equations describe the center of mass evolution, but to determine the electromagnetic force it is necessary to know the charge trajectory where the external fields are evaluated. It is important to realize that this kind of description is independent of the shape and size of the object and in the limit; if there is no multipolar structure, what we have are just two position vectors linked, respectively, to m and e and three orthogonal directions to describe orientation. These kinds of objects have obtained a Lagrangian description[1,2] in terms of charge position **r** and orientation α, but allow dependence of the Lagrangian on the charge acceleration $d^2\mathbf{r}/dt^2$, where the center of mass position **q** is a derived observable.

For relativistic particles, the center of mass **q** can never reach the speed of light, but for the charge position **r** we have found[2] three separate classes of particles for which the internal charge velocity $d\mathbf{r}/dt$ can be less, equal, or greater than c. In fact, it is the quantization of a particle whose charge is circling around the center of mass with velocity c that leads to Dirac's equation, as will be shown in Sec. V.

It is the aim of this contribution to quantize these Lagrangian systems and show the spin structure dependence on the internal variables, velocity and orientation. In particular half integer spins arise from the dependence of the wave function on the orientation while if spin is related only to the *Zitterbewegung* then half integer spins can never arise.

III. QUANTIZATION OF GENERALIZED LAGRANGIAN SYSTEMS

Let us consider a generalized Lagrangian system as described above whose evolution takes place on the kinematical space between points x_1 and x_2.

For quantizing these generalized Lagrangian systems we shall follow Feynman's path integral method.[9] The uncertainty principle is introduced in Feynman's approach by the condition that if no measurement is performed to determine the trajectory followed by the system from x_1 to x_2 then all paths are allowed with the same probability. But being that the different paths are interfering alternatives, their probabilities must be calculated from a probability amplitude. Thus, to every possible trajectory followed by the system, $x(\tau)$ in X space, Feynman associates a complex number $\phi[x(\tau)]$ called the probability amplitude of this alternative, given by

$$\phi[x(\tau)] = N \exp\left\{\frac{i}{\hbar}\int_{\tau_1}^{\tau_2} L(x(\tau),\dot{x}(\tau))d\tau\right\} = N \exp\left\{\frac{i}{\hbar} A_{[x]}(x_2,x_1)\right\}, \tag{8}$$

where N is a path independent normalization factor, and where the phase in units of $\hbar$ is the classical action of the system $A_{[x]}(x_2,x_1)$ along the path $x(\tau)$. This probability amplitude is clearly a function of the initial and final points x_1 and x_2 in X space, respectively.

The total probability amplitude that the system will arrive at x_2 coming from x_1, i.e., Feynman's kernel $K(x_2,x_1)$, is obtained as the sum or integration over all paths of terms of the form of Eq. (8). Then Feynman's kernel $K(x_2,x_1)$, will be in general a function or more precisely a distribution on the $X \times X$ manifold. If information concerning the initial point is lost, and the final point is left arbitrary, say x, the kernel reduces to the probability amplitude for finding the system at point x, i.e., the wave function $\Phi(x)$. By the above discussion we see that wave functions must be functions of the kinematical variables.

To see how the wave function transforms between inertial observers, let O and O' be two inertial observers related by means of a transformation $g \in G$, such that the kinematical variables transforms as

$$x'^i = f^i(g,x).$$

If points x_1 and x_2 are close enough, then the kernel reduces to the probability amplitude along the classical path joining them. If for O the system follows path $\bar{x}(\tau)$, it follows for O' the path $\bar{x}'(\tau) = f(g,\bar{x}(\tau))$ and because the action along classical paths transforms according to Eq. (5), then Feynman's kernel transforms as

$$K'(x_2',x_1') = K(x_2,x_1)\exp\left\{\frac{i}{\hbar}\left(\alpha(g;x_2) - \alpha(g;x_1)\right)\right\}$$

and if the information concerning the initial point is lost, the wave function transforms as the part related to the variables x_2

$$\Phi'(x'(x)) = \Phi'(gx) = \Phi(x)\exp\left\{\frac{i}{\hbar}\left(\alpha(g;x) + \theta(g)\right)\right\}$$

or in terms of unprimed x variables

$$\Phi'(x) = \Phi(g^{-1}x)\exp\left\{\frac{i}{\hbar}\left(\alpha(g;g^{-1}x) + \theta(g)\right)\right\}, \tag{9}$$

where $\theta(g)$ is some function defined on G but independent of x. Since $|\Phi(x)|^2 d\mu(x)$ where $d\mu(x)$ is the volume element in X space is interpreted as the probability of finding the system inside the volume element $d\mu(x)$ around point x, the probability of finding it anywhere in X must be unity, so that

$$\int_X |\Phi(x)|^2 \, d\mu(x) = 1.$$

Since from Eq. (9)

$$|\Phi'(x')|^2 = |\Phi(x)|^2 \tag{10}$$

it is sufficient for the conservation of probability to assume that the measure $\mu(x)$ be group invariant. Equation (10) implies also that inertial observers measure locally the same probability. Consequently, the Hilbert space $\mathcal{H}$ whose unit rays represent the pure states of the system is the space of squared-integrable functions $L^2(X,\mu)$ defined on the kinematical space X, with $\mu(x)$ being an invariant measure such that the scalar product on $\mathcal{H}$ is defined as

$$\langle \Phi | \Psi \rangle = \int_X \Phi^*(x)\Psi(x)d\mu(x), \tag{11}$$

with $\Phi^*(x)$ being the complex conjugate function of $\Phi(x)$.

The relativity principle and Wigner's theorem[10,11] imply that to every symmetry $g \in G$ of a continuous group, there exists a one to one mapping of unit rays into unit rays that is induced on $\mathcal{H}$ by a unitary operator $U(g)$ defined up to a phase that maps a wave function defined on x into an arbitrary wave function of the image unit ray in x'. If we interpret $\Phi(x)$ as the wave function that describes the state of the system for the O observer and $\Phi'(x)$ for O', then we have

$$U(g)\Phi(x) = \Phi'(x) = \Phi(g^{-1}x)\exp\left\{\frac{i}{\hbar}\,\alpha(g;g^{-1}x) + \theta(g)\right\}.$$

Since the $\theta(g)$ function gives rise to a constant phase we can neglect it and then we take as the definition of the unitary representation of the group G on Hilbert's space $\mathcal{H}$

$$\Phi'(x) = U(g)\Phi(x) = \Phi(g^{-1}x)\exp\left\{\frac{i}{\hbar}\,\alpha(g;g^{-1}x)\right\}. \tag{12}$$

Since the gauge functions satisfy Eq. (3), the phase term can be replaced by

$$\alpha(g;g^{-1}x) = -\alpha(g^{-1};x) + \alpha(0;x) + \xi(g,g^{-1}) = -\alpha(g^{-1};x) + \zeta(g)$$

because gauge functions can always be chosen such that $\alpha(0;x) = 0$ and the group function $\zeta(g)$, giving rise also to a constant phase, can be suppressed.

