Classical relativistic spinning particles

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An elementary particle is defined as a mechanical system whose kinematical space is a homogeneous space of the Poincaré group. Lagrangians for describing these systems depend on higher-order derivatives and some of them are analyzed. For bradyons the Lagrangian depends on the acceleration and angular velocity of the particle and is characterized by two parameters m and s, the rest mass and absolute value of spin, respectively. In general the spin is of kinematical nature, related to the rotation and internal orbital motion of the system. Two different kinds of bradyons appear according to the spin structure. One has a spin related to the generalized angular velocity while the other is a function of the generalized acceleration. Photons are massless particles with spin lying along the direction of motion and energy $h\nu$, where ν is the frequency of its rotational motion. Particles moving in circles with velocity c in their center of mass frame are also predicted, showing a Dirac-type Hamiltonian. There also appear particles with tachyonic orbital motion whose center of mass has bradyonic motion. Transformation properties under space and time reversal are also analyzed.

I. INTRODUCTION

Many attempts can be found in the literature to describe classical spinning particles, which in general endow the point particle with some additional degrees of freedom to give account of the spin structure.[1]

Recently,[2] the possible internal spaces that, in addition to the space-time position variables, describe spinning particles, have been completely classified and analyzed.

In the approach we present here we shall describe structured particles by adding some extra degrees of freedom and by allowing the Lagrangian to depend on higher-order derivatives. Lagrangian theory was generalized by Ostrogradsky[3] and since then several contributors have claimed to consider generalized Lagrangians for studying generalized electrodynamics[4] and the classical spinning particle.[5]

We are thus basically working in a generalized Lagrangian formalism, in which dependence on higher-order derivatives is assumed, and which is sketched in Sec. II. However, we assume that the dynamics is based upon the knowledge of the action function of the system, which is a real function defined on $X \times X$, where X is the kinematical space of the system that will be conveniently defined and that is in general different from the configuration space and the phase space of the system. This means that a particular path followed by the system can be expressed in terms of end-point conditions in X space, as in Feynman's approach. Physical considerations lead us to define in Sec. II C an elementary particle as a system for which X is a homogeneous space of the kinematic group G. This statement restricts the dependence on higher-order derivatives to the G structure. In this work G will be the Poincaré group $\mathscr{P}$, so that X is at most ten dimensional, implying that the Lagrangian dependence on the derivatives is at most on the acceleration and angular velocity of the particle.

The remainder of this work shows that the proposed formalism is not empty, by explicitly constructing several Lagrangians. In order to work out a specific X space we present in Sec. III a useful Poincaré group parametrization, where the parameters are the relative velocity and orientation and the space and time translation among inertial observers.

In Sec. IV we study the simplest case, that of a point particle, obtaining the habitual results, but preparing the ground for further applications. In Secs. V and VI we analyze two particular Poincaré homogeneous spaces: the most general bradyon and the kinematical space of particles that travel at the speed of light. In the first, two kinds of particles come out according to the kinematical structure of their spins. In the second group we have found the photon with its properties of having no transversal spin arriving at $H = h\nu$ for the expression of its energy, where ν is the frequency of its rotational motion along the spin direction.

However, we have also found particles that, although they travel at the speed of light, have a center of mass with a straight bradyonic motion with constant velocity below c. For these particles we have found a certain analogy between their Hamiltonian and Dirac's Hamiltonian, and a particular Lagrangian has been analyzed. In Sec. VII particles with internal orbital tachyonic motion are considered, having a center of mass that travels at velocity $u \leqslant c$. Section VIII is devoted to the analysis of the previous Lagrangians under the discrete symmetry operations of time and space inversion.

II. GENERAL FORMALISM

A. Generalized Lagrangian systems

Although the generalized Lagrangian formalism is well known, we shall sketch it briefly in order to enhance the role of the manifold X, the kinematical space of the system, and the action function on $X \times X$, which are defined later.

Let us consider those mechanical systems of n degrees of freedom that can be described by means of a generalized Lagrangian function $\mathbf{L}(t, q_i^{(s)}(t))$, $i = 1,...,n$, $s = 0,1,...,k$, which depends on the time t and on the n generalized coordinates $q_i(t)$, and their derivatives up to order k. Here $q_i^{(s)}(t) = d^s q_i(t)/dt^s$.

The action functional is defined as

$$A[q(t)] = \int_{t_1}^{t_2} L(t,q_i^{(s)}(t))dt, \tag{2.1}$$

where the condition that A be extremal for the class of paths $q_i(t)$ with fixed end points [i.e., with fixed values $q_i^{(s)}(t_1)$, $q_i^{(s)}(t_2)$, $i=1,...,n$, $s=0,1,...,k-1$] implies that the functions $q_i(t)$ must necessarily satisfy the Euler–Lagrange dynamical equations.[6]

$$\sum_{s=0}^{k}(-1)^s \frac{d^s}{dt^s}\left(\frac{\partial L}{\partial q_i^{(s)}}\right)=0, \quad i=1,...,n. \tag{2.2}$$

A generalization to systems for which the order k is different for each generalized coordinate q_i can be obtained easily, but in this work we shall consider for simplicity the same order k in all variables.

Existence and uniqueness theorems imply that a particular solution of this $2k$ th-order system (2.2) is determined by giving the $2kn$ values $q_i^{(s)}(t)$ $i=1,...,n$, $s=0,1,...,2k-1$, at the initial time t_1. If we fix end-point conditions, i.e., the values $q_i^{(s)}(t_1)$ and $q_i^{(s)}(t_2)$, $i=1,...,n$, $s=0,1,...,k-1$, there will not exist, in general, a solution of (2.2), although the variational problem (2.1) leads to the system (2.2) for the class of paths with fixed end-point conditions. However, if there exist solutions, perhaps nonunique, with fixed end points, this means in some sense that the above initial conditions at time t_1 can be expressed, perhaps in a nonuniform way, in terms of the end-point conditions. Thus a particular solution is finally expressed as a function $q_i(t;q_j^{(s)}(t_1), q_j^{(s)}(t_2))$, $j=1,...,n$, $s=0,1,...,k-1$, of time and of $2kn$ independent parameters, related to end-point conditions, and we shall consider from now on those mechanical systems for which this holds. A generalized Lagrangian formalism and the existence of solutions with fixed end-point conditions are the basic assumptions of the formalism we propose.

By considering this particular solution, the action function is defined as the value of the functional (2.1) for this particular path. Thus the action function becomes a function of $2(kn+1)$ independent variables

$$A(t_1,q_i^{(s)}(t_1);t_2,q_i^{(s)}(t_2))$$
$$\equiv A(x_1,x_2), \quad i=1,...,n, \quad s=0,1,...,k-1,$$

with the property $A(x,x)=0$.

Definition: We shall call kinematical variables of the system to the time t and the n generalized coordinates and their derivatives up to order $k-1$ $q_i^{(s)}$, $s=0,1,...,k-1$, and they will be denoted by x_j, $j=0,1,...,kn$. The $(kn+1)$-dimensional manifold spanned by the kinematical variables is called the kinematical space of the system X.

If the trajectories are written in parametric form $\{t(\tau),q(\tau)\}$, in terms of some evolution parameter τ, the Lagrangian can be expressed in terms of the kinematical variables and their first τ derivatives, and (2.1) appears:

$$A[(\tau),q(\tau)] = \int_{\tau_1}^{\tau_2} L\left(x(\tau),\frac{\dot{x}(\tau)}{\dot{t}(\tau)}\right)\dot{t}(\tau)d\tau$$
$$= \int_{\tau_1}^{\tau_2} \hat{L}(x(\tau),\dot{x}(\tau))d\tau, \tag{2.3}$$

where $\hat{L}=L\dot{t}(\tau)$.

Although (2.3) looks like a first-order system of $kn+1$ degrees of freedom, we see that there exist among the kinematical variables $(k-1)n$ nonholonomic differential constraints $q_i^{(s)}(\tau)=\dot{q}_i^{(s-1)}(\tau)/\dot{t}/(\tau)$, $i=1,...,n,s=1,...,k-1$, where the overdot means a τ derivative.

We can see[7] that the function $\hat{L}$ is τ independent and homogeneous of first degree in terms of the derivatives of the kinematical variables giving rise to a further constraint

$$\hat{L}=\frac{\partial \hat{L}}{\partial \dot{x}^j}\dot{x}^j = F_j(x,\dot{x})\dot{x}^j, \tag{2.4}$$

which, together with the $(k-1)n$ differential constraints, reduces to n the number of independent variables. The action functional in the form (2.3) is also independent with respect to parametric transformations, and the functions $F_j(x,\dot{x})$ are homogeneous functions of zero degree in the derivatives of the kinematical variables.

Conversely, if the system is described by the knowledge of the action function $A(x_1,x_2)$, which is assumed to be a continuous and differentiable function of the kinematical variables of the initial and final points, then the Lagrangian can be obtained by the limiting process:

$$\hat{L}=\lim_{y\to x}\frac{\partial A(x,y)}{\partial y^j}\dot{x}^j. \tag{2.5}$$

This can be seen from (2.3) by considering two close points and thus

$$A(x(\tau),x(\tau+d\tau))=A(x(\tau),x(\tau)+\dot{x}(\tau)d\tau)=\hat{L}\,d\tau.$$

By making a Taylor expansion of A, taking into account the condition $A(x,x)=0$, we get (2.5).

The function of the kinematical variables and their derivatives (2.5) together with the homogeneity condition (2.4) and the differential constraints among the kinematical variables reduces the problem to that of a system of n degrees of freedom, where its Lagrangian is a function of the derivatives of the generalized coordinates up to order k. From now on we shall delete the caret over the function $\hat{L}$, and we shall consider systems for which trajectories are written in parametric form.

What we want to emphasize is that the important dynamical object of the theory is the action function. Its knowledge determines by (2.5) the Lagrangian L, and thus the dynamical equations (2.2). Here $A(x_1,x_2)$ characterizes the dynamics globally. We have a similar situation in the quantum scattering theory, in which the dynamics is globally contained in the S matrix. The Feynman path integral approach links both formalisms by relating the action function for a particular path with the phase of the corresponding probability amplitude. In quantum mechanics all paths can be followed, so that we have to add the corresponding probability amplitudes; while in classical mechanics the variational formalism singles out just one path, and thus the action function for that path contains the required dynamical information.

B. The relativity principle

Let G be the kinematical group[8] that acts transitively on the space-time Y as a transformation group. The group G defines the class of equivalent observers, called inertial observers, for which the laws of physics are the same, and we shall assume that a realization of G on the kinematical space of the system X is known.

The invariance of the dynamical equations for two inertial observers O and O' related by a transformation $g \in G$, implies[9] that the action function must transform according to

$$A(gx_1,gx_2) = A(x_1,x_2) + \alpha(g;x_2) - \alpha(g;x_1), \quad (2.6)$$

where $\alpha(g;x)$ is a function defined on $G \times X$, which verifies, for all $g,g' \in G$, and all $x \in X$,

$$\alpha(g';gx) + \alpha(g;x) - \alpha(g'g;x) = \xi(g',g), \quad (2.7)$$

where $\xi(g',g)$ is an exponent of G.[10]

This function $\alpha(g;x)$ is called a gauge function for the group G and the kinematical space X. Different mechanical systems with the same kinematical space X can be characterized by different gauge functions.

From (2.5), the Lagrangian transforms

$$L\left(gx(\tau),\frac{d(gx(\tau))}{d\tau}\right) = L(x(\tau),\dot{x}(\tau)) + \frac{d\alpha(g;x(\tau))}{d\tau}, \quad (2.8)$$

which, together with the homogeneity condition (2.4), will lead to certain transformation properties for the functions F_j under the group G, giving us information about the structure of these functions. Expression (2.8) is the restriction imposed to the Lagrangian by the relativity principle.

