Space-time structure of classical and quantum mechanical spin

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The classical variables that define the state of a classical elementary particle are contained in the kinematical group of space-time transformations related to the Relativity Principle. Classical spin observables are expressed in terms of the velocity and orientation and their derivatives, the acceleration and angular velocity of the charge, whose motion around the center of mass produces the magnetic moment of the particle and also an oscillating electric dipole. Quantum mechanical spin operators also depend on the velocity and orientation variables and are differential operators with respect to these magnitudes. The spin structure of the photon and electron will be shown.

1 Introduction

Although it is possible to produce a quantum mechanical analysis of a system without reference to a previous classical description, it is clear that a richer classical analysis will produce a more detailed quantum mechanical description, because the quantum mechanical operators will inherit their differential structure from the classical variables we use. In general, the final theoretical description should be different if instead of producing a quantization of a point particle and afterwards we introduce its spin, for instance like in Pauli's equation, we produce first a classical description of a spinning particle and finally quantize this system. It is this second way we have taken to develop a classical description of elementary particles and their spin structure.

2 The classical and quantum mechanical interpretation of spin

We can find in the literature a classical and quantum mechanical interpretation of spin. From the classical viewpoint, according to Bargmann, Michel and Telegdi, BMT for short [1], the electron has an intrinsic spin $\mathbf{S}$ and a magnetic moment $\boldsymbol{\mu}$ where both vector magnitudes are related by

$$\boldsymbol{\mu} = g\,\frac{e}{2m}\,\mathbf{S},$$

where g is the gyromagnetic ratio. In the presence of an external electromagnetic field it satisfies

$$\frac{d\mathbf{S}}{dt} = \boldsymbol{\mu} \times \mathbf{B}. \tag{1}$$

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BMT equation is the relativistic generalization of this equation. This classical spin and its absolute value are constants of the motion for a free particle. Even with interaction S^2 is conserved.

In Dirac's theory the total angular momentum of the electron, is [2]

$$\mathbf{J} = \mathbf{r} \times \mathbf{P} + \mathbf{S}, \tag{2}$$

where

$$\mathbf{S} = \frac{\hbar}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

in terms of Pauli matrices $\boldsymbol{\sigma}$. For a free particle, $\mathbf{J}$ and $\mathbf{P}$ are constants of the motion and thus, taking the time derivative of (2) we get

$$\frac{d\mathbf{S}}{dt} = \mathbf{P} \times \mathbf{v}, \tag{3}$$

where $\mathbf{v} = c\boldsymbol{\alpha}$ is Dirac's velocity operator.

Dirac's spin operator is not a constant of the motion for a free particle. Only its absolute value remains constant.

It is clear from the dynamical viewpoint (1) and (3), that the quantum mechanical Dirac's spin operator represents a different observable than the classical BMT spin.

We shall review in what follows an alternative description of spin which is based on a formalism for describing classical elementary particles. The aim is to produce a group theoretical description of elementary particles as close as possible as the quantum mechanical one. Once the elementary particle is defined we shall see its spin structure and we will be able to identify spin observables related to both Dirac and BMT spin observables.

3 Kinematical theory of elementary particles

The kinematical theory of elementary particles developed by the author [3] defines a classical elementary particle as a Lagrangian system whose kinematical space is a homogeneous space of the kinematical group of space-time transformations related to the special Relativity Principle. The main highlights of the mentioned approach are:

– The classical variables that characterize the initial and final state of a classical elementary particle in a Lagrangian approach are precisely the parameters of the kinematical group of space-time symmetries or of any of its homogeneous spaces. Since any element of the Poincaré group can be parametrized in terms of the time and space translation and the relative velocity and orientation among inertial observers, then the most general relativistic spinning particle is described by the variables time t, position $\mathbf{r}$, velocity $\mathbf{v}$ and orientation $\boldsymbol{\alpha}$. We shall call these variables kinematical variables and the manifold they span the kinematical space of the system.

– A classical spinning particle is thus described as a point with orientation. The particle moves and rotates in space. The point $\mathbf{r}$ describes the position of the charge, which is a different point than its center of mass $\mathbf{q}$ and it describes a harmonic motion around it.