If the unitary operator is represented in terms of the corresponding self-adjoint generators of the group algebra

$$U(g) = \exp\left\{\frac{i}{\hbar}\,X_\sigma g^\sigma\right\}$$

then the self-adjoint operators X_σ when acting on the wave functions have the differential representation

$$X_\sigma = \frac{\hbar}{i}\,u_\sigma^j(x)\,\frac{\partial}{\partial x^j} - v_\sigma(x), \tag{13}$$

where

$$u_\sigma^j(x) = \frac{\partial f^j(g,x)}{\partial g^\sigma}\bigg|_{g=0}, \quad v_\sigma(x) = \frac{\partial \alpha(g;x)}{\partial g^\sigma}\bigg|_{g=0}.$$

Because of the presence of the $v_\sigma(x)$ term in Eq. (13), the X_σ generators do not satisfy in general the commutation relations of the group G, but rather the commutation relations of a central extension of G. The group representation is not a true representation but a projective representation of G as shown in Ref. 8.

In fact

$$U(g_1)\Phi(x) = \Phi(g_1^{-1}x)\exp\left\{\frac{i}{\hbar}\,\alpha(g_1;g_1^{-1}x)\right\}$$

acting on the left with $U(g_2)$

$$U(g_2)U(g_1)\Phi(x) = U(g_2)\Phi(g_1^{-1}x)\exp\left\{\frac{i}{\hbar}\,\alpha(g_1;g_1^{-1}x)\right\}$$

$$= \Phi((g_2g_1)^{-1}x)\exp\left\{\frac{i}{\hbar}\,\alpha(g_2;g_2^{-1}x)\right\}\exp\left\{\frac{i}{\hbar}\,\alpha(g_1;(g_2g_1)^{-1}x)\right\},$$

while acting on $\Phi(x)$ with $U(g_2g_1)$

$$U(g_2g_1)\Phi(x) = \Phi((g_2g_1)^{-1}x)\exp\left\{\frac{i}{\hbar}\,\alpha(g_2g_1;(g_2g_1)^{-1}x)\right\}.$$

If we define $(g_2g_1)^{-1}x = z$ then $g_1z = g_2^{-1}x$ and thus since gauge functions satisfy Eq. (3)

$$\alpha(g_2;g_1z) + \alpha(g_1;z) = \alpha(g_2g_1;z) + \xi(g_2,g_1)$$

we obtain

$$U(g_2)U(g_1)\Phi(x) = U(g_2g_1)\Phi(x)\exp\left\{\frac{i}{\hbar}\,\xi(g_2;g_1)\right\}$$

and if $\Phi(x)$ is arbitrary we have a projective representation of the group G.

For both Galilei and Poincaré particles the kinematical space is the ten-dimensional manifold spanned by the variables $(t,\mathbf{r},\mathbf{u},\boldsymbol{\alpha})$, t being the time, $\mathbf{r}$ the position, $\mathbf{u}$ the velocity, and $\boldsymbol{\alpha}$ the orientation of the particle. Thus in the quantum formalism the wave function of an elementary particle is a squared-integrable function $\Phi(t,\mathbf{r},\mathbf{u},\boldsymbol{\alpha})$ of these kinematical variables. For point particles, the kinematical space is just the four-dimensional space–time, so that wave functions are only functions of time and position, but spinning particles will have to depend on some additional variables like velocity and orientation giving rise to these additional variables to the spin structure.

IV. NONRELATIVISTIC SPINNING PARTICLES

Let $\mathscr{G}$ be the Galilei group. Let us first consider Galilei particles with (anti)orbital spin. This corresponds to systems for which $X = \mathscr{G}/SO(3)$ and thus the kinematical variables are time, position, and velocity. A particular classical example is given by the Lagrangian

$$L = \frac{m}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2 - \frac{m}{2\omega^2}\left(\frac{d\mathbf{u}}{dt}\right)^2,$$

with $\mathbf{u} = d\mathbf{r}/dt$. The particle trajectory is

$$\mathbf{r}(t) = \mathbf{A} + \mathbf{B}t + \mathbf{C} \cos \omega t + \mathbf{D} \sin \omega t = \mathbf{q}(t) + \mathbf{k}(t),$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are constants. The center of mass $\mathbf{q}(t) = \mathbf{A} + \mathbf{B}t$ has a straight motion while the relative position vector $\mathbf{k}(t) = \mathbf{C} \cos \omega t + \mathbf{D} \sin \omega t$ experiences an elliptic motion of angular frequency ω around its center of mass, the spin being the consequence of this internal motion.

The wave functions are functions on X and thus functions of the variables $(t, \mathbf{r}, \mathbf{u})$. On this kinematical space the gauge function is[7]

$$\alpha(g; x(\tau)) = m(\mathbf{v}^2 t(\tau)/2 + \mathbf{v} \cdot R(\boldsymbol{\alpha}) \mathbf{r}(\tau)),$$

where $\mathbf{v}$ and $\boldsymbol{\alpha}$ are group parameters, m defines the mass of the system and thus the ten self-adjoint generators of the projective unitary representation of the Galilei group $\mathscr{G}$ are given by

$$H = i\hbar \frac{\partial}{\partial t}, \quad \mathbf{P} = \frac{\hbar}{i} \nabla, \quad \mathbf{K} = \frac{\hbar}{i} t \nabla + \frac{\hbar}{i} \nabla_u - m\mathbf{r}, \quad \mathbf{J} = \frac{\hbar}{i} \mathbf{r} \times \nabla + \frac{\hbar}{i} \mathbf{u} \times \nabla_u = \mathbf{L} + \mathbf{S},$$

∇_u being the gradient operator with respect to the $\mathbf{u}$ variables.