Among the gauge functions there exists an equivalence relation.[9] Two gauge functions α_1 and α_2 are said to be equivalent if

$$\alpha_1(g;x) - \alpha_2(g;x) = \Phi(x) - \Phi(gx) + \sigma(g), \quad (2.9)$$

where Φ and σ are some functions defined on X and G, respectively. Thus with G and X fixed, to every $\alpha(g;x)$ solution of (2.7) up to an equivalence, the relativity principle in its form (2.8) will give us information about the Lagrangian mechanical systems whose dynamical laws are g invariant.

In particular if X is a homogeneous space of G then (2.7) has the solution[9]

$$\alpha(g;x) = \xi(g,h_x), \quad (2.10)$$

where $h_x \in G$ is any element of the equivalence class $x \in X$.

In this paper G will be the Poincaré group $\mathscr{P}$, and all its exponents are equivalent to zero[10] so that, for those mechanical systems for which X is a homogeneous space of $\mathscr{P}$, the action function and the Lagrangian can be taken strictly invariant.

C. Elementary systems

An elementary mechanical system will be defined as that system for which the evolution from the initial to the final point, if no interaction is present, is necessarily free.

Let us consider a system that is observed at instant τ by a certain inertial observer O. At instant $\tau + d\tau$ some physical observables will change their values as measured by O. However, if the system is elementary, then there will exist at instant $\tau + d\tau$ another inertial observer O' for which the measurements of physical observables will give the same values as those obtained by O at the earlier time τ.

These two inertial observers will be related by some infinitesimal transformation $\delta g(\tau)$ of the kinematical group G. If the evolution of the elementary system is free, this means that the corresponding infinitesimal transformation $\delta g(\tau)$ must be independent of τ. Otherwise we could distinguish one instant from another by looking at the different change in the physical observables, and thus concluding that this difference in the physical behavior of the system is produced by some interaction.

Thus if the evolution is free, the measurement of any observable by observer O at instant $\tau + d\tau$ will be obtained from its measurement at instant τ by acting with δg in the corresponding realization of the algebra of observables. Since δg is constant, it generates a one-parameter subgroup of G, such that the evolution of any observable is the action of this one-parameter subgroup on its initial value. In this way, the free or inertial motions are identified with the one-parameter subgroups of G.

We have seen in Sec. II A that for Lagrangian systems the dynamical information is contained in the action function, which is a function of the kinematical variables at the initial and final points. If the evolution is free, the final point x_2 is obtained by acting on x_1 with the corresponding one-parameter subgroup generated by δg, and thus there exists a finite group element g such that $x_2 = gx_1$.

Conversely, if we fix x_1 and x_2 and the evolution has to be free, then necessarily the kinematical space has to be a homogeneous space of G. Otherwise, if X is not a homogeneous space of G, then in general there will not exist any group element and any one-parameter subgroup of G that brings x_1 to x_2, and the evolution of the system will no longer be free.

Definition: An elementary classical particle is that mechanical system for which its kinematical space X is a homogeneous space of the corresponding kinematical group G.

III. THE POINCARE GROUP

The kinematical group for relativistic systems is the Poincaré group $\mathscr{P}$. The usual covariant parametrization of $\mathscr{P}$ is given by the four-vector a^μ of the space-time translations and the 16 elements $\Lambda^\mu{}_\nu$ of the Lorentz transformation, and we write it as (a,Λ). The composition law is given by

$$(a',\Lambda')(a,\Lambda) = (\Lambda' a + a',\Lambda'\Lambda),$$

i.e.,

$$a''^\mu = \Lambda'^\mu{}_\nu a^\nu + a'^\mu, \quad \Lambda''^\mu{}_\nu = \Lambda'^\mu{}_\sigma \Lambda^\sigma{}_\nu.$$

However, the elements $\Lambda^\mu{}_\nu$ are not independent and they verify the ten relations $\Lambda^\mu{}_\sigma \eta^{\sigma\lambda} \Lambda^\nu_\lambda = \eta^{\mu\nu}$, where $\eta^{\mu\nu}$ is the Minkowski metric tensor. This parametrization is used in Refs. 1 and 11.

Instead of this covariant parametrization we give here an essential parametrization in terms of ten independent parameters.[12] Since every Lorentz transformation can always be written as a product $\Lambda = LR$ of a boost L by a rotation R, we shall use the relative velocity vector $\mathbf{v}$ that characterizes L and the three angular variables for the rotation R as the six essential parameters without any further constraint. Now the formulation is not manifestly covariant but the physical interpretation of these parameters as velocity and orientation will be shared later by the variables of the corresponding homogeneous X spaces.

Then every element is parametrized in terms of the ten real parameters $g \equiv (b,\mathbf{a},\mathbf{v},\boldsymbol{\mu})$, where $b \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^3$ represent the time and space translation, $\mathbf{v} \in \mathbb{R}^3$, with $v < c$, is the relative velocity among observers, and $\boldsymbol{\mu} = \mathbf{e}\tan(\alpha/2)$ is the relative orientation of their spatial Cartesian coordinate frames. It is parametrized by the clockwise rotation around the direction given by the unit vector $\mathbf{e}$ and of value $\alpha \in [0,\pi]$ to get the O' frame from that of O. With this parametrization, $\boldsymbol{\mu}$ takes values on a real three-dimensional compact manifold which we shall denote by $\mathbb{R}_c^3$.

The orthogonal rotation matrix $R(\boldsymbol{\mu})$ is given by

$$R(\boldsymbol{\mu})_{ij} = [1/(1+\mu^2)][(1-\mu^2)\delta_{ij} + 2\mu_i\mu_j - 2\epsilon_{ijk}\mu^k]. \quad (3.1)$$

The composition of two rotations $R(\boldsymbol{\mu}')R(\boldsymbol{\mu}) = R(\boldsymbol{\mu}'')$ is

$$\boldsymbol{\mu}'' = (\boldsymbol{\mu}' + \boldsymbol{\mu} + \boldsymbol{\mu}' \times \boldsymbol{\mu})/(1 - \boldsymbol{\mu}'\cdot\boldsymbol{\mu}). \quad (3.2)$$

The action of a group element g on the Minkowski space-time $\mathscr{M}$ is

$$t' = \gamma t + \gamma(\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{r})/c^2 + b, \quad (3.3)$$

$$\mathbf{r}' = R(\boldsymbol{\mu})\mathbf{r} + \gamma\mathbf{v}t + \gamma^2(\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{r})\mathbf{v}/(1+\gamma)c^2 + \mathbf{a}, \quad (3.4)$$

where $(t,\mathbf{r})$ and $(t',\mathbf{r}')$ are the coordinates of the same space-time event for observers O and O', respectively, where $\gamma = (1 - v^2/c^2)^{-1/2}$.

The composition law of the Poincaré group in this parametrization $g'' = g'g$ can be expressed[12] as

$$b'' = \gamma' b + \gamma'(\mathbf{v}'\cdot R(\boldsymbol{\mu}')\mathbf{a})/c^2 + b', \quad (3.5)$$

$$\mathbf{a}'' = R(\boldsymbol{\mu}')\mathbf{a} + \gamma'\mathbf{v}'b + [\gamma'^2/(1+\gamma')c^2] \times (\mathbf{v}'\cdot R(\boldsymbol{\mu}')\mathbf{a})\mathbf{v}' + \mathbf{a}', \quad (3.6)$$

$$\mathbf{v}'' = \frac{R(\boldsymbol{\mu}')\mathbf{v} + \gamma'\mathbf{v}' + \gamma'^2 c^{-2}(\mathbf{v}'\cdot R(\boldsymbol{\mu}')\mathbf{v})\mathbf{v}'/(1+\gamma')}{\gamma'(1 + (\mathbf{v}'\cdot R(\boldsymbol{\mu}')\mathbf{v})/c^2)}, \quad (3.7)$$

$$\boldsymbol{\mu}'' = \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} + \boldsymbol{\mu}' \times \boldsymbol{\mu} + \mathbf{F}(\mathbf{v}',\boldsymbol{\mu}';\mathbf{v},\boldsymbol{\mu})}{1 - \boldsymbol{\mu}'\cdot\boldsymbol{\mu} + G(\mathbf{v}',\boldsymbol{\mu}';\mathbf{v},\boldsymbol{\mu})}, \quad (3.8)$$

where $\mathbf{F}(\mathbf{v}',\boldsymbol{\mu}';\mathbf{v},\boldsymbol{\mu})$ and $G(\mathbf{v}',\boldsymbol{\mu}';\mathbf{v},\boldsymbol{\mu})$ are the functions

$$\mathbf{F}(\mathbf{v}',\boldsymbol{\mu}';\mathbf{v},\boldsymbol{\mu}) = [\gamma'\gamma/(1+\gamma')(1+\gamma)c^2][\mathbf{v}\times\mathbf{v}' + \mathbf{v}(\mathbf{v}'\cdot\boldsymbol{\mu}') + \mathbf{v}'(\mathbf{v}\cdot\boldsymbol{\mu}) + \mathbf{v}\times(\mathbf{v}'\times\boldsymbol{\mu}') + (\mathbf{v}\times\boldsymbol{\mu})\times\mathbf{v}' + (\mathbf{v}\cdot\boldsymbol{\mu})(\mathbf{v}'\times\boldsymbol{\mu}') + (\mathbf{v}\times\boldsymbol{\mu})(\mathbf{v}'\cdot\boldsymbol{\mu}') + (\mathbf{v}\times\boldsymbol{\mu}) \times(\mathbf{v}'\times\boldsymbol{\mu}')], \quad (3.9)$$

$$G(\mathbf{v}',\boldsymbol{\mu}';\mathbf{v},\boldsymbol{\mu}) = [\gamma'\gamma/(1+\gamma')(1+\gamma)c^2] \times[\mathbf{v}\cdot\mathbf{v}' + \mathbf{v}\cdot(\mathbf{v}'\times\boldsymbol{\mu}') + \mathbf{v}'\cdot(\mathbf{v}\times\boldsymbol{\mu}) - (\mathbf{v}\cdot\boldsymbol{\mu})(\mathbf{v}'\cdot\boldsymbol{\mu}') + (\mathbf{v}\times\boldsymbol{\mu})\cdot(\mathbf{v}'\times\boldsymbol{\mu}')]. \quad (3.10)$$

The proper Lorentz group $\mathscr{L}$ is the set $\{(0,0,\mathbf{v},\boldsymbol{\mu})\,|\,\mathbf{v}\in\mathbb{R}^3,\ v<c,\ \boldsymbol{\mu}\in\mathbb{R}_c^3\}$ so that every Lorentz transformation is parametrized in terms of the two three-vectors $\mathbf{v}$ and $\boldsymbol{\mu}$, $\Lambda(\mathbf{v},\boldsymbol{\mu})$ is a product of a pure Lorentz transformation $L(\mathbf{v})$ by the rotation $R(\boldsymbol{\mu})$, and $\Lambda(\mathbf{v},\boldsymbol{\mu}) = L(\mathbf{v})R(\boldsymbol{\mu})$. The expression (3.7) is the relativistic addition of the two velocities $\mathbf{v}'$ and $R(\boldsymbol{\mu}')\mathbf{v}$, because

$$L(\mathbf{v}'')R(\boldsymbol{\mu}'') = L(\mathbf{v}')R(\boldsymbol{\mu}')L(\mathbf{v})R(\boldsymbol{\mu}) = L(\mathbf{v}')R(\boldsymbol{\mu}')L(\mathbf{v})R(-\boldsymbol{\mu}')R(\boldsymbol{\mu}')R(\boldsymbol{\mu}) = L(\mathbf{v}')L(R(\boldsymbol{\mu}')\mathbf{v})R(\boldsymbol{\mu}')R(\boldsymbol{\mu}),$$

since $R(\boldsymbol{\mu}')L(\mathbf{v})R(-\boldsymbol{\mu}') = L(R(\boldsymbol{\mu}')\mathbf{v})$ and the composition of the two boosts $L(\mathbf{v}')L(R(\boldsymbol{\mu}')\mathbf{v}) = L(\mathbf{v}'')R(\mathbf{w})$, where $R(\mathbf{w})$ is the corresponding Wigner rotation. The expression (3.8) comes from $R(\boldsymbol{\mu}'') = R(\mathbf{w})R(\boldsymbol{\mu}')R(\boldsymbol{\mu})$.

