# A note on the graphical representation of rotations

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**Abstract.** It is stressed that every rotation can be obtained as the composition of two rotations of angle $\pi$. This fact is used to give an alternative derivation of Hamilton's representation of rotations by great circle-oriented arcs of length $\alpha/2$ on the unit sphere. Composition of rotations corresponds to addition of consecutive arcs.

**Laburpena.** Edozein biraketa $\pi$ balioa duten beste bi biraketaren konposaketa dela azpimarratzen da. Emaitza honen ondorioz, Hamilton-ek asmatutako biraketen adierazpidea, unitate esferako zirkulu nagusi batetan $\alpha/2$ luzera duten arku norabidatuen bidez alegia, lortzen da aukerazko metodo baten bidez. Biraketen konposaketa ondoz-ondoko arkuen batuketari dagokio eredu honetan.

## 1. Introduction

The usual graphical representation of the composition of rotations, known as the Euler–Rodrigues construction (Misner *et al* 1973, Altmann 1986) is based upon the fact that every rotation of value $\alpha$ around an axis can always be decomposed into two reflections around two planes intersecting along the axis and separated by $\alpha/2$. As a consequence of this approach and based on Hamilton's work on quaternions (Hamilton 1853) one can define a new entity, the *turn*, as an ordered pair of points on the surface of the unit sphere (Biedenharn and Louck 1989). These two points define an oriented great circle arc. Addition on the unit sphere of these turns corresponds to the composition of rotations. We present here an alternative derivation of this graphical representation using the fact that every rotation of angle $\alpha$ can also be constructed from two rotations of value $\pi$ around two axes, orthogonal to the rotation axis, and separated by $\alpha/2$.

## 2. Composition of rotations

Let us consider a rotation of value $\alpha$ in the clockwise direction around an axis characterized by the unit vector $\boldsymbol{u}$. Its matrix representation is given by

$$R(\alpha, u)_{ij} = \delta_{ij}\cos\alpha + u_i u_j(1-\cos\alpha) - \varepsilon_{ijk}u_k \sin\alpha. \tag{1}$$

This is the normal parametrization of the rotation group where every rotation is represented by a vector $\boldsymbol{\alpha} = \alpha\boldsymbol{u}$ such that the corresponding orthogonal $3\times3$ matrix is obtained from $\boldsymbol{\alpha}$ and the three antisymmetric generators

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

by means of the exponential operation $R(\boldsymbol{\alpha}) \equiv R(\alpha, \boldsymbol{u}) = \exp(\boldsymbol{\alpha}\cdot\boldsymbol{J})$.

The composition of two rotations $R(\alpha, \boldsymbol{u})R(\beta, \boldsymbol{v}) = R(\gamma, \boldsymbol{w})$ is again another rotation whose parameters $(\gamma, \boldsymbol{w})$ can be expressed in terms of $(\alpha, \boldsymbol{u})$ and $(\beta, \boldsymbol{v})$ in a rather involved way (see appendix)

$$\boldsymbol{w}\tan\gamma/2 = \frac{\boldsymbol{u}\tan\alpha/2 + \boldsymbol{v}\tan\beta/2 + \tan\alpha/2\tan\beta/2(\boldsymbol{u}\times\boldsymbol{v})}{1-\tan\alpha/2\tan\beta/2(\boldsymbol{u}\cdot\boldsymbol{v})}. \tag{2}$$

If $\alpha = \beta = \pi$, expression (2) leads to

$$\boldsymbol{w}\tan\frac{\gamma}{2} = \frac{\boldsymbol{v}\times\boldsymbol{u}}{\boldsymbol{u}\cdot\boldsymbol{v}} \tag{3}$$

so that $\boldsymbol{w}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$ and $\tan(\gamma/2) = \sin\phi/\cos\phi = \tan\phi$, where $\phi$ is the angle subtended by $\boldsymbol{u}$ and $\boldsymbol{v}$ and thus $\gamma = 2\phi$.

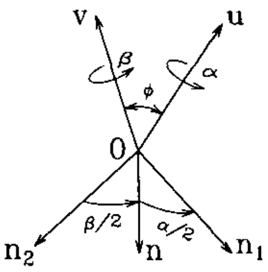



**Figure 1.** Composition of two rotations.

Conversely, we see from (3) that every rotation can always be expressed as a product of two rotations of value $\pi$, $R(\gamma, \boldsymbol{w}) = R(\pi, \boldsymbol{u})R(\pi, \boldsymbol{v})$, $\boldsymbol{u}$ and $\boldsymbol{v}$ lying arbitrarily in the plane orthogonal to $\boldsymbol{w}$ and separated by an angle $\gamma/2$ from $\boldsymbol{v}$ to $\boldsymbol{u}$.