– When expressed the Lagrangian in terms of the kinematical variables it becomes a homogeneous function of first degree in terms of the derivatives of the kinematical variables and consequently it also depends on the acceleration of point $\mathbf{r}$ and on the angular velocity $\boldsymbol{\omega}$. It turns out that it can be written as

$$L = T\dot{t} + \mathbf{R}\cdot\dot{\mathbf{r}} + \mathbf{V}\cdot\dot{\mathbf{v}} + \mathbf{W}\cdot\boldsymbol{\omega}, \tag{4}$$

where $T = \partial L/\partial\dot{t}$, $\mathbf{R} = \partial L/\partial\dot{\mathbf{r}}$, $\mathbf{V} = \partial L/\partial\dot{\mathbf{v}}$ and $\mathbf{W} = \partial L/\partial\boldsymbol{\omega}$.

– For a free relativistic particle, when analyzing the invariance under the different one-parameter subgroups of the Poincaré group, Noether's theorem determines the usual constants of the motion which take the following form in terms of the above magnitudes: Energy,

$$H = -T - \mathbf{v}\cdot\frac{d\mathbf{V}}{dt},$$

linear momentum,

$$\mathbf{P} = \mathbf{R} - \frac{d\mathbf{V}}{dt}, \tag{5}$$

kinematical momentum

$$\mathbf{K} = \frac{H}{c^2}\mathbf{r} - \mathbf{P}t - \frac{1}{c^2}\mathbf{S}\times\mathbf{v}, \tag{6}$$

and angular momentum

$$\mathbf{J} = \mathbf{r}\times\mathbf{P} + \mathbf{S}. \tag{7}$$

where the observable $\mathbf{S}$, takes the form

$$\mathbf{S} = \mathbf{v}\times\mathbf{V} + \mathbf{W}. \tag{8}$$

– The linear momentum (5) is not lying along the velocity $\mathbf{v}$ of point $\mathbf{r}$. Point $\mathbf{r}$ does not represent the center of mass position. If in terms of the last term in (6) we define the position vector

$$\mathbf{k} = \frac{1}{H}\mathbf{S}\times\mathbf{v},$$

then the center of mass position can be defined as $\mathbf{q} = \mathbf{r} - \mathbf{k}$, such that the kinematical momentum can be written as

$$\mathbf{K} = \frac{H}{c^2}\mathbf{q} - \mathbf{P}t.$$

Observable $\mathbf{k}$ is the relative position of point $\mathbf{r}$ with respect to the center of mass. Taking the time derivative of this expression leads for the linear momentum to

$$\mathbf{P} = \frac{H}{c^2}\frac{d\mathbf{q}}{dt},$$

which is the usual relativistic expression of the linear momentum in terms of the center of mass velocity.

- The observable $\mathbf{S}$ is the classical equivalent of Dirac's spin operator, because it satisfies the free dynamical equation

$$\frac{d\mathbf{S}}{dt} = \mathbf{P}\times\mathbf{v}.$$

- The structure of the spin observable (8) is twofold. One, $\mathbf{v}\times\mathbf{V}$, is related to the dependence of the Lagrangian on the acceleration which produces a separation between the center of mass and center of charge. The motion of the charge around the center of mass is known as the zitterbewegung. The other part $\mathbf{W}$ is related to the rotation of the particle.
- The classical equivalent of BMT spin observable is the angular momentum of the system in the center of mass frame, or

$$\mathbf{S}_{BMT} = \mathbf{J} - \mathbf{q}\times\mathbf{P} = \mathbf{k}\times\mathbf{P} + \mathbf{S}.$$

If $\mathbf{P} = 0$, both spin observables coincide. Therefore the difference between BMT and Dirac spin is the orbital angular momentum of the motion of point $\mathbf{r}$ with respect to the center of mass.

- The magnetic moment of the particle is produced by the charge motion and is thus related to the zitterbewegung part of the spin. It is because the spin has another contribution coming from the rotation that a pure kinematical interpretation of the gyromagnetic ratio has been given [4].

4 The classical structure of the electron

- The classical system that when quantized satisfies Dirac's equation[5] corresponds to a particle whose charge is moving at the speed of light, and therefore if $v = c$ is constant its acceleration is always orthogonal to the velocity.
- if we take in (6) the time derivative of this expression and afterwards its scalar product with vector $\mathbf{v}$, since $v = c$ we get the relationship

$$H - \mathbf{P}\cdot\mathbf{v} - \frac{1}{c^2}\left(\mathbf{S}\times\frac{d\mathbf{v}}{dt}\right)\cdot\mathbf{v} = 0$$

which is the classical equivalent of Dirac's equation. This expression is relativistically invariant and therefore to obtain a linear relation between H and $\mathbf{P}$ it is not necessary to perform any square root of the Klein–Gordon operator.