The general continuous subgroups of $\mathscr{P}$ were classified by Patera *et al.*[13] and thus the homogeneous spaces of $\mathscr{P}$ can be obtained as the corresponding quotient structures. However, we are interested in those homogeneous spaces that describe particles with the maximum structure.

We devote the remaining sections of this work to analyze different homogeneous spaces; we begin with the Minkowski space-time to describe the simplest case, that of a point particle.

Later on we will be interested in those homogeneous spaces with higher dimenson, giving rise to systems with the highest number of degrees of freedom. We shall start with the Poincaré group itself, for describing general bradyons; the nine-dimensional manifold X_c, which describes particles that travel at the speed of light, defined as $X_c = \mathscr{P}/\mathscr{V}$, where $\mathscr{V}$ is the one-parameter subgroup of pure Lorentz transformations in a given direction; and finally the seven-dimensional manifold $X_T = \mathscr{P}/\mathrm{SO}(3)$ for particles with tachyonic internal orbital motion.

IV. POINT PARTICLES

Let us consider first those mechanical systems for which the kinematical space $X = \mathscr{P}/\mathscr{L}$ is the Minkowski space-time $\mathscr{M}$. The purpose is to illustrate the method for obtaining their Lagrangians for further generalizations.

An element $x \in X$ is characterized by the four real numbers $(t(\tau),\mathbf{r}(\tau))$ that transform under $\mathscr{P}$ according to the formulas (3.3) and (3.4), and which are physically interpreted as the time and position of the system, respectively. There are no constraints among these four kinematical variables and only the homogeneity condition (2.4) will reduce the number of degrees of freedom to 3. The general Lagrangian for these systems can be written as

$$L = -H\dot{t} + \mathbf{p}\cdot\dot{\mathbf{r}}, \quad (4.1)$$

where H and $\mathbf{p}$ are defined by $H = -\partial L/\partial\dot{t}$ and $p_i = \partial L/\partial\dot{r}^i$, are functions of $t,\mathbf{r}$, and are homogeneous of zero degree in terms of the derivatives $\dot{t}(\tau)$ and $\dot{\mathbf{r}}(\tau)$.

If we assume that the parameter τ is invariant under the group $\mathscr{P}$, the derivatives transform as

$$\dot{t}'(\tau) = \gamma\dot{t}(\tau) + \gamma(\mathbf{v}\cdot R(\boldsymbol{\mu})\dot{\mathbf{r}}(\tau))/c^2, \quad (4.2)$$

$$\dot{\mathbf{r}}'(\tau) = R(\boldsymbol{\mu})\dot{\mathbf{r}}(\tau) + \gamma\mathbf{v}\dot{t}(\tau) + [\gamma^2/(1+\gamma)c^2](\mathbf{v}\cdot R(\boldsymbol{\mu})\dot{\mathbf{r}}(\tau))\mathbf{v}, \quad (4.3)$$

and the invariance of L under $\mathscr{P}$ leads for $\mathbf{p}$ and H to the transformation equations

$$H'(\tau) = \gamma H(\tau) + \gamma(\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{p}(\tau)), \tag{4.4}$$

$$\mathbf{p}'(\tau) = R(\boldsymbol{\mu})\mathbf{p}(\tau) + \gamma\mathbf{v}H(\tau)/c^2 + [\gamma^2/(1+\gamma)c^2](\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{p}(\tau))\mathbf{v}, \tag{4.5}$$

and thus since H and $\mathbf{p}$ are invariant under translations they are functions independent of t and $\mathbf{r}$ depending only on $\dot{t}$ and $\dot{\mathbf{r}}$.

From (4.2) and (4.3) we see that $c^2\dot{t}^2(\tau) - \dot{\mathbf{r}}^2(\tau)$ is group invariant and we shall consider those systems for which this invariant remains either greater than, equal to, or less than zero during the evolution.

If $c^2\dot{t}^2(\tau) - \dot{\mathbf{r}}^2(\tau) > 0$, for $\tau\in[\tau_1,\tau_2]$, then $\dot{t}(\tau)\neq 0$ for every inertial observer and then $t(\tau)$ can be inverted to obtain $\tau(t)$ and thus we can make a time evolution description $\mathbf{r}(t)$. The velocity is defined as $\mathbf{u}(\tau) = \dot{\mathbf{r}}(\tau)/\dot{t}(\tau)$, and H and $\mathbf{p}$ are only functions of $\mathbf{u}$, with $u < c$. This particle is called a spinless bradyon.

If $c^2\dot{t}^2(\tau) - \dot{\mathbf{r}}^2(\tau) = 0$, for $\tau\in[\tau_1,\tau_2]$, then $\dot{t}(\tau)$ and $\dot{\mathbf{r}}(\tau)$ are different from zero for every observer. The velocity of this system $\mathbf{u} = \dot{\mathbf{r}}/\dot{t}$ is such that $u = c$, and this particle is called a spinless photon.

If $c^2\dot{t}^2(\tau) - \dot{\mathbf{r}}(\tau) < 0$, for $\tau\in[\tau_1,\tau_2]$, then there exist observers for whom $\dot{t}(\tau) = 0$ and it is not possible, in general, to make a time evolution description. However, $|\dot{\mathbf{r}}|\neq 0$ for every observer, so that the magnitude $\mathbf{l} = \dot{\mathbf{r}}\dot{t}/\dot{\mathbf{r}}^2$ is homogeneous of zero degree and well defined. Here H and $\mathbf{p}$ are only functions of $\mathbf{l}$, and the velocity of this system $\mathbf{u} = \dot{\mathbf{r}}/\dot{t}$ is always greater than c, and for some observers can become infinite. This system is a spinless tachyon.

Since the action function is invariant under $\mathscr{P}$, Noether's theorem defines the following constants of the motion:

under time translation, the energy

$$H = -\frac{\partial L}{\partial \dot{t}}; \tag{4.6}$$

under space translation, the linear momentum

$$p_i = \frac{\partial L}{\partial \dot{r}^i}; \tag{4.7}$$

under a Lorentz boost, the Poincaré momentum

$$\boldsymbol{\pi} = -H\mathbf{r}/c^2 + \mathbf{p}t; \tag{4.8}$$

under a rotation, the angular momentum

$$\mathbf{J} = \mathbf{r}\times\mathbf{p}. \tag{4.9}$$

From (4.4) and (4.5) $(H/c)^2 - p^2$ is group invariant and from (4.6) and (4.7) it is also a constant of the motion, and thus has to be independent of $\dot{\mathbf{r}}$ and $\dot{t}$. Consequently it defines a constant parameter that will be used to characterize the system.

Taking the τ derivative in (4.8), $\dot{\boldsymbol{\pi}} = 0 = -H\dot{\mathbf{r}}/c^2 + \mathbf{p}\dot{t}$ and then $\mathbf{p} = H\mathbf{u}/c^2$. In the bradyonic case, $u < c$ and thus $(H/c)^2 - p^2 = m^2c^2$ is positive and defines the constant parameter m, the rest mass of the system. Substituting the expression for $\mathbf{p}$ leads to $H = mc^2/(1 - u^2/c^2)^{1/2}$ and the Lagrangian (4.1) becomes

$$L = -mc^2(1 - c^2\dot{\mathbf{r}}^2/\dot{t}^2)^{1/2}\dot{t}$$
$$= -mc(c^2 - u^2)^{1/2}\dot{t} \tag{4.10}$$

with the action function

$$A(x_1,x_2) = -mc(c^2(t_2 - t_1)^2 - (\mathbf{r}_2 - \mathbf{r}_1)^2)^{1/2}. \tag{4.11}$$

In the photonic case the parameter $m = 0$ and the Lagrangian and the action function are also identically zero. This formalism does not predict spinless photons.

For the pointlike tachyon we get $(H/c)^2 = p^2c^2l^2$, the invariant is negative and we write it as $-\alpha^2 = (H/c)^2 - p^2$. This α is the absolute value of the linear momentum carried by the particle for the observer for which $H = 0$, which corresponds to infinite speed. Thus the energy H is given by

$$H = \mathbf{p}\cdot\mathbf{l}c^2, \tag{4.12}$$

the Lagrangian for the spinless tachyon is

$$L = \alpha(1 - c^2\dot{t}^2/\dot{\mathbf{r}}^2)^{1/2}|\dot{\mathbf{r}}|, \tag{4.13}$$

and the action function is

$$A(x_1,x_2) = \alpha((\mathbf{r}_2 - \mathbf{r}_1)^2 - c^2(t_2 - t_1)^2)^{1/2}. \tag{4.14}$$

V. GENERAL POINCARÉ BRADYONS

Let us consider the mechanical system for which $X = \mathscr{P}$. An element x of X will be given by the ten real numbers, $x\equiv(t(\tau),\mathbf{r}(\tau),\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))$ with domains $t\in\mathbb{R}$, $\mathbf{r}\in\mathbb{R}^3$, $\mathbf{u}\in\mathbb{R}^3$, $u < c$, and $\boldsymbol{\rho}\in\mathbb{R}^3_c$. Taking into account (3.5)–(3.8), they transform under $\mathscr{P}$ as follows:

$$t'(\tau) = \gamma t(\tau) + \gamma(\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{r}(\tau))/c^2 + b, \tag{5.1}$$

$$\mathbf{r}'(\tau) = R(\boldsymbol{\mu})\mathbf{r}(\tau) + \gamma\mathbf{v}t(\tau) + [\gamma^2/(1+\gamma)c^2] \times(\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{r}(\tau))\mathbf{v} + \mathbf{a}, \tag{5.2}$$

$$\mathbf{u}'(\tau) = \frac{R(\boldsymbol{\mu})\mathbf{u}(\tau) + \gamma\mathbf{v} + \gamma^2c^{-2}(\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{u}(\tau))\mathbf{v}/(1+\gamma)}{\gamma(1 + (\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{u}(\tau))/c^2)}, \tag{5.3}$$

$$\boldsymbol{\rho}'(\tau) = \frac{\boldsymbol{\mu} + \boldsymbol{\rho}(\tau) + \boldsymbol{\mu}\times\boldsymbol{\rho}(\tau) + \mathbf{F}(\mathbf{v},\boldsymbol{\mu};\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))}{1 - \boldsymbol{\mu}\cdot\boldsymbol{\rho}(\tau) + G(\mathbf{v},\boldsymbol{\mu};\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))}. \tag{5.4}$$

The way they transform allows us to interpret $t(\tau)$ as the time, $\mathbf{r}(\tau)$ as the position, $\mathbf{u}(\tau)$ as the velocity, and $\boldsymbol{\rho}(\tau)$ as the orientation of the system.

There are three additional constraints among the x variables. The velocity $\mathbf{u}(\tau) = \dot{\mathbf{r}}(\tau)/\dot{t}(\tau)$, together with the homogeneity condition (2.4), reduces to six the number of degrees of freedom of the system. The six independent variables are $\mathbf{r}(t)$ and $\boldsymbol{\rho}(t)$ and the Lagrangian will depend up to the second derivative of $\mathbf{r}$ and only on the first derivative of $\boldsymbol{\rho}$. Since $u < c$ the system is called a bradyon.