One Casimir operator of this realization is the internal energy $H - \mathbf{P}^2/2m$. We see that the spin operator only differentiates with respect to the velocity variables, and consequently commutes with H and $\mathbf{P}$, so that we can find simultaneously the eigenstates of the three commuting operators $H - \mathbf{P}^2/2m$, S^2 and S_3. Because the spin operators only affect the wave function in its dependence on the $\mathbf{u}$ variables we can choose functions with the variables separated in the form $\Phi(t, \mathbf{r}, \mathbf{u}) = \Sigma_i \ \psi_i(t, \mathbf{r}) \chi_i(\mathbf{u})$ so that

$$(H - \mathbf{P}^2/2m) \psi_i(t, \mathbf{r}) = E \psi_i(t, \mathbf{r}), \quad S^2 \chi_i(\mathbf{u}) = s(s+1)\hbar^2 \chi_i(\mathbf{u}), \quad S_3 \chi_i(\mathbf{u}) = m_s \hbar \chi_i(\mathbf{u})$$

and thus the space–time dependent wave function is uncoupled with the spin part and satisfies Schrödinger's equation. Due to the S^2 structure in terms of the $\mathbf{u}$ variables, which is that of an orbital angular momentum, the spin part of the wave function is of the form

$$\chi(\mathbf{u}) = f(u) Y_s^{m_s}(\theta, \phi),$$

with $f(u)$ being an arbitrary function of the modulus of $\mathbf{u}$ and $Y_s^{m_s}(\theta, \phi)$ the spherical harmonics on the direction of $\mathbf{u}$.

Classical spinning particles with spin of orbital nature do not lead to half integer spin values.

Another examples of spinning particles are those which have orientation and thus angular velocity. For instance, if $X = \mathscr{G}/R_v^3$, R_v^3 being the subgroup of pure Galilei transformations, then the kinematical space is spanned by the variables $(t, \mathbf{r}, \boldsymbol{\alpha})$. This corresponds, for instance, to the Lagrangian system

$$L = \frac{m}{2} \left(\frac{d\mathbf{r}}{dt} \right)^2 + \frac{I}{2} \boldsymbol{\omega}^2.$$

The particle travels at constant velocity while it rotates with constant angular velocity $\boldsymbol{\omega}$. The spin is just $\mathbf{S} = I\boldsymbol{\omega}$, and the center of charge and center of mass represent the same point.

To describe orientation we can think on the three orthogonal unit vectors $\mathbf{e}_i$, $i = 1, 2, 3$ linked to the body, similarly as in a rigid rotator. If initially they are taken parallel to the spatial Cartesian axis, then their nine components considered by columns define an orthogonal rotation matrix $R_{ij}(\boldsymbol{\alpha})$ that describes the triad evolution with the initial condition $R_{ij}(t=0) = \delta_{ij}$. This matrix is usually parameterized in terms of Euler's angles. In this work we shall use two alternative parameterizations: One in terms of a three-vector $\boldsymbol{\rho} = \tan(\alpha/2)\mathbf{a}$, with $\mathbf{a}$ being a unit vector along the rotation axis and α the rotated angle, and the normal or canonical representation in terms of

another three-vector $\boldsymbol{\alpha} = \alpha \mathbf{a}$. Rotations around the coordinate axis are treated in these parameterizations in a more symmetrical way. For the normal parameterization and in the laboratory frame the body axis has Cartesian components given by

$$(\mathbf{e}_i)_j = R_{ji}(\boldsymbol{\alpha}) = \delta_{ji} \cos \alpha + a_j a_i (1 - \cos \alpha) - \epsilon_{jik} a_k \sin \alpha, \tag{14}$$

and in the $\boldsymbol{\rho}$ parameterization by

$$(\mathbf{e}_i)_j = R_{ji}(\boldsymbol{\rho}) = \frac{1}{1+\rho^2} ((1-\rho^2)\delta_{ji} + 2\rho_j\rho_i - 2\epsilon_{jik}\rho_k), \tag{15}$$

and where the Cartesian components of the rotation axis unit vector $\mathbf{a}$ are

$$a_1 = \sin \theta \cos \phi, \quad a_2 = \sin \theta \sin \phi, \quad a_3 = \cos \theta,$$

where θ is the polar angle and ϕ the usual azimuthal angle.

On the corresponding Hilbert space, the Galilei generators are given by

$$H = i\hbar \frac{\partial}{\partial t}, \quad \mathbf{P} = \frac{\hbar}{i} \boldsymbol{\nabla}, \quad \mathbf{K} = \frac{\hbar}{i} t \boldsymbol{\nabla} - m\mathbf{r}, \tag{16}$$

$$\mathbf{J} = \frac{\hbar}{i} \mathbf{r} \times \boldsymbol{\nabla} + \frac{\hbar}{2i} [\boldsymbol{\nabla}_\rho + \boldsymbol{\rho} \times \boldsymbol{\nabla}_\rho + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \boldsymbol{\nabla}_\rho)] = \mathbf{L} + \mathbf{S}, \tag{17}$$

with $\boldsymbol{\nabla}_\rho$ being the gradient operator with respect to the $\boldsymbol{\rho}$ variables and in the $\boldsymbol{\rho}$ parameterization of the rotation group.

Here again the spin operator commutes with H and $\mathbf{P}$ and the wave function can be separated, $\Phi(t,\mathbf{r},\boldsymbol{\rho}) = \Sigma_i \psi_i(t,\mathbf{r})\chi_i(\boldsymbol{\rho})$, leading to the equations

$$(H - \mathbf{P}^2/2m)\psi_i(t,\mathbf{r}) = E\psi_i(t,\mathbf{r}),$$

$$S^2\chi_i(\boldsymbol{\rho}) = s(s+1)\hbar^2\chi_i(\boldsymbol{\rho}), \tag{18}$$

$$S_3\chi_i(\boldsymbol{\rho}) = m_s\hbar\chi_i(\boldsymbol{\rho}). \tag{19}$$

Bopp and Haag[3] succeeded in finding $s = 1/2$ solutions for the system of equations (18) and (19). They are called Wigner's functions.[12] Solutions of Eq. (18) for arbitrary spin s are but a linear combination of the matrix elements of a $(2s+1) \times (2s+1)$ irreducible matrix representation of the rotation group as derived from the Peter–Weyl theorem on finite representations of compact groups.[13,14] We shall deal with the $s = 1/2$ functions in the next section where explicit expressions will be given.

V. RELATIVISTIC PARTICLES

We can similarly quantize classical relativistic particles. We shall pay attention however to the kinematical space of particles traveling at the speed of light.

In Ref. 2, when dealing with Poincaré particles, we found a nine-dimensional homogeneous space of the Poincaré group, spanned by the ten variables $(t,\mathbf{r},\mathbf{u},\boldsymbol{\alpha})$ similarly as in the Galilei case, but now with $\mathbf{u}$ restricted to $u = c$. For this system since $\mathbf{u} \cdot \dot{\mathbf{u}} = 0$, they describe particles with a circular internal orbital motion at the constant speed c.