We shall take advantage of this last conclusion to graphically obtain the composition of two arbitrary rotations.

Let us consider again the rotations $R(\alpha, \boldsymbol{u})$ and $R(\beta, \boldsymbol{v})$ which will be expressed in terms of the corresponding pair of $\pi$ rotations. We see (figure 1) that the orthogonal planes to $\boldsymbol{u}$ and $\boldsymbol{v}$ passing through the point O, intersect each other along a straight line given by the unit vector $\boldsymbol{n}$. Thus, if in the plane orthogonal to $\boldsymbol{u}$ and separated by $\alpha/2$ from $\boldsymbol{n}$ we define the unit vector $\boldsymbol{n}_1$ then according to (3), $R(\alpha, \boldsymbol{u}) = R(\pi, \boldsymbol{n}_1)R(\pi, \boldsymbol{n})$. Similarly, if in the plane orthogonal to $\boldsymbol{v}$ we represent the unit vector $\boldsymbol{n}_2$ separated by $\beta/2$ in the opposite direction from $\boldsymbol{n}$ then $R(\beta, \boldsymbol{v}) = R(\pi, \boldsymbol{n})R(\pi, \boldsymbol{n}_2)$ and thus

$$R(\alpha, \boldsymbol{u})R(\beta, \boldsymbol{v}) = R(\pi, \boldsymbol{n}_1)R(\pi, \boldsymbol{n})R(\pi, \boldsymbol{n})R(\pi, \boldsymbol{n}_2)$$

$$= R(\pi, \boldsymbol{n}_1)R(\pi, \boldsymbol{n}_2)$$

since $R(\pi, \boldsymbol{n})R(\pi, \boldsymbol{n}) = 1$, so that the composition of the two rotations $R(\alpha, \boldsymbol{u})R(\beta, \boldsymbol{v})$ appears again as the product of two rotations of value $\pi$ so that it is a rotation around an axis orthogonal to $\boldsymbol{n}_2$ and $\boldsymbol{n}_1$ and twice the angle subtended between $\boldsymbol{n}_2$ and $\boldsymbol{n}_1$.

This composition can be pictured if we represent a rotation in the following way. Let us consider the unit sphere (see figure 2). To every unit vector $\boldsymbol{u}$ we associate the corresponding great circle, obtained by the intersection with the sphere of the plane orthogonal to $\boldsymbol{u}$ passing through the centre. Now, if we draw on this circle, at any place, an oriented arc of length $\alpha/2$, this arc will represent the rotation $R(\alpha, \boldsymbol{u})$. Points A and B are the end points of the corresponding unit vectors of the couple of $\pi$ rotations that generate $R(\alpha, \boldsymbol{u})$. This oriented arc representation is the *turn* of Biedenharn and Louck.

We can pass from figure 1 to figure 3, by realising that point A represents the end point of unit vector $\boldsymbol{n}$ and B and C respectively the end points of $\boldsymbol{n}_2$ and $\boldsymbol{n}_1$, so that the great circle arc from B to C corresponds to the composite rotation.

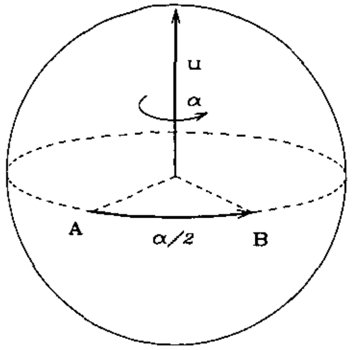



**Figure 2.** Representation of a rotation by a great circle oriented arc on the unit sphere.

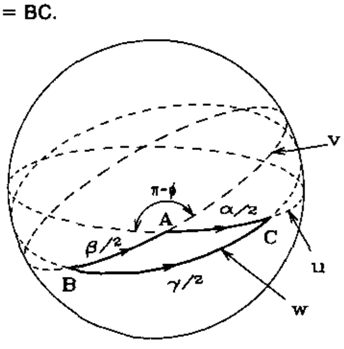
The non-commutative property of rotations is shown, and the fact that the composite rotation in the reverse order is of the same angle but about a different axis.