Again assuming that the τ parameter is group invariant, taking the τ derivative in both sides of (5.1) (5.4) we get that $\dot{t}(\tau)$ and $\dot{\mathbf{r}}(\tau)$ transform like (4.2)–(4.3) and $\dot{\mathbf{u}}(\tau)$ and $\dot{\boldsymbol{\rho}}(\tau)$ in a very complicated way. However, instead of the derivatives $\dot{\mathbf{u}}(\tau)$ and $\dot{\boldsymbol{\rho}}(\tau)$, we shall define two other three-vectors

$\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ (see the Appendix)

$$\boldsymbol{\alpha}(\tau) = \frac{\gamma(u)}{c} \times \left[\dot{\mathbf{u}}(\tau) + \frac{\gamma(u)^2}{(1+\gamma(u))} \frac{\mathbf{u}\cdot\dot{\mathbf{u}}}{c^2} \mathbf{u}(\tau) + \mathbf{u}(\tau) \times \boldsymbol{\omega}_0(\tau) \right], \tag{5.5}$$

$$\boldsymbol{\omega}(\tau) = \gamma(u)\boldsymbol{\omega}_0 - \frac{\gamma(u)^2}{(1+\gamma(u))} \frac{\mathbf{u}\cdot\boldsymbol{\omega}_0}{c^2} \mathbf{u} + \boldsymbol{\omega}_{\mathrm{T}} \tag{5.6}$$

that are the strict components of an antisymmetric tensor $\Omega^{\mu\nu}$, $\alpha_i = \Omega^{i0}$, and $\omega_i = \frac{1}{2}\epsilon_{ijk}\Omega^{jk}$, and where the variables $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_{\mathrm{T}}$ are given by

$$\boldsymbol{\omega}_0 = 2(\dot{\boldsymbol{\rho}} + \boldsymbol{\rho}\times\dot{\boldsymbol{\rho}})/(1+\rho^2), \tag{5.7}$$

$$\boldsymbol{\omega}_{\mathrm{T}} = -[\gamma(u)^2/(1+\gamma(u))](\mathbf{u}\times\dot{\mathbf{u}})/c^2. \tag{5.8}$$

If τ is the time, $\boldsymbol{\omega}_0$ is the instantaneous angular velocity and $\boldsymbol{\omega}_{\mathrm{T}}$ the Thomas angular velocity. The new varibles $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ transform under $\mathscr{P}$:

$$\boldsymbol{\alpha}'(\tau) = \gamma R(\boldsymbol{\mu})\boldsymbol{\alpha}(\tau) - [\gamma^2/(1+\gamma)c^2](\mathbf{v}\cdot R(\boldsymbol{\mu})\boldsymbol{\alpha}(\tau))\mathbf{v} + (\gamma/c)(\mathbf{v}\times R(\boldsymbol{\mu})\boldsymbol{\omega}(\tau)), \tag{5.9}$$

$$\boldsymbol{\omega}'(\tau) = \gamma R(\boldsymbol{\mu})\boldsymbol{\omega}(\tau) - [\gamma^2/(1+\gamma)c^2](\mathbf{v}\cdot R(\boldsymbol{\mu})\boldsymbol{\omega}(\tau))\mathbf{v} - (\gamma/c)(\mathbf{v}\times R(\boldsymbol{\mu})\boldsymbol{\alpha}(\tau)). \tag{5.10}$$

The homogeneity condition in terms of the variables $(\dot{t},\dot{\mathbf{r}},\dot{\mathbf{u}},\dot{\boldsymbol{\rho}})$ allows us to write L in the form

$$L = -T\dot{t} + \mathbf{Q}\cdot\dot{\mathbf{r}} + \mathbf{U}\cdot\dot{\mathbf{u}} + \mathbf{V}\cdot\dot{\boldsymbol{\rho}}, \tag{5.11}$$

and in terms of the variables $(\dot{t},\dot{\mathbf{r}},\boldsymbol{\alpha},\boldsymbol{\omega})$,

$$L = -T\dot{t} + \mathbf{Q}\cdot\dot{\mathbf{r}} + \mathbf{D}\cdot\boldsymbol{\alpha} + \mathbf{S}\cdot\boldsymbol{\omega}, \tag{5.12}$$

where the functions are defined by

$$T = -\frac{\partial L}{\partial \dot{t}}, \quad Q_i = \frac{\partial L}{\partial \dot{r}^i}, \quad D_i = \frac{\partial L}{\partial \alpha^i},$$

$$S_i = \frac{\partial L}{\partial \omega^i}, \quad U_i = \frac{\partial L}{\partial \dot{u}^i}, \quad V_i = \frac{\partial L}{\partial \dot{\rho}^i},$$

and they are functions of the kinematical variables $(t,\mathbf{r},\mathbf{u},\boldsymbol{\rho})$ and homogeneous functions of zero degree in the variables $(\dot{t},\dot{\mathbf{r}},\boldsymbol{\alpha},\boldsymbol{\omega})$. The observable T has the dimensions of energy, $\mathbf{Q}$ of linear momentum and $\mathbf{D}$ and $\mathbf{S}$ of angular momentum, and, being the Lagrangian invariant under $\mathscr{P}$, they transform

$$T'(\tau) = \gamma T(\tau) + \gamma(\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{Q}(\tau)), \tag{5.13}$$

$$\mathbf{Q}'(\tau) = R(\boldsymbol{\mu})\mathbf{Q}(\tau) + \gamma\mathbf{v}T(\tau)/c^2 + [\gamma^2/(1+\gamma)c^2](\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{Q}(\tau))\mathbf{v}, \tag{5.14}$$

$$\mathbf{D}'(\tau) = \gamma R(\boldsymbol{\mu})\mathbf{D}(\tau) - [\gamma^2/(1+\gamma)c^2](\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{D}(\tau))\mathbf{v} - (\gamma/c)(\mathbf{v}\times R(\boldsymbol{\mu})\mathbf{S}(\tau)), \tag{5.15}$$

$$\mathbf{S}'(\tau) = \gamma R(\boldsymbol{\mu})\mathbf{S}(\tau) - [\gamma^2/(1+\gamma)c^2] \times (\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{S}(\tau))\mathbf{v} + (\gamma/c)(\mathbf{v}\times R(\boldsymbol{\mu})\mathbf{D}(\tau)). \tag{5.16}$$

The observables $\mathbf{D}$ and $\mathbf{S}$ are the strict components of an antisymmetric tensor $S^{\mu\nu} = -S^{\nu\mu}$, $D_i = S^{0i}$, $S_i = \frac{1}{2}\epsilon_{ijk}S^{jk}$. We see that these functions are invariant under translations and thus independent of t and $\mathbf{r}$, and since, for the bradyonic case, $\dot{t}\neq 0$ for every observer, they only have to be functions of $(\mathbf{u},\boldsymbol{\rho},\boldsymbol{\alpha}/\dot{t},\boldsymbol{\omega}/\dot{t})$. Since for the observer for which the system has zero velocity, $\mathbf{u}=0$, $c\boldsymbol{\alpha}/\dot{t}$ reduces to $d\mathbf{u}/dt$ and $\boldsymbol{\omega}/\dot{t}$ to $\boldsymbol{\omega}_0/t$, we say that $c\boldsymbol{\alpha}/\dot{t}$ and $\boldsymbol{\omega}/\dot{t}$ are, respectively, the generalized acceleration and generalized angular velocity of the system.

Noether's theorem defines the following constants of the motion:

energy

$$H = T - \frac{d\mathbf{U}}{dt}\cdot\mathbf{u}; \tag{5.17}$$

linear momentum,

$$\mathbf{p} = \mathbf{Q} - \frac{d\mathbf{U}}{dt}; \tag{5.18}$$

Poincaré momentum

$$\boldsymbol{\pi} = -H\mathbf{r}/c^2 + \mathbf{p}t + \mathbf{D}/c; \tag{5.19}$$

angular momentum,

$$\mathbf{J} = \mathbf{r}\times\mathbf{p} + \mathbf{S}; \tag{5.20}$$

where the function $\mathbf{U}(\tau)$ is given in terms of $\mathbf{D}$ and $\mathbf{S}$ by

$$\mathbf{U}(\tau) = \frac{\gamma(u)}{c}\left[\mathbf{D}(\tau) + \frac{\gamma(u)^2}{(1+\gamma(u))}\frac{\mathbf{u}\cdot\mathbf{D}}{c^2}\mathbf{u}(\tau) + \frac{\gamma(u)}{(1+\gamma(u))c}\mathbf{u}(\tau)\times\mathbf{S}(\tau)\right].$$

Energy and linear momentum transform as in (4.4) and (4.5). The center of mass observer is defined as that observer for which $\mathbf{p}=0$ and $\boldsymbol{\pi}=0$. These six conditions do not define uniquely an observer, but rather the class of observers for which the center of mass is at rest and placed at the origin of the coordinate frame. They are thus defined up to an arbitrary rotation and to an arbitrary time translation. We shall call to this class the center of mass observer as is usually done.

The observable $\mathbf{S}$ is called the spin of the system and is a constant of the motion only for the center of mass observer. Since we consider systems for which $H>0$ we can define the observable $\mathbf{k} = c\mathbf{D}/H$ with dimensions of length such that taking the τ derivative in (5.19) we get

$$\dot{\boldsymbol{\pi}} = (\dot{\mathbf{r}} - \dot{\mathbf{k}})H/c^2 + \mathbf{p}\dot{t} = 0.$$

Thus $\mathbf{p} = (H/c^2)d(\mathbf{r}-\mathbf{k})/dt$ and defines for every observer the position of a point $\mathbf{q} = \mathbf{r}-\mathbf{k}$ that moves at constant velocity $d\mathbf{q}/dt$. We see that $\mathbf{q}$ is the position of the center of mass and thus $\mathbf{k}$ is the relative position of the system with respect to its center of mass. The absolute value of $\mathbf{k}$ gives information about a lower bound for the radius of the particle.

From these constants of the motion other constants can be defined:

$$w^0 = \mathbf{p}\cdot\mathbf{J} = \mathbf{p}\cdot\mathbf{S}, \tag{5.21}$$

$$\mathbf{W} = H\mathbf{J}/c - c\mathbf{p}\times\boldsymbol{\pi} = H(\mathbf{S}+\mathbf{k}\times\mathbf{p})/c. \tag{5.22}$$

These can be expressed in terms of the four-vector $p^\mu \equiv (H/c,\mathbf{p})$ and the antisymmetric tensor $S^{\mu\nu}$ in the form $w_\sigma = \frac{1}{2}\epsilon_{\sigma\mu\nu\lambda}S^{\mu\nu}p^\lambda$. Equations (5.21) and (5.22) are the components of the Pauli–Lubanski four-vector. From (5.21), taking the time derivative we obtain $\mathbf{p}\cdot d\mathbf{S}/dt = 0$, and the spin variation is orthogonal to the center of mass motion, the helicity, or spin projection on that direction, remaining constant.

Thus $p_\mu p^\mu = m^2c^2$ and $-w^\mu w_\mu = m^2c^2s^2$ are two functionally independent invariants, constants of the motion, that define two constant parameters m and s called, respectively, the rest mass and the absolute value of the spin for the center of mass observer and we expect that the Lagrangian will be an explicit function of them.