– The center of mass observer is defined by the conditions $\mathbf{P} = 0$ and $\mathbf{K} = 0$. For this observer $H = mc^2$ and from (7) the total angular momentum reduces to the spin $\mathbf{S}$ which is a constant of the motion. From (6) we get

$$m\mathbf{r} = \frac{1}{c^2}\mathbf{S} \times \mathbf{v},$$

so that the charge of the particle is describing circles at the speed of light in a plane orthogonal to $\mathbf{S}$. The radius of this motion is

$$R_o = \frac{S}{mc}.$$

The frequency of this motion is

$$\omega = \frac{mc^2}{S} = \frac{2mc^2}{\hbar} = 1.55 \times 10^{21}\,\mathrm{s}^{-1}.$$

– It turns out that although the particle is pointlike, because of the zitterbewegung it has a localized region of influence of size $2R_o$. For the electron $S = \hbar/2$ and therefore $2R_o \simeq 7.6 \times 10^{-11}$cm is Compton's wave length. The latest LEP experiments establish an upper bound of 10^{-17}cm for the radius of the charge of the electron, which is consistent with this pointlike interpretation, while its quantum mechanical behaviour is produced for distances of its Compton wave length, six orders of magnitude larger.

– The magnetic moment of the electron is not an intrinsic property like the charge. It is produced by the motion of the charge and therefore is orthogonal to the zitterbewegung plane. But at the same time, the electron has an oscillating electric dipole of magnitude eR_o lying on the zitterbewegung plane. Its time average value vanishes and in low energy interactions we can neglect the effect of this electric dipole but in high energy physics we have to take into account the detailed position of the interacting charges and thus the electric dipole contribution is not negligible. This electric dipole is not related to a loss of spherical symmetry of some charge distribution. The charge distribution is spherically symmetric because it is just a point. It is the instantaneous electric dipole with respect to the center of mass.

As an approximation we can consider the electron as a point, its center of mass, where we also locate the charge, but at the same time we have to assign to this point two electromagnetic vector quantities, a magnetic moment lying along the spin direction and an oscillating electric dipole on a plane orthogonal to the spin.

5 The classical structure of the photon

– Since the photon is moving at a constant speed, the $\mathbf{V} \cdot \dot{\mathbf{v}}$ term does not appear in the general Lagrangian (4). Therefore $H = -T$, $\mathbf{P} = \mathbf{R}$ and only $\mathbf{S} = \mathbf{W}$. The spin is produced by the rotation of the body frame that lies along the velocity $\mathbf{v}$.

– A general Lagrangian for describing the photon is given by

$$L = \epsilon \frac{S}{c} \frac{\dot{\mathbf{r}} \cdot \boldsymbol{\omega}}{\dot{t}},$$

where $S = \hbar$ is the magnitude of spin and $\epsilon = \pm 1$ is its helicity.

– It turns out that $\mathbf{S} = \mathbf{W} = \epsilon \hbar \mathbf{v}/c$, and therefore the spin is not tranversal. The energy $H = -T = \mathbf{S} \cdot \boldsymbol{\omega}/\dot{t}$, and being definite positive implies that $\mathbf{S}$ and $\boldsymbol{\omega}$ have the same direction and thus $\hbar\omega = h\nu$. The frequency of a photon is the frequency of its rotational motion along the direction of motion.

– The linear momentum $\mathbf{P} = \mathbf{R} = \epsilon \hbar \boldsymbol{\omega}/\dot{t} = \hbar \mathbf{k}$, where $\mathbf{k}$ is the wave number.

The classical photon is thus a point moving in a straight line at a constant velocity c which is rotating with some angular velocity along the direction of motion leftwards ($\epsilon = -1$) or rightwards ($\epsilon = +1$), with the spin and angular velocity pointing in the same direction although they are not directly related because S is an invariant property while ω is not.

6 Quantization

Feynman's quantization of the above Lagrangian formalism leads to the following conclusions

– The wave function is a complex squared integrable function defined on the kinematical variables, $\psi(t, \mathbf{r}, \mathbf{v}, \boldsymbol{\alpha})$.