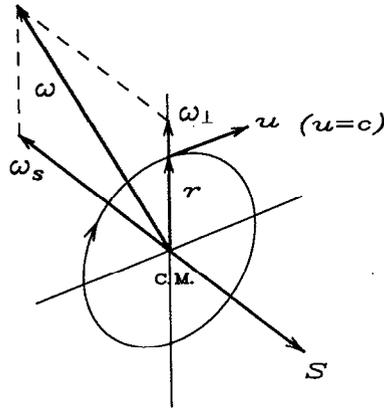The following Poincaré invariant Lagrangian, built from that proposed in Ref. 2

FIG. 1. Motion of the center of change of the electron in the center of mass frame.

$$L = \frac{1}{2}\, mc^3 \, \frac{\boldsymbol{\omega} \cdot \dot{\mathbf{u}}}{(d\mathbf{u}/dt + \mathbf{u} \times \boldsymbol{\omega})^2}$$

is defined on this kinematical space where $\boldsymbol{\omega}$ is the angular velocity of the body conveniently expressed in terms of the orientation $\boldsymbol{\alpha}$ and its time derivative, $u = c$ is the center of charge velocity and $d\mathbf{u}/dt = \dot{\mathbf{u}}/\dot{t}$, the charge acceleration.

When solved in the center of mass frame, (see Fig. 1) the center of charge describes a circle of radius $R_0 = S/mc$ at the constant speed c, being the spin

$$\mathbf{S} = \mathbf{u} \times \mathbf{U} + \mathbf{Z} = \frac{1}{2}\, mc^3 \, \frac{d\mathbf{u}/dt + \mathbf{u} \times \boldsymbol{\omega}}{(d\mathbf{u}/dt + \mathbf{u} \times \boldsymbol{\omega})^2} \,,$$

orthogonal to the charge trajectory plane and a constant of the motion in this frame.

The angular velocity in this frame has two components: one, $\omega_s = mc^2/S$, in the opposite direction to the constant spin $\mathbf{S}$ and the other orthogonal to it, $\omega_\perp = \omega_s/2$, lying along the direction from the center of mass to the center of charge and which is half of the other component. This implies that the internal motion is periodic, of frequency $\omega_s/4\pi$, such that when the body comes back to its initial position, the center of charge has exactly had two turns around the center of mass. Thus, this internal electric current contributes with a two turn loop to the particle magnetic moment, supplying a naive interpretation of the $g = 2$ gyromagnetic ratio.

When quantized, the wave function of the system is a function $\Psi(t, \mathbf{r}, \mathbf{u}, \boldsymbol{\alpha})$ of these kinematical variables. For the Poincaré group all exponents and thus all gauge functions on homogeneous spaces are equivalent to zero, and the Lagrangians for free particles can thus be taken strictly invariant. Projective representations reduce to true representations so that the ten generators on the Hilbert space are given by

$$H = i\hbar \, \frac{\partial}{\partial t}, \quad \mathbf{P} = \frac{\hbar}{i}\, \boldsymbol{\nabla}, \quad \mathbf{K} = \frac{\hbar}{i}\, \frac{\mathbf{r}}{c}\, \frac{\partial}{\partial t} + \frac{\hbar}{i}\, ct\boldsymbol{\nabla} - \frac{1}{c}\, \mathbf{u} \times \mathbf{S}, \quad \mathbf{J} = \frac{\hbar}{i}\, \mathbf{r} \times \boldsymbol{\nabla} + \mathbf{S},$$

where the spin is the differential operator

$$\mathbf{S} = \frac{\hbar}{i}\, \mathbf{u} \times \boldsymbol{\nabla}_u + \frac{\hbar}{2i}\, (\boldsymbol{\nabla}_\rho + \boldsymbol{\rho} \times \boldsymbol{\nabla}_\rho + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \boldsymbol{\nabla}_\rho)) = \mathbf{S}_u + \mathbf{S}_\rho$$

and where the differential operators ∇_u and ∇_ρ are the corresponding gradient operators with respect to the $\mathbf{u}$ and $\boldsymbol{\rho}$ variables as in the Galilei case.

To obtain the complete commuting set of observables we start with the Casimir invariant operator, or Klein–Gordon operator

$$KG \equiv H^2 - c^2\mathbf{P}^2 - m^2c^4 = 0 \tag{20}$$

such that in the above representation only differentiates the wave function with respect to position $\mathbf{r}$ and time t. Since the spin only operates on the velocity and orientation variables, it commutes with the Klein–Gordon operator (20). Thus, we can find simultaneous eigenfunctions of Eq. (20), S^2, and S_3. This allows us to try solutions in separate variables so that the wave function can be written as

$$\Psi(t,\mathbf{r},\mathbf{u},\boldsymbol{\rho}) = \sum_i \psi_i(t,\mathbf{r})\Phi_i(\mathbf{u},\boldsymbol{\rho}),$$

where $\psi_i(t,\mathbf{r})$ are the space–time components and the $\Phi_i(\mathbf{u},\boldsymbol{\rho})$ represent the internal spin structure. Consequently

$$(H^2 - c^2\mathbf{P}^2 - m^2c^4)\psi_i(t,\mathbf{r}) = 0, \tag{21}$$

i.e., space–time components satisfy the Klein–Gordon equation, while the internal structure part satisfies

$$S^2\Phi_i(\mathbf{u},\boldsymbol{\rho}) = s(s+1)\hbar^2\Phi_i(\mathbf{u},\boldsymbol{\rho}), \tag{22}$$

$$S_3\Phi_i(\mathbf{u},\boldsymbol{\rho}) = m_s\hbar\,\Phi_i(\mathbf{u},\boldsymbol{\rho}). \tag{23}$$

To find solutions of Eqs. (22) and (23) we see that the spin can be separated into two commuting angular momentum parts $\mathbf{S} = \mathbf{S}_u + \mathbf{S}_\rho$: one, $\mathbf{S}_u$, that differentiates with respect to the $\mathbf{u}$ variables and related to the *Zitterbewegung* and the other, $\mathbf{S}_\rho$, that only acts on the orientation variables and is thus related to the internal rotational motion. The total spin squared

$$S^2 = \mathbf{S}_u^2 + \mathbf{S}_\rho^2 + 2\mathbf{S}_u\cdot\mathbf{S}_\rho \tag{24}$$

is expressed as the sum of three commuting terms and its eigenvectors are obtained as the simultaneous eigenvectors of the three commuting operators on the right hand side of Eq. (24). Thus, each $\Phi_i(\mathbf{u},\boldsymbol{\rho})$ can again be separated as

$$\Phi_i(\mathbf{u},\boldsymbol{\rho}) = \sum_j U_{ij}(\mathbf{u})V_{ij}(\boldsymbol{\rho}).$$

Functions $U_{ij}(\mathbf{u})$ are spherical harmonics defined on the orientation of the velocity vector $\mathbf{u}$ because the $\mathbf{S}_u$ operator has the structure of an orbital angular momentum in terms of the $\mathbf{u}$ variables, and thus its eigenvalues are integer numbers.