Let H_0 and $\mathbf{p}_0$ be the energy and linear momentum for the observer for which the variables $\mathbf{u} = \boldsymbol{\rho} = 0$. For this observer they are only functions of $\boldsymbol{\alpha}/\dot{t}$ and $\boldsymbol{\omega}/\dot{t}$. For the general observer for which $\mathbf{u}$ and $\boldsymbol{\rho}$ are different from zero, H and $\mathbf{p}$ are obtained from H_0 and $\mathbf{p}_0$ by the transformation equations (4.4) and (4.5) and thus

$$H(\mathbf{u},\boldsymbol{\rho},\boldsymbol{\alpha}/\dot{t},\boldsymbol{\omega}/\dot{t}) = \gamma(u)H_0 + \gamma(u)(\mathbf{u}\cdot R(\boldsymbol{\rho})\mathbf{p}_0), \qquad (5.23)$$

$$\mathbf{p}(\mathbf{u},\boldsymbol{\rho},\boldsymbol{\alpha}/\dot{t},\boldsymbol{\omega}/\dot{t}) = R(\boldsymbol{\rho})\mathbf{p}_0 + \gamma(u)\mathbf{u}H_0/c^2 + [\gamma(u)^2/(1+\gamma(u))c^2](\mathbf{u}\cdot R(\boldsymbol{\rho})\mathbf{p}_0)\mathbf{u}. \qquad (5.24)$$

Since the first half of the Lagrangian $-T\dot{t} + \mathbf{Q}\cdot\dot{\mathbf{r}} = -H\dot{t} + \mathbf{p}\cdot\dot{\mathbf{r}}$ is Poincaré invariant, substituting H and $\mathbf{p}$ in terms of (5.23) and (5.24) reduces it to $-H_0\dot{t}/\gamma(u)$. Because $\dot{t}/\gamma(u) = (c^2\dot{t}^2 - \dot{\mathbf{r}}^2)^{1/2}/c$ is a Poincaré-invariant function, H_0 must also be an invariant function of their arguments.

Similarly, the second half of the Lagrangian $\mathbf{D}\cdot\boldsymbol{\alpha} + \mathbf{S}\cdot\boldsymbol{\omega} = \frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu}$ is itself Poincaré invariant and we have to choose for $\mathbf{D}$ and $\mathbf{S}$ functions of $(\mathbf{u},\boldsymbol{\rho},\boldsymbol{\alpha}/\dot{t},\boldsymbol{\omega}/\dot{t})$ in order that this holds. From $\Omega^{\mu\nu}$ we can form the two invariants $\epsilon_{\mu\nu\sigma\lambda}\Omega^{\mu\nu}\Omega^{\sigma\lambda}$ and $\Omega_{\mu\nu}\Omega^{\mu\nu}$, which reduce, respectively, to $\boldsymbol{\alpha}\cdot\boldsymbol{\omega}$ and $\alpha^2 - \omega^2$. Expressed in terms of the kinematical variables and their derivatives these invariants are

$$\alpha^2 - \omega^2 = \frac{2\gamma(u)^2[\dot{\mathbf{u}}^2 + \dot{\mathbf{u}}\cdot(\mathbf{u}\times\boldsymbol{\omega}_0)]}{(1+\gamma(u))c^2} + \frac{(2+2\gamma(u)+\gamma(u)^2)\gamma(u)^4}{(1+\gamma(u))^2c^4} \times (\mathbf{u}\cdot\dot{\mathbf{u}})^2 - \omega_0^2, \qquad (5.25)$$

$$\boldsymbol{\alpha}\cdot\boldsymbol{\omega} = \frac{\gamma(u)}{c}\left[\dot{\mathbf{u}}\cdot\boldsymbol{\omega}_0 + \frac{\gamma(u)^2}{(1+\gamma(u))}\frac{(\mathbf{u}\cdot\dot{\mathbf{u}})(\mathbf{u}\cdot\boldsymbol{\omega}_0)}{c^2}\right]. \qquad (5.26)$$

Thus two elementary choices for $\mathbf{D}$ and $\mathbf{S}$: First, to choose $\mathbf{D}$ and $\mathbf{S}$ proportional to $-\gamma(u)\boldsymbol{\alpha}/\dot{t}$ and $\gamma(u)\boldsymbol{\omega}/\dot{t}$, respectively, with the same proportionality coefficient, which has to be an invariant function. In this case the spin is proportional to the generalized angular velocity (suggesting an intrinsic angular momentum of a rotating nature) and the momentum $\mathbf{D}$ (and thus the relative position vector $\mathbf{k}$) has opposite direction to the generalized acceleration, suggesting for the *Zitterwebegung* a certain kind of generalized central motion.

On the other side we can choose $\mathbf{D}$ proportional to the function $\gamma(u)\boldsymbol{\omega}/\dot{t}$ and $\mathbf{S}$ to $\gamma(u)\boldsymbol{\alpha}/\dot{t}$ and we see in this case that $\mathbf{S}$ is by no means related to the angular velocity and the internal motion is no longer a generalized central motion.

These two possible elementary Lagrangians expressed in terms of the two invariants A and B will reduce to

$$L_B = -A(1 - c^{-2}\dot{\mathbf{r}}^2/\dot{t}^2)^{1/2}\dot{t} - B(\alpha^2-\omega^2)(\dot{t}^2 - \dot{\mathbf{r}}^2/c^2)^{-1/2}, \qquad (5.27)$$

$$L_F = -A(1 - c^{-2}\dot{\mathbf{r}}^2/\dot{t}^2)^{1/2}\dot{t} + B(\boldsymbol{\alpha}\cdot\boldsymbol{\omega})(\dot{t}^2 - \dot{\mathbf{r}}^2/c^2)^{-1/2}, \qquad (5.28)$$

where $\alpha^2 - \omega^2$ and $\boldsymbol{\alpha}\cdot\boldsymbol{\omega}$ are given, respectively, in (5.25) and (5.26).

Lagrangians of the first type (5.27) can be found in the literature. Constantelos[14] quotes a Lagrangian in which $\boldsymbol{\omega}_0 = 0$ and thus the particle has internal orbital motion but no rotation. The Lagrangian depends on the velocity and acceleration of the particle but not on the angular variables. On the other hand, Hanson and Regge[11] when discussing the relativistic spherical top, assume $\dot{\mathbf{u}} = 0$ and thus the invariant $\alpha^2 - \omega^2$ reduces to $-\omega_0^2$. The particle has no internal orbital motion; its position coincides with its center of mass but it rotates with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}_0/\dot{t}$ since this rotation is responsible for the existence of spin. However, to the best of our knowledge, no Lagrangians of the form (5.28) have been studied before. These two Lagrangians lead to nonlinear dynamical equations.

The Lagrangian (5.27), for the center of mass observer, gives rise to the equations

$$\mathbf{r} = -\frac{2B^2}{mc^2}\gamma(u)^2\left[\frac{d\mathbf{u}}{dt} + \mathbf{u}\times\boldsymbol{\Omega} + \frac{\gamma(u)^2}{(1+\gamma(u))} \times \left(\frac{\mathbf{u}}{c}\cdot\frac{d\mathbf{u}}{dt}\right)\frac{\mathbf{u}}{c}\right], \qquad (5.29)$$

$$\mathbf{S} = 2B^2\gamma(u)\left[\gamma(u)\boldsymbol{\Omega} - \frac{\gamma(u)^2}{(1+\gamma(u))c^2} \times \left((\mathbf{u}\cdot\boldsymbol{\Omega})\mathbf{u} + \mathbf{u}\times\frac{d\mathbf{u}}{dt}\right)\right], \qquad (5.30)$$

where the spin $\mathbf{S}$ is constant. Solutions, with constant absolute value of velocity and angular velocity and $\boldsymbol{\Omega}\cdot\mathbf{u} = 0$, exist and show the motion displayed in Fig. 1.

Similarly, the Lagrangian (5.28) also describes motions with constant absolute value of velocity and angular velocity, where $\boldsymbol{\Omega}$ is orthogonal to $\mathbf{u}$, and in the center of mass

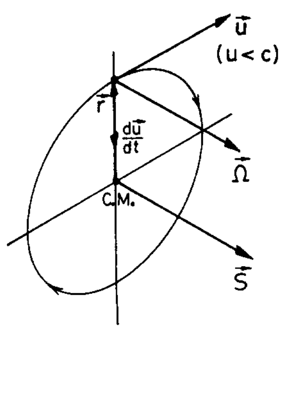

FIG. 1. Motion in the center of mass (C.M.) frame of particle (5.27).

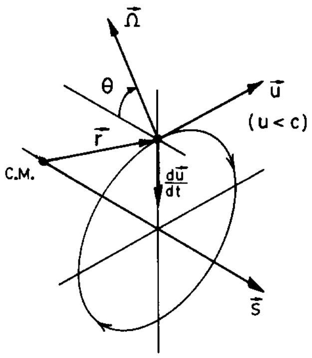

FIG. 2. Motion in the C.M. frame of particle (5.28).

frame they are given by

$$\mathbf{r}=\frac{B}{mc}\gamma(u)\left[\gamma(u)\boldsymbol{\Omega}-\frac{\gamma(u)^2}{(1+\gamma(u))c^2}\times\left((\mathbf{u}\cdot\boldsymbol{\Omega})\mathbf{u}+\mathbf{u}\times\frac{d\mathbf{u}}{dt}\right)\right], \tag{5.31}$$

$$\mathbf{S}=\frac{B}{c}\gamma(u)^2\left[\frac{d\mathbf{u}}{dt}+\mathbf{u}\times\boldsymbol{\Omega}+\frac{\gamma(u)^2}{(1+\gamma(u))}\times\left(\frac{\mathbf{u}}{c}\cdot\frac{d\mathbf{u}}{dt}\right)\frac{\mathbf{u}}{c}\right]. \tag{5.32}$$

One possible motion that verifies these is depicted in Fig. 2.

VI. LUXONS

Now let us consider those mechanical systems whose kinematical space X_C is the nine-dimensional manifold spanned by the variables $(t,(\tau),\mathbf{r}(\tau),\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))$ with domains $t\in\mathbb{R}$, $\mathbf{r}\in\mathbb{R}^3$, and $\boldsymbol{\rho}\in\mathbb{R}^3_c$ (as before) and $\mathbf{u}\in\mathbb{R}^3$, but with $u=c$. These particles are usually called luxons.

This manifold X_c is a homogeneous space isomorphic to $\mathscr{P}/\mathscr{V}$, where $\mathscr{V}$ is the one-parameter subgroup of pure Lorentz transformations on a given direction. In fact, let $x\equiv(0,\mathbf{0},\mathbf{u},\mathbf{0})$, where $u=c$ is a point of this manifold X_c The stabilizer group of this point is the subgroup of pure Lorentz transformations in the direction given by $\mathbf{u}$, $\mathscr{V}_u$. Thus $X_c\approx\mathscr{P}/\mathscr{V}_u$.