## 3. An example

The operations of crystal point groups (Hammermesh 1964) are orthogonal transformations. Let us consider for instance, a body that has two rotational symmetries of 90° around the O$Y$ and O$Z$ axes. Then, the composition is again a symmetry. If we display in figure 4 the corresponding arcs of length $\pi/4$ and the $\boldsymbol{n}$, $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ unit vectors, the great circle arc AB represents another symmetry. We see that $\boldsymbol{n} = (1, 0, 0)$, $\boldsymbol{n}_1 = (1/\sqrt{2}, 1/\sqrt{2}, 0)$ and $\boldsymbol{n}_2 = (1/\sqrt{2}, 0, 1/\sqrt{2})$ and thus $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 1/2 = \cos(\pi/3)$ and $\boldsymbol{n}_2 \times \boldsymbol{n}_1$ is a vector in the direction of $\boldsymbol{w} \equiv (-1, 1, 1)$, then that body also has a symmetry of 120° around the $\boldsymbol{w}$ axis. Similarly, if we take a 180° rotation around O$Z$ and the same 90° around O$Y$, then arc AC will represent a symmetry of 180° around the (1, 0, 1) axis.

More detailed symmetries concerning crystal point groups can be developed in the same way but are out of the scope of the present paper.

**Figure 3.** Composition of rotations on the unit sphere. BA + AC = BC.



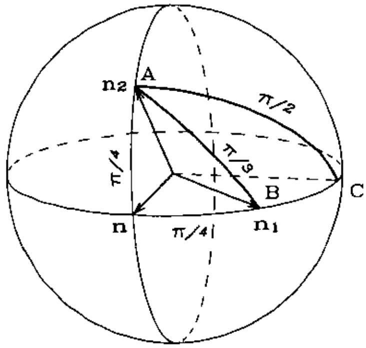



**Figure 4.** Two orthogonal 90° symmetries imply a 120° symmetry around the (1, 1, 1) axis.

## Appendix

If we define the following three vectors

$$\boldsymbol{\rho} = \tan\frac{\alpha}{2}\boldsymbol{w} \qquad \boldsymbol{\mu} = \tan\frac{\beta}{2}\boldsymbol{v} \qquad \boldsymbol{\nu} = \tan\frac{\gamma}{2}\boldsymbol{w}$$

then equation (2) reads

$$\boldsymbol{\nu} = \frac{\boldsymbol{\rho} + \boldsymbol{\mu} + \boldsymbol{\rho} \times \boldsymbol{\mu}}{1 - \boldsymbol{\rho} \cdot \boldsymbol{\mu}}. \qquad \text{(A.1)}$$

Instead of obtaining (A.1) by the 3 × 3 matrix product, we shall use the spinor representation of rotations, such that every rotation can be expressed (Misner *et al* 1973) as:

$$R(\alpha, \boldsymbol{u}) = \cos(\alpha/2) - \mathrm{i}\sin(\alpha/2)(\boldsymbol{u} \cdot \boldsymbol{\sigma}) \qquad \text{(A.2)}$$

$\sigma_i$ being the Pauli matrices, in such a way that the above definition becomes:

$$R(\boldsymbol{\rho}) \equiv R(\alpha, \boldsymbol{u}) = \frac{1}{\sqrt{1 + \rho^2}}(1 - \mathrm{i}\boldsymbol{\rho} \cdot \boldsymbol{\sigma}). \qquad \text{(A.3)}$$

It is now easier to obtain (A.1) from $R(\boldsymbol{\nu}) = R(\boldsymbol{\rho})R(\boldsymbol{\mu})$ with the matrices written in the form (A.3) and with the help of the identity

$$(\boldsymbol{a} \cdot \boldsymbol{\sigma})(\boldsymbol{b} \cdot \boldsymbol{\sigma}) = (\boldsymbol{a} \cdot \boldsymbol{b})1 + \mathrm{i}(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{\sigma} \qquad \text{(A.4)}$$

In fact

$$R(\boldsymbol{\rho})R(\boldsymbol{\mu}) = \frac{1}{\sqrt{(1 + \rho^2)(1 + \mu^2)}} \times ((1 - \boldsymbol{\rho} \cdot \boldsymbol{\mu}) - \mathrm{i}(\boldsymbol{\rho} + \boldsymbol{\mu} + \boldsymbol{\rho} \times \boldsymbol{\mu}) \cdot \boldsymbol{\sigma}) \qquad \text{(A.5)}$$

and

$$\frac{1}{\sqrt{1 + \nu^2}} = \frac{(1 - \boldsymbol{\rho} \cdot \boldsymbol{\mu})}{\sqrt{(1 + \rho^2)(1 + \mu^2)}}$$

which leads to the desired result.

## References

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Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman) ch 41