For spin-1/2 particles, if we first take for simplicity the eigenfunctions $V_{ij}(\boldsymbol{\rho})$ of S_ρ^2 with eigenvalue 1/2, and then since the total spin has to be 1/2, the orbital $\mathbf{S}_u$ part can only contribute with spherical harmonics of value $l_u = 0$ and $l_u = 1$.

Finally if the $\Phi_i(\mathbf{u},\boldsymbol{\rho})$ functions have to be eigenfunctions of the operator $\mathbf{S}_u\cdot\mathbf{S}_\rho$ this leads through a straightforward calculation to where only the $l_u = 0$ case contributes and it turns out that the Φ_i functions are independent of the velocity variables.

Then spin-1/2 functions $\Phi_i^{1/2}(\rho)$ are linear combinations of the four Wigner's functions[12,15] so that the Hilbert space that describes the internal structure of this particle is isomorphic to the four-dimensional Hilbert space $\mathbf{C}^4$.

Several parameterizations have been used for these functions. In the work of Bopp and Haag[3] and in Dahl's contribution,[6] they are expressed in terms of Euler's angles. Bacry and Kihlberg[5] use another parameterization. In what follows we shall use both the normal or canonical parameterization of the rotation group $\boldsymbol{\alpha}=\alpha\mathbf{a}$ and the $\boldsymbol{\rho}=\tan(\alpha/2)\mathbf{a}$ parameterization.

In terms of the parameterization $\boldsymbol{\rho}$, the spin operators are given by

$$\mathbf{S}=\frac{\hbar}{2i}\left(\nabla_\rho+\boldsymbol{\rho}\times\nabla_\rho+\boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)\right) \tag{25}$$

as in Eq. (17) and in the normal parameterization $\boldsymbol{\alpha}$ by

$$
\begin{aligned}
S_1=\frac{\hbar}{2i}\Bigg\{ & 2\,\sin\,\theta\,\cos\,\phi\,\frac{\partial}{\partial\alpha}+\left(\frac{\cos\,\theta\,\cos\,\phi}{\tan(\alpha/2)}-\sin\,\phi\right)\frac{\partial}{\partial\theta}\\
& -\left(\frac{\sin\,\phi}{\sin\,\theta\,\tan(\alpha/2)}+\frac{\cos\,\theta\,\cos\,\phi}{\sin\,\theta}\right)\frac{\partial}{\partial\phi}\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
S_2=\frac{\hbar}{2i}\Bigg\{ & 2\,\sin\,\theta\,\sin\,\phi\,\frac{\partial}{\partial\alpha}+\left(\frac{\cos\,\theta\,\sin\,\phi}{\tan(\alpha/2)}+\cos\,\phi\right)\frac{\partial}{\partial\theta}\\
& +\left(\frac{\cos\,\phi}{\sin\,\theta\,\tan(\alpha/2)}-\frac{\cos\,\theta\,\sin\,\phi}{\sin\,\theta}\right)\frac{\partial}{\partial\phi}\Bigg\},
\end{aligned}
$$

$$
S_3=\frac{\hbar}{2i}\left(2\,\cos\,\theta\,\frac{\partial}{\partial\alpha}-\frac{\sin\,\theta}{\tan(\alpha/2)}\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\phi}\right).
$$

These operators satisfy the commutation relations

$$[S_i,\,S_j]=i\hbar\,\epsilon_{ijk}S_k. \tag{26}$$

If we define the spin projections on the body axis $\mathbf{e}_i$, $Z_i=\mathbf{e}_i\cdot\mathbf{S}$ then they satisfy[15]

$$[Z_i,\,Z_j]=-i\hbar\,\epsilon_{ijk}Z_k, \tag{27}$$

$$[Z_i,\,S_j]=0 \tag{28}$$

so that the Z_i spin operators satisfy the so-called "anomalous" commutation relations (27) while they commute with the spin components S_i. They are explicitly given by

$$\mathbf{Z}=\frac{\hbar}{2i}\left(\nabla_\rho-\boldsymbol{\rho}\times\nabla_\rho+\boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)\right)$$

in the $\boldsymbol{\rho}$ parameterization and in terms of canonical variables by

$$
\begin{aligned}
Z_1=\frac{\hbar}{2i}\Bigg\{ & 2\,\sin\,\theta\,\cos\,\phi\,\frac{\partial}{\partial\alpha}+\left(\frac{\cos\,\theta\,\cos\,\phi}{\tan(\alpha/2)}+\sin\,\phi\right)\frac{\partial}{\partial\theta}\\
& -\left(\frac{\sin\,\phi}{\sin\,\theta\,\tan(\alpha/2)}-\frac{\cos\,\theta\,\cos\,\phi}{\sin\,\theta}\right)\frac{\partial}{\partial\phi}\Bigg\},
\end{aligned}
$$

$$Z_2 = \frac{\hbar}{2i} \left\{ 2 \sin \theta \sin \phi \frac{\partial}{\partial \alpha} + \left(\frac{\cos \theta \sin \phi}{\tan(\alpha/2)} - \cos \phi \right) \frac{\partial}{\partial \theta} \right.$$

$$\left. + \left(\frac{\cos \phi}{\sin \theta \tan(\alpha/2)} + \frac{\cos \theta \sin \phi}{\sin \theta} \right) \frac{\partial}{\partial \phi} \right\},$$

$$Z_3 = \frac{\hbar}{2i} \left(2 \cos \theta \frac{\partial}{\partial \alpha} - \frac{\sin \theta}{\tan(\alpha/2)} \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \phi} \right).$$

If we have two arbitrary directions in space characterized by the unit vectors $\mathbf{u}$ and $\mathbf{v}$, respectively, and $S_\mathbf{u}$ and $S_\mathbf{v}$ are the corresponding spin projections $S_\mathbf{u} = \mathbf{u} \cdot \mathbf{S}$ and $S_\mathbf{v} = \mathbf{v} \cdot \mathbf{S}$, then $S_{-\mathbf{u}} = -S_\mathbf{u}$, and $[S_\mathbf{u}, S_\mathbf{v}] = i\hbar S_{\mathbf{u} \times \mathbf{v}}$. In the case of the anomalous commutation relations, we have, for instance, $[Z_1, Z_2] = -i\hbar Z_3$, suggesting that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = -\hat{\mathbf{e}}_3$ and thus $\hat{\mathbf{e}}_i$ vectors behave in the quantum case as a left handed system. In this case $\mathbf{e}_i$ vectors are not arbitrary vectors in space, but rather vectors linked to the rotating body and thus they are not compatible observables in the sense that any measurement to determine, say, the components of $\mathbf{e}_i$, will produce a body motion that will shadow the measurement of the others. We shall use this interpretation of a left handed system later.