Again there are three constraints between the kinematical variables, $\mathbf{u}=\dot{\mathbf{r}}/\dot{t}$. The kinematical variables $t,\mathbf{r},\mathbf{u}$ transform like (5.1)–(5.3), while the transformation of $\boldsymbol{\rho}$ is obtained from (5.4) taking the limit $u=c$ on the right-hand side:

$$\boldsymbol{\rho}'(\tau)=\frac{\boldsymbol{\mu}+\boldsymbol{\rho}(\tau)+\boldsymbol{\mu}\times\boldsymbol{\rho}(\tau)+\mathbf{F}_c(\mathbf{v},\boldsymbol{\mu};\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))}{1-\boldsymbol{\mu}\cdot\boldsymbol{\rho}(\tau)+G_c(\mathbf{v},\boldsymbol{\mu};\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))}, \tag{6.1}$$

where

$$\mathbf{F}_c(\mathbf{v},\boldsymbol{\mu};\mathbf{u},\boldsymbol{\rho})=[\gamma/(1+\gamma)c^2][\mathbf{u}\times\mathbf{v}+\mathbf{u}(\mathbf{v}\cdot\boldsymbol{\mu})+\mathbf{v}(\mathbf{u}\cdot\boldsymbol{\rho})+\mathbf{u}\times(\mathbf{v}\times\boldsymbol{\mu})+(\mathbf{u}\times\boldsymbol{\rho})\times\mathbf{v}+(\mathbf{u}\cdot\boldsymbol{\rho})(\mathbf{v}\times\boldsymbol{\mu})+(\mathbf{u}\times\boldsymbol{\rho})(\mathbf{v}\cdot\boldsymbol{\mu})+(\mathbf{u}\times\boldsymbol{\rho})\times(\mathbf{v}\times\boldsymbol{\mu})], \tag{6.2}$$

$$G_c(\mathbf{v},\boldsymbol{\mu};\mathbf{u},\boldsymbol{\rho})=[\gamma/(1+\gamma)c^2[\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot(\mathbf{v}\times\boldsymbol{\mu})+\mathbf{v}\cdot(\mathbf{u}\times\boldsymbol{\rho})-(\mathbf{u}\cdot\boldsymbol{\rho})(\mathbf{v}\cdot\boldsymbol{\mu})+(\mathbf{u}\times\boldsymbol{\rho})\cdot(\mathbf{v}\times\boldsymbol{\mu})]. \tag{6.3}$$

Since $u'=u=c$, Eq. (5.3), implies that $\mathbf{u}'$ is obtained from $\mathbf{u}$ by means of an orthogonal transformation:

$$\mathbf{u}=\;=R(\boldsymbol{\phi})\mathbf{u}, \tag{6.4}$$

where $\boldsymbol{\phi}$ is given by

$$\boldsymbol{\phi}=\frac{\boldsymbol{\mu}+\mathbf{F}_c(\mathbf{v},\boldsymbol{\mu};\mathbf{u}(\tau),\mathbf{0})}{1+G_c(\mathbf{v},\boldsymbol{\mu};\mathbf{u}(\tau),\mathbf{0})}. \tag{6.5}$$

Equation (6.1) also corresponds to

$$R(\boldsymbol{\rho}')=R(\boldsymbol{\phi})R(\boldsymbol{\rho}), \tag{6.6}$$

with the same $\boldsymbol{\phi}$ as in (6.5).

Since $u(\tau)=c$, we have to distinguish two different kinds of systems. Taking the τ derivative of this identity we get $\mathbf{u}(\tau)\cdot\dot{\mathbf{u}}(\tau)=0$, and we shall next consider systems for which $\dot{\mathbf{u}}(\tau)=0$ and those for which $\dot{\mathbf{u}}(\tau)\neq0$ and is orthogonal to $\mathbf{u}$.

A. Massless particles

If $\dot{\mathbf{u}}(\tau)=0$ then $\mathbf{u}(\tau)$ is constant, the system follows a straight trajectory with velocity c, and the kinematical space reduces to the seven-dimensional manifold $(t(\tau),\mathbf{r}(\tau),\boldsymbol{\rho}(\tau))$, with $t\in\mathbb{R}$, $\mathbf{r}\in\mathbb{R}^3$, $\boldsymbol{\rho}\in\mathbb{R}^3_c$.

The derivatives $\dot{t}$ and $\dot{\mathbf{r}}$ transform as in (4.2) and (4.3) and, instead of the variable $\dot{\boldsymbol{\rho}}$, we define the linear function of it $\boldsymbol{\omega}_0$ given by (5.7), which transforms under $\mathscr{P}$, as

$$\boldsymbol{\omega}_0'(\tau)=R(\boldsymbol{\phi})\boldsymbol{\omega}_0(\tau), \tag{6.7}$$

where $\boldsymbol{\phi}$ is given again by (6.5). The invariant $c^2\dot{t}^2-\dot{\mathbf{r}}^2=0$, and $\dot{t}\neq0$ and $\dot{\mathbf{r}}\neq0$ for every observer.

For this system there are no differential constraints between the kinematical variables, the Lagrangian will only depend on the first derivatives of the variables $\mathbf{r}$ and $\boldsymbol{\rho}$, and the homogeneity condition (2.4) reduces to 6 the number of degrees of freedom of the system. This condition leads to a Lagrangian of the form

$$L=-H\dot{t}+\mathbf{p}\cdot\dot{\mathbf{r}}+\mathbf{S}\cdot\boldsymbol{\omega}_0, \tag{6.8}$$

where the functions

$$H=-\frac{\partial L}{\partial\dot{t}},\quad p_i=\frac{\partial L}{\partial\dot{r}^i},\quad S_i=\frac{\partial L}{\partial\omega_0^i}$$

will be functions of $(t,\mathbf{r},\boldsymbol{\rho})$, homogeneous of zero degree of $(\dot{t},\dot{\mathbf{r}},\boldsymbol{\omega}_0)$, and since $\dot{t}\neq0$ they can be expressed as functions of $\mathbf{u}=\dot{\mathbf{r}}/\dot{t}$ and $\boldsymbol{\Omega}=\boldsymbol{\omega}_0/\dot{t}$, which are, respectively, the velocity and angular velocity of the system.

The invariance of (6.8) under $\mathscr{P}$ implies that these functions transform as

$$H'\tau=\gamma H(\tau)+\gamma(\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{p}(\tau)), \tag{6.9}$$

$$\mathbf{p}'(\tau)=R(\boldsymbol{\mu})\mathbf{p}(\tau)+\gamma\mathbf{v}H(\tau)/c^2+[\gamma^2/(1+\gamma)c^2](\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{p}(\tau))\mathbf{v}, \tag{6.10}$$

$$\mathbf{S}'(\tau)=R(\boldsymbol{\phi})\mathbf{S}(\tau), \tag{6.11}$$

and being invariant under space and time translations they are only functions of $(\boldsymbol{\rho},\mathbf{u},\boldsymbol{\Omega})$ with the condition $u=c$.

Noether's theorem gives rise in this case to the following constants of the motion:

energy, H; (6.12)

linear momentum, $\mathbf{p}$; (6.13)

Poincaré momentum,

$$\boldsymbol{\pi} = -H\mathbf{r}/c^2 + \mathbf{p}t + \mathbf{S}\times\mathbf{u}/c^2; \tag{6.14}$$

angular momentum, $\mathbf{J} = \mathbf{r}\times\mathbf{p} + \mathbf{S}$. (6.15)

By analogy to the above case we say that $\mathbf{S}$ is the spin of the system. Taking in (6.15) the τ derivative we obtain $\dot{\mathbf{r}}\times\mathbf{p} + \dot{\mathbf{S}} = 0$ and thus $d\mathbf{S}/dt = \mathbf{p}\times\mathbf{u}$. Since $\mathbf{p}$ and $\mathbf{u}$ are constant vectors, the spin $\mathbf{S}$ has a constant time derivative. A continuously increasing spin system is far from being what we shall understand as an elementary particle, so that we shall only consider that system for which the constant $d\mathbf{S}/dt$ is zero. The spin is then a constant of the motion and due to(6.11) its absolute value is also a Poincaré-invariant parameter. In this case $\mathbf{p}$ and $\mathbf{u}$ are parallel vectors. In fact, by taking in (6.14) the τ derivative we get $\dot{\boldsymbol{\pi}} = 0 = -H\dot{\mathbf{r}}/c^2 + \mathbf{p}\dot{t}$ and thus $\mathbf{p} = H\mathbf{u}/c^2$.

Another group invariant and constant of the motion is $(H/c)^2 - p^2$, which vanishes identically; thus the mass of the particle is zero. Also the first two terms of the Lagrangian cancel out $-H\dot{t} + \mathbf{p}\cdot\dot{\mathbf{r}} = 0$, and L reduces to the third term $L = \mathbf{S}\cdot\boldsymbol{\omega}_0$, where $\mathbf{S}$ is only a function of $\boldsymbol{\rho}$, $\mathbf{u}$, and $\boldsymbol{\Omega}$. We see that $\epsilon = \mathbf{S}\cdot\mathbf{u}$ is another group invariant and constant of the motion, and we expect that the Lagrangian will be dependent on these two parameters S and ϵ. If we take into account (6.4), (6.6), (6.7), and (6.11) the general solution for $\mathbf{S}$ is a vector function of $R(\boldsymbol{\rho})\mathbf{z}$, $\mathbf{u}$, and $\boldsymbol{\omega}_0/\omega_0$, where $\mathbf{z}$ is a constant unit vector.

A system with a nontransversal spin will be such that $\mathbf{S} = \epsilon S\mathbf{u}/c$, where $\epsilon = \pm 1$, and thus the Lagrangian becomes

$$L = \epsilon S(\dot{\mathbf{r}}\cdot\boldsymbol{\omega}_0)/c\dot{t}. \tag{6.16}$$

From this particular Lagrangian we get $H = -\partial L/\partial t = \mathbf{S}\cdot\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the angular velocity of the particle. The linear momentum $p_i = \partial L/\partial \dot{r}^i = \epsilon S\Omega_i/c$ and the angular velocity lies in the direction of $\mathbf{u}$. Since H has to be definite positive, $\boldsymbol{\Omega} = \epsilon\Omega\mathbf{u}/c$, leading to the expression $H = S\Omega$ for the energy. Experimentally $S = \hbar$ and $H = \hbar\Omega = h\nu$. We say that system (6.16) is a photon of spin S and helicity ϵ. Thus the frequency of a photon appears as the frequency of its rotational motion, causing the rotation axis to lie parallel to the velocity. We cannot define any size associated with the photon as we did before in connection with the general Poincaré bradyon. It must be remarked that, although the spin and the angular velocity are not related, they have the same direction.

B. Massive particles

Now considering systems with $\dot{\mathbf{u}} \neq 0$ but orthogonal to $\mathbf{u}$, we have that the kinematical variables and the derivatives $\dot{t}$ and $\dot{\mathbf{r}}$ transform as before, and for $\dot{\mathbf{u}}$ and $\boldsymbol{\omega}_0$ we obtain

$$\dot{\mathbf{u}}' = R(\boldsymbol{\phi})\dot{\mathbf{u}} + \dot{R}(\boldsymbol{\phi})\mathbf{u}, \tag{6.17}$$

$$\boldsymbol{\omega}_0' = R(\boldsymbol{\phi})\boldsymbol{\omega}_0 + \boldsymbol{\omega}_\phi, \tag{6.18}$$

where $\boldsymbol{\phi}$ is again given by (6.5) and $\boldsymbol{\omega}_\phi$:

$$\boldsymbol{\omega}_\phi = \left(\gamma \frac{R\mathbf{u}\times\mathbf{v}}{c^2} - (\gamma - 1)\frac{R(\mathbf{u}\times\dot{\mathbf{u}})}{c^2} + \frac{2\gamma^2}{(1+\gamma)} \times \frac{[v\cdot R(\mathbf{u}\times\dot{\mathbf{u}})]}{c^3}\frac{\mathbf{v}}{c}\right)\left[\gamma\left(1 + \frac{\mathbf{v}\cdot R\mathbf{u}}{c^2}\right)\right]^{-1}. \tag{6.19}$$

Expression (6.17) can be rewritten in the form

$$\dot{\mathbf{u}}' = R(\boldsymbol{\phi})\dot{\mathbf{u}}/\gamma(1 + \mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{u}/c^2), \tag{6.20}$$

and the homogeneity condition leads to

$$L = -T\dot{t} + \mathbf{Q}\cdot\dot{\mathbf{r}} + \mathbf{U}\cdot\dot{\mathbf{u}} + \mathbf{Z}\cdot\boldsymbol{\omega}_0, \tag{6.21}$$

where

$$T = -\frac{\partial L}{\partial \dot{t}}, \quad Q_i = \frac{\partial L}{\partial \dot{r}^i}, \quad U_i = \frac{\partial L}{\partial \dot{u}^i}, \quad Z_i = \frac{\partial L}{\partial \omega_0^i}$$

and Noether's theorem again defines the constants of the motion:

energy, $H = T - \dfrac{d\mathbf{U}}{dt}\cdot\mathbf{u}$; (6.22)

linear momentum, $\mathbf{p} = \mathbf{Q} - \dfrac{d\mathbf{U}}{dt}$; (6.23)

Poincaré momentum,

$$\boldsymbol{\pi} = -H\mathbf{r}/c^2 + \mathbf{p}t + (\mathbf{S}\times\mathbf{u})/c^2; \tag{6.24}$$

angular momentum, $\mathbf{J} = \mathbf{r}\times\mathbf{p} + \mathbf{S}$; (6.25)

where the spin $\mathbf{S}$ is defined by

$$\mathbf{S} = \mathbf{u}\times\mathbf{U} + \mathbf{Z}. \tag{6.26}$$

The above definitions (6.22)–(6.25) lead for H and $\mathbf{p}$ to the transformation equations of a four-momentum as in (4.4)–(4.5) and for the spin to

$$\mathbf{S}'(\tau) = \gamma R(\boldsymbol{\mu})\mathbf{S}(\tau) - [\gamma^2/(1+\gamma)c^2](\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{S}(\tau))\mathbf{v} + (\gamma/c^2)(\mathbf{v}\times R(\boldsymbol{\mu})(\mathbf{S}(\tau)\times\mathbf{u})), \tag{6.27}$$

which corresponds to the transformation equations of an antisymmetric tensor $S^{\mu\nu}$ with strict components $S^{0i} = (\mathbf{S}\times\mathbf{u})^i/c$, $S^{ij} = \epsilon^{ijk}S_k$.