– The angular momentum of the system is

$$\mathbf{J} = \mathbf{r} \times \frac{\hbar}{i} \nabla + \mathbf{S}, \tag{9}$$

where quantum mechanical spin operator $\mathbf{S}$, equivalent to Dirac's spin operator, takes the form

$$\mathbf{S} = \mathbf{v} \times \frac{\hbar}{i} \nabla_v + \mathbf{D}_\alpha = \mathbf{S}_v + \mathbf{S}_\alpha. \tag{10}$$

∇_v is the gradient operator with respect to the velocity variables and $\mathbf{D}_\alpha$ is a linear differential operator of first order with respect to the orientation variables. Its explicit form depends on the particular parametrization of the rotation group we use to characterize the orientation of the system. For instance, if we parameterize every rotation of angle θ by the three-vector $\boldsymbol{\alpha} = \mathbf{n} \tan \theta/2$, where $\mathbf{n}$ represents a unit vector along the rotation axis, $\mathbf{S}_\alpha$ is written as

$$\mathbf{S}_\alpha = \frac{\hbar}{2i} \left[\nabla_\alpha + \boldsymbol{\alpha} \times \nabla_\alpha + \boldsymbol{\alpha}(\boldsymbol{\alpha} \cdot \nabla_\alpha) \right].$$

∇_α is the gradient operator with respect to $\boldsymbol{\alpha}$ variables. The first part of (10), $\mathbf{S}_v$, has integer eigenvalues because it has the form of an orbital angular momentum in terms of the $\mathbf{v}$ variables. Half-integer eigenvalues come only from operator $\mathbf{S}_\alpha$.

The first term that depends on the velocity variables is related to the zitterbewegung while the second, $\mathbf{S}_\alpha$, takes into account the change of orientation, i.e., the rotation of the particle.

– The quantum mechanical structure of spin operator (10) only involves the variables $\mathbf{v}$ and $\boldsymbol{\alpha}$ and linear differential operators on these variables. We see in fact that these variables are the additional variables we use to distinguish this particle from the point particle case.

7 Electron motion in Dirac's theory

It is convenient to remember some of the features that Dirac obtained for the motion of a free electron [6]. Let point $\mathbf{r}$ be the position vector on which Dirac's spinor $\psi(t, \mathbf{r})$ is defined. When computing the velocity of point $\mathbf{r}$, Dirac arrives to:

1. The velocity $\mathbf{v} = i/\hbar[H, \mathbf{r}] = c\boldsymbol{\alpha}$, in terms of $\boldsymbol{\alpha}$ matrices and says, '*. . . a measurement of a component of the velocity of a free electron is certain to lead to the result* $\pm c$'.

2. The linear momentum does not have the direction of this velocity $\mathbf{v}$, but must be related to some average value of it: . . . '*the* x_1 *component of the velocity,* $c\alpha_1$*, consists of two parts, a constant part* $c^2 p_1 H^{-1}$*, connected with the momentum by the classical relativistic formula, and an oscillatory part, whose frequency is at least* $2mc^2/h$*, . . .*'.

3. About the position $\mathbf{r}$: '*The oscillatory part of* x_1 *is small, . . ., which is of the order of magnitude* $\hbar/mc$*, . . .*'.

   And when analyzing in his original 1928 paper [2], the interaction of the electron with an external electromagnetic field, after performing the square of Dirac's operator he obtains two new interaction terms:

$$\frac{e\hbar}{2m}\boldsymbol{\Sigma}\cdot\mathbf{B} + \frac{ie\hbar}{2mc}\boldsymbol{\alpha}\cdot\mathbf{E},$$

   where the total electron spin is written $\mathbf{S} = \hbar\boldsymbol{\Sigma}/2$ and $\mathbf{E}$ and $\mathbf{B}$ are the external electric and magnetic fields, respectively. He says:

4. '*The electron will therefore behave as though it has a magnetic moment* $\frac{e\hbar}{2m}\boldsymbol{\Sigma}$ *and an electric moment* $\frac{ie\hbar}{2mc}\boldsymbol{\alpha}$*. The magnetic moment is just that assumed in the spinning electron model*' (Pauli model). '*The electric moment, being a pure imaginary, we should not expect to appear in the model.*' It is surprising this Dirac attitude concerning the electric dipole when compared with his attitude

about the negative energy states of the electron. Both electric and magnetic moments are obtained on an equal footing, simmilarly as the positive and negative energy states. In this latest case the acceptance of negative energy states lead to the interpretation of these states as the states of the antiparticle. However considering that the electron has an electric dipole it seems to be a loss of spherical symmetry in its charge distribution.

It is this electric dipole which in addition to the magnetic moment we claim is justified by the zitterbewegung and has a clear classical interpretation.

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