The operator $S^2 = Z^2$ has the differential representation

$$S^2 = Z^2 = -\hbar^2 \left[\frac{\partial^2}{\partial \alpha^2} + \frac{1}{\tan(\alpha/2)} \frac{\partial}{\partial \alpha} + \frac{1}{4 \sin^2(\alpha/2)} \left\{ \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right\} \right].$$

(29)

Wigner's functions, because Eqs. (28) and (29) can be taken as simultaneous eigenfunctions of the three commuting S^2, S_3, and Z_3 operators[15]

$$S^2 D_{m,n}^{1/2}(\alpha, \theta, \phi) = \tfrac{1}{2}(\tfrac{1}{2} + 1)\hbar^2 D_{m,n}^{1/2}(\alpha, \theta, \phi),$$

$$S_3 D_{m,n}^{1/2}(\alpha, \theta, \phi) = m\hbar D_{m,n}^{1/2}(\alpha, \theta, \phi),$$

$$Z_3 D_{m,n}^{1/2}(\alpha, \theta, \phi) = n\hbar D_{m,n}^{1/2}(\alpha, \theta, \phi)$$

are explicitly given by the following functions:

$$\Phi_1(\alpha, \theta, \phi) = D_{1/2, 1/2}^{1/2}(\alpha, \theta, \phi) = \sqrt{2}(\cos(\alpha/2) + i \cos \theta \sin(\alpha/2)),$$

$$\Phi_2(\alpha, \theta, \phi) = D_{-1/2, 1/2}^{1/2}(\alpha, \theta, \phi) = i\sqrt{2} \sin(\alpha/2)\sin \theta e^{-i\phi},$$

$$\Phi_3(\alpha, \theta, \phi) = D_{1/2, -1/2}^{1/2}(\alpha, \theta, \phi) = i\sqrt{2} \sin(\alpha/2)\sin \theta e^{i\phi},$$

$$\Phi_4(\alpha, \theta, \phi) = D_{-1/2, -1/2}^{1/2}(\alpha, \theta, \phi) = \sqrt{2}(\cos(\alpha/2) - i \cos \theta \sin(\alpha/2)).$$

(30)

They form an orthonormal set with respect to the normalized invariant measure

$$d\mu(\alpha, \theta, \phi) = \frac{1}{4\pi^2} \sin \theta \sin^2(\alpha/2) d\alpha \, d\theta \, d\phi$$

such that the scalar product is defined as

$$\langle\Phi|\Psi\rangle=\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\int_0^{2\pi}d\alpha\ \Phi^*(\alpha,\theta,\phi)\Psi(\alpha,\theta,\phi)\ \frac{1}{4\pi^2}\sin\theta\ \sin^2(\alpha/2).$$

The matrix representation of any internal observable A is obtained as $A_{ij}=\langle\Phi_i|A\Phi_j\rangle$, $i,j=1,2,3,4$. Once these four basis vectors are fixed, when acting on the subspace they span, the differential operators S_i and Z_i have the matrix representation

$$\hat{\mathbf{S}}=\frac{\hbar}{2}\begin{pmatrix}\boldsymbol{\sigma}&0\\0&\boldsymbol{\sigma}\end{pmatrix},\tag{31}$$

$$\hat{Z}_1=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\hat{Z}_2=\frac{\hbar}{2}\begin{pmatrix}0&i1\\-i1&0\end{pmatrix},\quad\hat{Z}_3=\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\tag{32}$$

where $\boldsymbol{\sigma}$ are the three Pauli matrices and 1 represents the 2×2 unit matrix.

If we similarly compute the matrix elements of the nine components of the unit vectors $(\mathbf{e}_i)_j$, $i,j=1,2,3$ given by Eq. (14) we obtain the nine traceless Hermitian matrices

$$\hat{\mathbf{e}}_1=\frac{1}{3}\begin{pmatrix}0&\boldsymbol{\sigma}\\\boldsymbol{\sigma}&0\end{pmatrix},\quad\hat{e}_2=\frac{1}{3}\begin{pmatrix}0&i\boldsymbol{\sigma}\\-i\boldsymbol{\sigma}&0\end{pmatrix},\quad\hat{e}_3=\frac{1}{3}\begin{pmatrix}\boldsymbol{\sigma}&0\\0&-\boldsymbol{\sigma}\end{pmatrix}\tag{33}$$

that together with the above six spin components, form a set of 15 traceless linearly independent Hermitian matrices. It is easily checked that $\hat{\mathbf{e}}_i\cdot\hat{\mathbf{S}}=\hat{\mathbf{S}}\cdot\hat{\mathbf{e}}_i=\hat{Z}_i$, and in the quantum case, observables $\hat{\mathbf{e}}_i$ are not commuting unit vector operators (see Appendix). Even more, their eigenvalues are $\pm1/3$.

We finally write the wave function for spin-1/2 particles in the following form:

$$\Psi(t,\mathbf{r},\mathbf{u},\boldsymbol{\alpha})=\sum_i\ \psi_i(t,\mathbf{r})\Phi_i(\alpha,\theta,\phi).$$

Then, once the Φ_i functions that describe the internal structure are identified with the four orthogonal unit vectors of the internal Hilbert space $\mathbb{C}^4$, the wave function becomes a four-component space–time wave function, and the six spin components S_i and Z_j and the nine vector components $(\hat{\mathbf{e}}_i)_j$, together the 4×4 unit matrix, completely exhaust the 16 linearly independent 4×4 Hermitian matrices. They form a basis of Dirac's algebra, such that any other internal observable that describes the internal structure, for instance, internal velocity and acceleration, must necessarily be expressed as a real linear combination of the mentioned 16 Hermitian matrices. This is perhaps one of the reasons why in this spin-1/2 system the wave function does not show dependence on the velocity $\mathbf{u}$ and this variable becomes a function of the orientation. In fact the internal electron orientation completely characterizes its internal structure (see Appendix).