From (6.24), by taking the τ derivative and doing the dot product with $\mathbf{u}$, we obtain

$$H = \mathbf{p}\cdot\mathbf{u} + \mathbf{S}\cdot\left(\frac{d\mathbf{u}}{dt}\times\mathbf{u}\right)c^{-2}. \tag{6.28}$$

In a certain sense this Hamiltonian looks like Dirac's Hamiltonian for a fermion $H = c\mathbf{p}\cdot\boldsymbol{\alpha} + \beta mc^2$, where $\boldsymbol{\alpha}$ and β are Dirac's matrices, and, since $c\boldsymbol{\alpha}$ is identified with the local velocity $\mathbf{u}$, we have finally $H = \mathbf{p}\cdot\mathbf{u} + \beta mc^2$. In the identification the spin term gives rise to the mass term, suggesting a mass–spin relation. However, we shall not discuss any quantization procedure in this work and we delay these considerations to a subsequent paper.

From (6.25) we have that $d\mathbf{S}/dt = \mathbf{p}\times\mathbf{u}$ and we can again define the center of mass observer by $\mathbf{p} = \boldsymbol{\pi} = 0$. For this observer $\mathbf{S}$ is a constant of the motion and the energy does not necessarily vanish, defining the rest mass of the system. From (6.24), $\mathbf{r} = (\mathbf{S}\times\mathbf{u})/H$, and thus the internal motion lies on a plane orthogonal to $\mathbf{S}$. The velocity is then orthogonal to $\mathbf{S}$, and since S, $u = c$, and $H = mc^2$ are constants in this frame, the internal motion is a circle of radius $R_0 = S/mc$.

Equation (6.27) can also be written

$$\mathbf{S}' = \gamma(1 + \mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{u}/c^2)R(\boldsymbol{\phi})\mathbf{S}, \tag{6.29}$$

with the same $R(\boldsymbol{\phi})$ as in (6.6) and thus

$$\mathbf{S}'\cdot\mathbf{u}' = \gamma(1 + \mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{u}/c^2)\mathbf{S}\cdot\mathbf{u}$$

and $\mathbf{S}\cdot\dot{\mathbf{u}} = \mathbf{S}'\cdot\dot{\mathbf{u}}'$. Since, for the center of mass observer $\mathbf{S}\cdot\mathbf{u} = 0$ and $\mathbf{S}\cdot d\mathbf{u}/dt = 0$, the spin remains orthogonal to $\mathbf{u}$ and $d\mathbf{u}/dt$ for every observer.

The particle has mass and spin and moves in circles with velocity c in a plane orthogonal to $\mathbf{S}$ for the center of mass observer. All these features are general and independent of the particular Lagrangian (6.21). The only thing that remains to be described is the angular motion. All the above considerations do not give information about the angular velocity of the system. Therefore the different kinds of Lagrangians of the form (6.21) will differ from each other by describing different angular motions, which will be related to other kinds of observables, such as, for instance, electromagnetic multipole momenta. However, since we are describing here free particles, we do not expect that such properties will appear in this setup.

Coming back to the general situation we see that the term $-T\dot{t} + \mathbf{Q}\cdot\dot{\mathbf{r}} = -H\dot{t} + \mathbf{p}\cdot\dot{\mathbf{r}}$ is a Poincaré-invariant term and then $\mathbf{U}\cdot\dot{\mathbf{u}} + \mathbf{Z}\cdot\boldsymbol{\omega}_0$ also has to be invariant. Thus the general Lagrangian seems to be the sum of two invariant terms depending on the two constant parameters m and s that are functions of the variables $(\mathbf{u},\boldsymbol{\rho})$ and homogeneous of first degree in the derivatives $(\dot{t},\dot{\mathbf{r}},\dot{\mathbf{u}},\dot{\boldsymbol{\rho}})$.

We find that the first degree term $\dot{\mathbf{u}} + \mathbf{u}\times\boldsymbol{\omega}_0 = (d\mathbf{u}/dt + \mathbf{u}\times\boldsymbol{\Omega})\dot{t} = \mathbf{y}$ transforms under $\mathscr{P}$ in the form $\mathbf{y}' = R(\boldsymbol{\phi})\mathbf{y}$, where $\boldsymbol{\phi}$ is given in (6.5), so that $y^2 = (\dot{\mathbf{u}} + \mathbf{u}\times\boldsymbol{\omega}_0)^2$ is a second degree invariant term. Similarly $(\boldsymbol{\omega}_0\cdot\dot{\mathbf{u}})\dot{t}$ and $\dot{\mathbf{u}}^2\dot{t}^2$ are, respectively, third and fourth degree invariant terms. We can thus find several first degree invariant terms, and among others we quote

$$mc\frac{\dot{\mathbf{u}}^2\dot{t}}{\boldsymbol{\omega}_0\cdot\dot{\mathbf{u}}},\quad mc^2\dot{t}\left(\frac{\dot{\mathbf{u}}^2}{\mathbf{y}^2}\right)^{1/2},\quad \frac{mc^3(\boldsymbol{\omega}_0\cdot\dot{\mathbf{u}})\dot{t}}{y^2},$$
$$\frac{m^2c^5}{S^2}\frac{\dot{\mathbf{u}}^2\dot{t}^2}{y^3},\quad \frac{S\boldsymbol{\omega}_0\cdot\dot{\mathbf{u}}}{(\dot{\mathbf{u}}^2)^{1/2}},\quad S\frac{y}{c},$$

where the parameters m and S have been introduced by dimensionality considerations.

For instance, the first degree invariant Lagrangian

$$L = mc^3\frac{(\boldsymbol{\Omega}\cdot\dot{\mathbf{u}})}{(d\mathbf{u}/dt + \mathbf{u}\times\boldsymbol{\Omega})^2} - \frac{S}{c}\left[\left(\frac{d\mathbf{u}}{dt} + \mathbf{u}\times\boldsymbol{\Omega}\right)^2\right]^{1/2}\dot{t} \tag{6.30}$$

leads for the spin $\mathbf{S} = \mathbf{u}\times\mathbf{U} + \mathbf{Z}$ to

$$\mathbf{S} = mc^3\frac{(d\mathbf{u}/dt + \mathbf{u}\times\boldsymbol{\Omega})}{(d\mathbf{u}/dt + \mathbf{u}\times\boldsymbol{\Omega})^2}, \tag{6.31}$$

and we see that $\mathbf{S}\cdot\mathbf{u} = 0$. For the center of mass observer, $\mathbf{p} = 0$, where $\mathbf{S}$ is constant, and $\boldsymbol{\pi} = 0$ implies that (6.24) reads $mc^2\mathbf{r} = \mathbf{S}\times\mathbf{u}$. We can eliminate $\mathbf{u}$ from these two equations obtaining $\mathbf{u} = -mc^2S^{-2}\mathbf{S}\times\mathbf{r} = \boldsymbol{\Omega}_l\times\mathbf{r}$, where the orbital angular velocity $\boldsymbol{\Omega}_l$ has direction opposite to the spin and constant value mc^2/S.

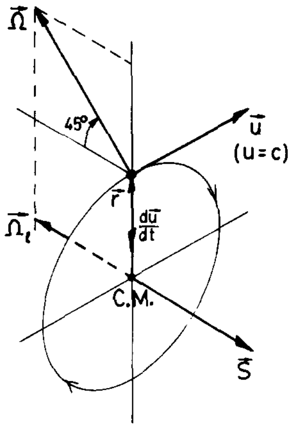

FIG. 3. Relative orientation of observables in the C.M. frame.

Since $\mathbf{S}\cdot d\mathbf{u}/dt = 0$ we get from (6.31) that

$$\left(\frac{d\mathbf{u}}{dt}\right)^2 + \left(\frac{d\mathbf{u}}{dt}\right)\cdot(\mathbf{u}\times\boldsymbol{\Omega}) = 0.$$

If we consider a body fixed coordinate frame that rotates with angular velocity $\boldsymbol{\Omega}_l$ with respect to the center of mass frame and we call Ω_s, Ω_u, and $\Omega_{\dot{u}}$ the components of $\boldsymbol{\Omega}$ along the three orthogonal directions of $\mathbf{S}$, $\mathbf{u}$, and the $d\mathbf{u}/dt$, respectively, we obtain $\Omega_l^2c^2 + \Omega_l\Omega_sc^2 = 0$, i.e., $\Omega_s = -\Omega_l$, since $\boldsymbol{\Omega}_l\times\mathbf{u} = d\mathbf{u}/dt$.

Taking the absolute value of (6.31) we get that $|(d\mathbf{u}/dt + \mathbf{u}\times\boldsymbol{\Omega})| = \Omega_lc$. Then its projection on the direction of $\mathbf{S}$ gives $S = mc^3(-c\Omega_{\dot{u}})/\Omega_l^2c^2$ and thus $\Omega_{\dot{u}} = -\Omega_l$.

Finally the condition $\mathbf{p} = \mathbf{Q} - d\mathbf{U}/dt = 0$ leads in this frame to

$$\mathbf{u}(\Omega^2) - \boldsymbol{\Omega}(\boldsymbol{\Omega}\cdot\mathbf{u}) - 2\frac{d\mathbf{u}}{dt}\times\boldsymbol{\Omega} - \frac{d^2\mathbf{u}}{dt^2} - \mathbf{u}\times\frac{d\boldsymbol{\Omega}}{dt} - c\frac{d\boldsymbol{\Omega}}{dt} = 0,$$

and, since $d^2\mathbf{u}/dt^2 = -\Omega_l^2\mathbf{u}$ and $d\boldsymbol{\Omega}/dt = (d\boldsymbol{\Omega}/dt)_b + \boldsymbol{\Omega}_l\times\boldsymbol{\Omega}$, where $(d\boldsymbol{\Omega}/dt)_b$ is the derivative of $\boldsymbol{\Omega}$ in the body fixed frame, we have

$$\mathbf{u}(3\Omega_l^2 + \Omega_u^2) - (\boldsymbol{\Omega} + \boldsymbol{\Omega}_l)c\Omega_u - 2\frac{d\mathbf{u}}{dt}\times\boldsymbol{\Omega} - \mathbf{u}\times\left(\frac{d\boldsymbol{\Omega}}{dt}\right)_b - c\left(\frac{d\boldsymbol{\Omega}}{dt}\right)_b - c(\boldsymbol{\Omega}_l\times\boldsymbol{\Omega}) = 0.$$

If we take the projection of this expression along the $\mathbf{S}$ direction, taking into account that $\Omega_s = -\Omega_l$ is constant, we obtain $\Omega_u = 0$, and thus the angular velocity $\boldsymbol{\Omega}$ is of constant value $\sqrt{2}\Omega_l$ and in the center of mass frame it rotates around the spin direction with angular velocity $\boldsymbol{\Omega}_l$. We see that for this system $\mathbf{S}$ and $\boldsymbol{\Omega}$ are not directly related (Fig. 3).

VII. TACHYONS

If we consider the manifold spanned by the variables $(t,\mathbf{r},\mathbf{u},\boldsymbol{\rho})$ with domains $t\in\mathbf{R}$, $\mathbf{r}\in\mathbf{R}^3$, $\boldsymbol{\rho}\in\mathbf{R}_c^3$ as before and $\mathbf{u}\in\mathbf{R}^3$, $u > c$, we see that the transformation equations for $\boldsymbol{\rho}$ [(5.4)] do not give any real limit when $u > c$ because the $\gamma(u)$ terms involved in the F and G functions are imaginary. Thus for tachyons we are left with the seven-dimensional manifold X_T spanned by the variables $(t(\tau),\mathbf{r}(\tau),\mathbf{u}(\tau))$, with $t\in\mathbb{R}$, $\mathbf{r}\in\mathbb{R}^3$, $\mathbf{u}\in\mathbb{R}^3$, and $u>c$, which is isomorphic to the homogeneous space $\mathscr{P}/\mathrm{SO}(3)$, and thus no angular variables can be defined. The kinematical variables transform as in (5.1)–(5.3). There exists the constraint $\mathbf{u}=\dot{\mathbf{r}}/\dot{t}$ and the homogeneity condition (2.4) allows us to write the Lagrangian

$$L=-T\dot{t}+\mathbf{Q}\cdot\dot{\mathbf{r}}+\mathbf{U}\cdot\dot{\mathbf{u}}, \tag{7.1}$$

where

$$T=-\frac{\partial L}{\partial \dot{t}},\quad Q_i=\frac{\partial L}{\partial \dot{r}^i},\quad U_i=\frac{\partial L}{\partial \dot{u}^i}.$$

Invariance of L leads for T and $\mathbf{Q}$ to the transformation equations (5.13) and (5.14) and for $\mathbf{U}$ to

$$\mathbf{U}'=\gamma\left(1+\frac{\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{u}}{c^2}\right)\left[R(\boldsymbol{\mu})\mathbf{U}+\frac{\gamma^2}{(1+\gamma)c^2}\times(\mathbf{v}\cdot R(\boldsymbol{\mu})\mathbf{U})\mathbf{v}+\frac{\gamma}{c^2}(\mathbf{u}\cdot\mathbf{U})\mathbf{v}\right] \tag{7.2}$$

and thus they are functions of $(\mathbf{u},\dot{t},\dot{\mathbf{r}},\dot{\mathbf{u}})$, being homogeneous of zero degree in terms of the derivatives.

Noether's theorem defines the constants of the motion to be

energy, $$H=T-\frac{d\mathbf{U}}{dt}\cdot\mathbf{u}; \tag{7.3}$$

linear momentum, $$\mathbf{p}=\mathbf{Q}-\frac{d\mathbf{U}}{dt}; \tag{7.4}$$

Poincaré momentum,

$$\boldsymbol{\pi}=-H\mathbf{r}/c^2+\mathbf{p}t+\mathbf{U}-[\mathbf{U}\cdot\mathbf{u}/c^2]\mathbf{u}; \tag{7.5}$$

angular momentum, $$\mathbf{J}=\mathbf{r}\times\mathbf{p}+\mathbf{u}\times\mathbf{U}. \tag{7.6}$$

The observable $\mathbf{U}-[(\mathbf{U}\cdot\mathbf{u})/c^2]\mathbf{u}$ is always different from zero since $u>c$, and if we define the relative position vector $\mathbf{k}$ as before by $H\mathbf{k}/c^2=\mathbf{U}-[(\mathbf{U}\cdot\mathbf{u})/c^2]\mathbf{u}$ then H has also to be different from zero for every observer. This implies that the invariant and constant of the motion $(H/c)^2-p^2$ cannot be negative and thus the system has a tachyonic internal orbital motion while its center of mass has a velocity $\leqslant c$. The spin of the particle $\mathbf{S}=\mathbf{u}\times\mathbf{U}=H\mathbf{u}\times\mathbf{k}/c^2$ is a constant of the motion in the center of mass frame for nonzero mass particles, while for massless particles it precesses around the linear momentum, always being orthogonal to the velocity.

The invariant Lagrangian for $u>c$ particles,

$$L_T=A\dot{t}(u^2-c^2)^{1/2}-\frac{B\dot{t}}{(u^2-c^2)^{3/2}}\times\left[\left(\frac{d\mathbf{u}}{dt}\right)^2-\frac{(\mathbf{u}\cdot d\mathbf{u}/dt)^2}{u^2-c^2}\right], \tag{7.7}$$

leads in the center of mass frame to the dynamical equations

$$m\mathbf{r}=\frac{-2B}{(u^2-c^2)^{3/2}}\frac{d^2\mathbf{r}}{dt^2}. \tag{7.8}$$

The internal motion is a central motion, and, being the spin constant, this gives rise to a first integral $\mathbf{S}=-\mathbf{r}\times m\mathbf{u}$, the motion lying in a plane orthogonal to the constant vector $\mathbf{S}$. In polar coordinates (r,θ) in this plane Eqs. (7.8) become

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2+\frac{m}{2B}\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2-c^2\right]^{3/2}r=0, \tag{7.9}$$

$$2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}=0, \tag{7.10}$$

the first integral $d\theta/dt=S/mr^2$, and thus the radial equation (7.9) becomes

$$\frac{d^2r}{dt^2}-\frac{S^2}{m^2r^3}+\frac{m}{2B}\left[\left(\frac{dr}{dt}\right)^2+\frac{S^2}{m^2r^2}-c^2\right]^{3/2}r=0. \tag{7.11}$$

We see that this equation has solutions with constant $r\neq 0$ and the system in the center of mass frame follows a circle with constant velocity $u>c$. A general solution of (7.11) has not yet been obtained.

VIII. INVERSIONS

Since space and time reversal are automorphisms of $\mathscr{P}$ given by

P: $(b,\mathbf{a},\mathbf{v},\boldsymbol{\mu})\to(b,-\mathbf{a},-\mathbf{v},\boldsymbol{\mu})$,

T: $(b,\mathbf{a},\mathbf{v},\boldsymbol{\mu})\to(-b,\mathbf{a},-\mathbf{v},\boldsymbol{\mu})$,

we extend this action on the general kinematical space $X=\mathscr{P}$ by

P: $(t(\tau),\mathbf{r}(\tau),\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))\to(t(\tau),-\mathbf{r}(\tau),-\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))$,

T: $(t(\tau),\mathbf{r}(\tau),\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))\to(-t(\tau),\mathbf{r}(\tau),-\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))$.

If we assume that τ parameter remains invariant under inversions, then we can define the P and T action on the derivatives as

P: $(\dot{t}(\tau),\dot{\mathbf{r}}(\tau),\dot{\mathbf{u}}(\tau),\dot{\boldsymbol{\rho}}(\tau))\to(\dot{t}(\tau),-\dot{\mathbf{r}}(\tau),-\dot{\mathbf{u}}(\tau),\dot{\boldsymbol{\rho}}(\tau))$,

T: $(\dot{t}(\tau),\dot{\mathbf{r}}(\tau),\dot{\mathbf{u}}(\tau),\dot{\boldsymbol{\rho}}(\tau))\to(-\dot{t}(\tau),\dot{\mathbf{r}}(\tau),-\dot{\mathbf{u}}(\tau),\dot{\boldsymbol{\rho}}(\tau))$,

and thus

P: $(\boldsymbol{\alpha}(\tau),\boldsymbol{\omega}(\tau))\to(-\boldsymbol{\alpha}(\tau),\boldsymbol{\omega}(\tau))$,

T: $(\boldsymbol{\alpha}(\tau),\boldsymbol{\omega}(\tau))\to(-\boldsymbol{\alpha}(\tau),\boldsymbol{\omega}(\tau))$.

Lagrangian (4.10) remains invariant under P and changes its sign under T so that dynamical equations are invariant under inversions. The Lagrangian (4.13) is itself invariant.

Similarly, Lagrangians (5.27) L_B and (7.7) L_T are invariant under P and change sign under T, and the photonic Lagrangian (6.16) is affected by a minus sign under both inversions.

However, the Lagrangian (5.28) L_F under parity operation has the first term invariant while the second one changes in sign. Under time reversal we have the opposite situation: the first term is affected by a minus sign but the second is left invariant, so that inversions alter this system.

Finally, the Lagrangian (6.30) has the same behavior as L_F under inversions. Its two terms are separately affected by a minus sign by a different inversion, and we can see from Fig. 3 that the internal motion is reversed but the spin remains unchanged.

APPENDIX: DEFINITION OF VARIOUS FUNCTIONS

If Λ is a Lorentz transformation, then $\Lambda G \Lambda^T = G$ holds, where G is the Minkowski metric tensor written in matrix form, i.e., diag$(-1,1,1,1)$, and Λ^T is the transpose of Λ.

If we form the matrix $\Lambda(\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))$ it also fulfills this condition. Taking the τ derivative,

$$\dot{\Lambda}(\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))G\Lambda^T(\mathbf{u}(\tau),\boldsymbol{\rho}(\tau)) + \Lambda(\mathbf{u}(\tau),\boldsymbol{\rho}(\tau))G\dot{\Lambda}^T(\mathbf{u}(\tau),\boldsymbol{\rho}(\tau)) = 0,$$

i.e., $\Omega + \Omega^T = 0$, and the antisymmetric matrix Ω is a linear function of the derivatives $\dot{\mathbf{u}}$ and $\dot{\boldsymbol{\rho}}$.

Under a Poincaré transformation, the variables $\mathbf{u}$ and $\boldsymbol{\rho}$ transform according to (5.3) and (5.4), which is equivalent to

$$\Lambda(\mathbf{u}',\boldsymbol{\rho}') = \Lambda(\mathbf{v},\boldsymbol{\mu})\Lambda(\mathbf{u},\boldsymbol{\rho}),$$

and thus

$$\dot{\Lambda}(\mathbf{u}',\boldsymbol{\rho}') = \Lambda(\mathbf{v},\boldsymbol{\mu})\dot{\Lambda}(\mathbf{u},\boldsymbol{\rho}),$$

so that

$$\begin{aligned}\Omega' &= \dot{\Lambda}(\mathbf{u}',\boldsymbol{\rho}')G\Lambda(\mathbf{u}',\boldsymbol{\rho}')\\ &= \Lambda(\mathbf{v},\boldsymbol{\mu})\dot{\Lambda}(\mathbf{u},\boldsymbol{\rho})G\Lambda(\mathbf{u},\boldsymbol{\rho})\Lambda^T(\mathbf{v},\boldsymbol{\mu})\\ &= \Lambda(\mathbf{v},\boldsymbol{\mu})\Omega\Lambda^T(\mathbf{v},\boldsymbol{\mu}),\end{aligned}$$

which corresponds to the transformation properties of a tensor $\Omega^{\mu\nu}$.

Similarly, if R is an orthogonal 3×3 matrix, $RR^T = \mathbb{1}$. In particular, if we define the orthogonal matrix $R(\boldsymbol{\rho}(\tau))$, then taking the τ derivative we get $\dot{R}(\tau)R^T(\tau) + R(\tau)\dot{R}^T(\tau) = 0$, i.e., $\Omega + \Omega^T = 0$, and if we call $\boldsymbol{\omega}_0$ the nondiagonal components of this Ω we get expression (5.7).

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