One of the constants of the motion, obtained from invariance of the Lagrangian under pure Lorentz transformations,[2] is the Poincaré momentum (6), such that its time derivative and the scalar product with $\mathbf{u}$ leads to the Poincaré invariant operator (Dirac's operator)

$$D\equiv H-\mathbf{P}\cdot\mathbf{u}-\frac{1}{c^2}\left(\frac{d\mathbf{u}}{dt}\times\mathbf{u}\right)\cdot\mathbf{S}=0.\tag{34}$$

When Dirac's operator D acts on a general wave function, we know that H and $\mathbf{P}$ have the differential representation given by Eq. (16) and the spin the differential representation (25), or the equivalent matrix representation (31), but we do not know how to represent the action of the velocity $\mathbf{u}$ and the $(d\mathbf{u}/dt)\times\mathbf{u}$ observable. However, we know that for this particle $\mathbf{u}$ and $d\mathbf{u}/dt$ are orthogonal vectors and together with vector $\mathbf{u}\times d\mathbf{u}/dt$ they form an orthogonal right handed system, and in the center of mass frame the particle describes a circle of radius $R_0=\hbar/2mc$ for
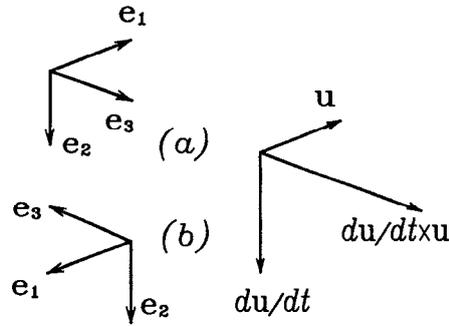
FIG. 2. Identification of observables leading to the Pauli–Dirac representation.

spin-1/2 particles in the plane spanned by $\mathbf{u}$ and $d\mathbf{u}/dt$ (see Fig. 1). Since they are translationally invariant observables they will be elements of Dirac's algebra, and it turns out that if we relate these three vectors with the orthogonal left handed system formed by vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ as shown in part (a) of Fig. 2 we have $\mathbf{u}=a\hat{\mathbf{e}}_1$ and $d\mathbf{u}/dt\times\mathbf{u}=b\hat{\mathbf{e}}_3$, where a and b are positive real numbers. Then the third term in the D operator is $-(b/c^2)\hat{\mathbf{e}}_3\cdot\hat{\mathbf{S}}=-(b/c^2)\hat{Z}_3$ and the D operator becomes

$$D=H-a\mathbf{P}\cdot\hat{\mathbf{e}}_1-\frac{b}{c^2}\hat{Z}_3=0, \tag{35}$$

while if we identify with the orthonormal system of part (b) of Fig. 2, we get

$$D=H+a\mathbf{P}\cdot\hat{\mathbf{e}}_1+\frac{b}{c^2}\hat{Z}_3=0. \tag{36}$$

Multiplying Eq. (36) by Eq. (35) we obtain

$$H^2-\frac{a^2}{9}\mathbf{P}^2\mathbf{1}-\frac{b^2\hbar^2}{4c^4}\mathbf{1}=0$$

and identification of this expression with the Klein–Gordon operator (20), leads to $a=3c$ and $b=2mc^4/\hbar=c^3/R_0$ and we obtain Dirac's operator

$$H-c\mathbf{P}\cdot\boldsymbol{\alpha}-\beta mc^2=0,$$

where Dirac's matrices $\boldsymbol{\alpha}$ and β are represented by

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}, \quad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

i.e., the Pauli–Dirac representation.

This representation is compatible with the vector $d\mathbf{u}/dt$ lying along the third vector $\hat{\mathbf{e}}_2$. In fact, in the center of mass frame, Dirac's Hamiltonian is $H=\beta mc^2$, and the time derivative of any observable A is obtained as

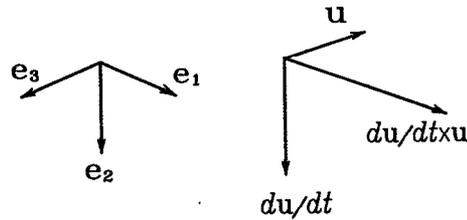$$\frac{dA}{dt}=\frac{i}{\hbar}[H,A]$$

FIG. 3. Identification of observables leading to Weyl's representation.

such that for the velocity operator $\mathbf{u} = c\boldsymbol{\alpha}$

$$\frac{d\mathbf{u}}{dt} = \frac{i}{\hbar}[mc^2\beta, c\boldsymbol{\alpha}] = \frac{2mc^3}{\hbar}\begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix} = \frac{c^2}{R_0}3\hat{\mathbf{e}}_2,$$

with c^2/R_0 being the constant modulus of the acceleration.

The time derivative of this Cartesian system is

$$\frac{d\hat{\mathbf{e}}_1}{dt} = \frac{i}{\hbar}[\beta mc^2, \hat{\mathbf{e}}_1] = \frac{c}{R_0}\hat{\mathbf{e}}_2, \quad \frac{d\hat{\mathbf{e}}_2}{dt} = \frac{i}{\hbar}[\beta mc^2, \hat{\mathbf{e}}_2] = -\frac{c}{R_0}\hat{\mathbf{e}}_1,$$

$$\frac{d\hat{\mathbf{e}}_3}{dt} = \frac{i}{\hbar}[\beta mc^2, \hat{\mathbf{e}}_3] = 0$$

since $\mathbf{e}_3$ is orthogonal to the trajectory plane and does not change, and where $c/R_0 = \omega_s$ is the angular velocity of the internal orbital motion. This time evolution of the observables $\mathbf{e}_i$ is the correct one if it is assumed to be a rotating left handed system of vectors as shown in Fig. 2(a).

Similarly

$$\frac{d\hat{\mathbf{S}}}{dt} = \frac{i}{\hbar}[\beta mc^2, \hat{\mathbf{S}}] = 0$$

since the spin is constant for the center of mass observer. Only the Z_3 spin component on the body axis remains constant while the other two, Z_1 and Z_2, change because of the rotation of the corresponding axis

$$\frac{d\hat{Z}_1}{dt} = \frac{i}{\hbar}[\beta mc^2, \hat{Z}_1] = \frac{c}{R_0}\hat{Z}_2, \quad \frac{d\hat{Z}_2}{dt} = \frac{i}{\hbar}[\beta mc^2, \hat{Z}_2] = -\frac{c}{R_0}\hat{Z}_1,$$

$$\frac{d\hat{Z}_3}{dt} = \frac{i}{\hbar}[\beta mc^2, \hat{Z}_3] = 0.$$

When analyzed from the point of view of an arbitrary observer, the classical motion is a helix of elliptic cross section and the acceleration is not of constant modulus c^2/R_0, and the spin, that remains orthogonal to both $\mathbf{u}$ and $d\mathbf{u}/dt$, is no longer a constant of the motion, because it is the total angular momentum $\mathbf{J} = \mathbf{r} \times \mathbf{P} + \mathbf{S}$ that is conserved.

Identification of the internal variables with different real linear combinations of the $\hat{\mathbf{e}}_i$ matrices lead to different equivalent representations of Dirac's matrices, because Dirac's operator D is rotationally invariant.

For instance, if we make the identification suggested by Fig. 3, $\mathbf{u} = -a\hat{\mathbf{e}}_3$ and the observable

$d\mathbf{u}/dt \times \mathbf{u} = b\hat{\mathbf{e}}_1$, we obtain by the same method

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} -\sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

and thus the gamma matrices

$$\gamma^0 = \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \gamma^0 \alpha = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix},$$

i.e., Weyl's representation.

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APPENDIX: DIRAC'S ALGEBRA

The three spatial spin components S_i, the three body spin projections Z_j, and the nine components of the body frame unit vectors $(\mathbf{e}_i)_j$, $i,j=1,2,3$, whose matrix representations are given in Eqs. (31)–(33), together with the 4×4 unit matrix 1, form a set of 16 linearly independent Hermitian matrices. They are a linear basis of Dirac's algebra, and satisfy the following commutation relations:

$$[S_i, S_j] = i\hbar\,\epsilon_{ijk}S_k, \quad [Z_i, Z_j] = -i\hbar\,\epsilon_{ijk}Z_k, \quad [S_i, Z_j] = 0, \tag{A1}$$

$$[S_i, (\mathbf{e}_j)_k] = i\hbar\,\epsilon_{ikr}(\mathbf{e}_j)_r, \quad [Z_i, (\mathbf{e}_j)_k] = -i\hbar\,\epsilon_{ijr}(\mathbf{e}_r)_k, \tag{A2}$$

$$[(\mathbf{e}_i)_k, (\mathbf{e}_j)_l] = \frac{4i}{9\hbar}\,[\delta_{ij}\epsilon_{klr}S_r - \delta_{kl}\epsilon_{ijr}Z_r] \tag{A3}$$

showing that the $\mathbf{e}_i$ operators transform like vectors under rotations but they are not commuting observables.

If we fix the couple of indices i, and j, then the set of four operators S^2, S_i, Z_j, and $(\mathbf{e}_j)_i$ form a complete commuting set since the algebra of 4×4 matrices admits four diagonal and linearly independent matrices. In fact, the wave functions given in Eq. (30) are simultaneous eigenfunctions of S^2, S_3, Z_3, and $(\mathbf{e}_3)_3$ with eigenvalues $s=1/2$ and for s_3, z_3, and e_{33} the following ones:

$$\Phi_1 = |1/2,1/2,1/3\rangle, \quad \Phi_2 = |-1/2,1/2,-1/3\rangle,$$

$$\Phi_3 = |1/2,-1/2,-1/3\rangle, \quad \Phi_2 = |-1/2,-1/2,1/3\rangle.$$

The basic observables satisfy the following anticommutation relations:

$$\{S_i, S_j\} = \{Z_i, Z_j\} = \frac{\hbar^2}{2}\,\delta_{ij}1, \tag{A4}$$

$$\{S_i, Z_j\} = \frac{3\hbar^2}{2}\,(\mathbf{e}_j)_i, \tag{A5}$$

$$\{S_i, (\mathbf{e}_j)_k\} = \tfrac{2}{3}\delta_{ik}Z_j, \quad \{Z_i, (\mathbf{e}_j)_k\} = \tfrac{2}{3}\delta_{ij}S_k, \tag{A6}$$

$$\{(\mathbf{e}_i)_j , (\mathbf{e}_k)_l\} = \tfrac{2}{9}\delta_{ik}\delta_{jl}\mathbb{1} + \tfrac{2}{3}\epsilon_{ikr}\epsilon_{jls}(\mathbf{e}_r)_s . \tag{A7}$$

If we define the dimensionless normalized matrices

$$a_{ij} = 3(\mathbf{e}_i)_j , \quad s_i = \frac{2}{\hbar}S_i , \quad z_i = \frac{2}{\hbar}Z_i \tag{A8}$$

together with the 4×4 unit matrix 1, they form a set of 16 matrices Γ_λ, $\lambda = 1,...,16$, that are Hermitian, unitary, linearly independent, and of unit determinant.

The set of 64 unitary matrices of determinant +1, $\pm\Gamma_\lambda$, $\pm i\Gamma_\lambda$, $\lambda = 1,...,16$ form a finite subgroup of SU(4). Its composition law can be obtained from

$$a_{ij}a_{kl} = \delta_{ik}\delta_{jl}\mathbb{1} + i\,\delta_{ik}\epsilon_{jlr}s_r - i\,\delta_{jl}\epsilon_{ikr}z_r + \epsilon_{ikr}\epsilon_{jls}a_{rs} , \tag{A9}$$

$$a_{ij}s_k = i\epsilon_{jkl}a_{il} + \delta_{jk}z_i , \tag{A10}$$

$$a_{ij}z_k = -i\epsilon_{ikl}a_{lj} + \delta_{ik}s_j , \tag{A11}$$

$$s_i a_{jk} = i\epsilon_{ikl}a_{jl} + \delta_{ik}z_j , \tag{A12}$$

$$s_i s_j = i\epsilon_{ijk}s_k + \delta_{ij}\mathbb{1}, \tag{A13}$$

$$s_i z_j = z_j s_i = a_{ji} , \tag{A14}$$

$$z_i a_{jk} = -i\epsilon_{ijl}a_{lk} + \delta_{ij}s_k , \tag{A15}$$

$$z_i z_j = -i\epsilon_{ijk}z_k + \delta_{ij}\mathbb{1} \tag{A16}$$

and similarly we can use these expressions to derive the commutation and anticommutation relations (A1)–(A7).

Dirac's algebra is generated by the four Dirac's gamma matrices γ^μ, $\mu = 0,1,2,3$ that satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}, \tag{A17}$$

with $\eta^{\mu\nu}$ being Minkowski's metric tensor.

Similarly it can be generated by the following four observables, for instance, S_1, S_2, Z_1, and Z_2. In fact by Eqs. (A13) and (A16) we obtain S_3 and Z_3, respectively, and by Eq. (A14) the remaining elements.

Classically, the internal orientation of an electron is characterized by the knowledge of the components of the body frame $(\mathbf{e}_i)_j$, $i,j = 1,2,3$ that altogether constitute an orthogonal matrix. To completely characterize in a unique way this orthogonal matrix we need at least four of these components. In the quantum version, the knowledge of four $(\mathbf{e}_i)_j$ matrices and by making use of Eqs. (A9)–(A16) allows us to recover the remaining elements of the complete Dirac's algebra. It is in this sense that *internal* orientation of the electron completely characterizes its internal structure.

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