Magnetic braking revisited

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The braking force acting on a conducting disk rotating under the influence of an external magnetic field of axial symmetry is calculated in a quasi-static approximation and the role played by the charge distributions induced in the disk is shown. The two cases of infinite and finite radius are considered to analyze the influence of edge effects and we obtain a general expression for the braking torque when the magnetic field has axial symmetry. The particular case of a uniform external magnetic field is used to show the working of a simplified model of a cylindrical battery. Analytical results are compared with those obtained by other authors.

I. INTRODUCTION

It is well known that, when a conductor is moving in a stationary magnetic field, electric currents are induced within it and dissipate energy by the Joule effect. This loss of energy manifests itself by the existence of a force on the body that produces a braking of the motion. This force is produced by the action of the external magnetic field on the induced currents. Though this effect is the basis of some technological applications,[1] most textbooks on electromagnetism[2] either do not mention this subject or discuss it only at a qualitative level or as suggested problems. One exception is the book by Smythe,[3] in which a quantitative analysis can be found.

Wiederick *et al.*[1] analyzed in detail the motion of a thin aluminum disk of very large radius rotating in an almost uniform magnetic field of rectangular cross section. Later, Heald[4] improved these calculations by suppressing the earlier hypothesis that the induced current is uniform in the rectangular ''footprint'' of the magnetic field.

Marcuso *et al.*[5,6] made use of a method of successive approximations to solve Maxwell's equations in order to compute the braking force on a rotating disk under the action of a static external nonuniform magnetic field. The braking torque was compared with the experimental results for disks of aluminum and copper, obtaining very good agreement except near the disk border. Cadwell[7] has recently analyzed the effect of magnetic damping on an aluminum plate moving on a horizontal air track as it passes between the poles of a horseshoe magnet.

Related works are those of Saslow[8] and MacLatchy *et al.*,[9] who calculated, among other things, the braking force acting on a magnetic dipole falling inside a cylindrical conductor.

The aim of this paper is to study the braking effect on a thin conducting disk rotating in an external, static, nonuniform magnetic field. This is done in the quasi-static approach and in the reference frame where the electromagnet that generates the magnetic field is at rest. We want to simplify and generalize previous calculations and to emphasize the role played by the charge densities that arise within the material. The induced currents may be explained in terms of a circuit with an electromotive force (emf) source connected to a resistor. In this case, the nature of the nonelectromagnetic force and the crucial role of the induced charge distributions are easily understood.

In Sec. II we study the case of the disk of infinite radius. Border effects are computed in Sec. III and we compute a general expression for the torque that reduces to Smythe's expressions[3] in the particular case in which the magnetic field is uniform. In Sec. IV a model for a cylindrical battery is proposed and we make some general comments in Sec. V.

II. INFINITE DISK

Let us consider a disk of conductivity σ and large radius R_d which is rotating around a perpendicular axis passing through the point O'. There is an external perpendicular magnetic field of axial symmetry applied around the point O, which is located at a distance R from the rotation axis (see Fig. 1). Inside the disk, the applied magnetic field vanishes outside a circular region of radius a, not necessarily small.

We will choose a coordinate system around O in such a way that the XOY plane coincides with the middle plane of the disk. Let $\mathbf{B}(r,z)$ be an external applied magnetic field of cylindrical symmetry. It can be represented as

$$\mathbf{B}(r,z)=B_r(r,z)\hat{\mathbf{r}}+B_z(r,z)\hat{\mathbf{z}}, \tag{1}$$

where $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ are unit vectors in the radial direction and along the OZ axis. In the cylindrical coordinates (r,ϕ) around O, the velocity of a point on the disk appears as

$$\mathbf{v}=\omega[R\sin\phi\hat{\mathbf{r}}+(R\cos\phi+r)\hat{\boldsymbol{\phi}}], \tag{2}$$

where $\hat{\boldsymbol{\phi}}$ is azimuthal unit vector.

In the laboratory frame, where the magnetic field is at rest, the magnitude of the angular velocity, ω, is considered constant, in such a way that we can assume that the problem is quasi-static. In particular, the current density satisfies the following conservation law:

$$\mathrm{div}\ \mathbf{j}=0. \tag{3}$$

As we shall discuss in detail in Sec. V [see Eq. (59)], we are also assuming that ω is small enough to warrant the validity of our first-order approximation.

Since the material satisfies Ohm's law and in the quasi-static approach the convective terms can be neglected, we have

$$\mathbf{j}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{4}$$

and the continuity equation (3) implies the presence inside the disk of a charge density given by

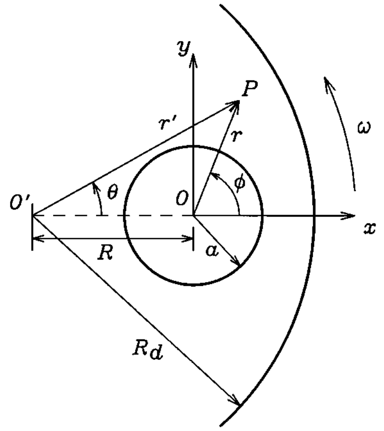

Fig. 1. Rotating disk and the coordinates used in different calculations. The influence area of the external magnetic field is the circle $r \leqslant a$.

$$\rho = \epsilon_0 \text{ div } \mathbf{E} = -\epsilon_0 \text{ div}(\mathbf{v} \times \mathbf{B}) = -\epsilon_0 \mathbf{B} \cdot (\text{curl } \mathbf{v})$$
$$= -2\epsilon_0 \omega B_z(r,z), \tag{5}$$

where use has been made of the fact that inside the disk $\mathbf{B}$ is sourceless: curl $\mathbf{B}=0$. We are also assuming that the electric permittivity and magnetic permeability of the material are those of vacuum, ϵ_0 and μ_0.

To analyze the braking effect, we have to compute the electric field $\mathbf{E}$ inside the disk. In the quasi-static approximation the electric field is conservative, $\mathbf{E} = -\text{grad } V$, and the potential satisfies Poisson's equation, that in cylindrical coordinates is written as follows:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho(r,\phi,z)}{\epsilon_0}. \tag{6}$$

Since we are assuming that the disk is very thin, $j_z = \sigma(E_z - v_\phi B_r) = 0$ and, by using again curl $\mathbf{B}=0$, we get

$$\frac{\partial^2 V}{\partial z^2} = -\frac{\partial E_z}{\partial z} = -v_\phi \frac{\partial B_r}{\partial z} = -v_\phi \frac{\partial B_z}{\partial r}, \tag{7}$$

and, rearranging terms in the expression for $\nabla^2 V$,

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 2\omega B_z + v_\phi \frac{\partial B_z}{\partial r}. \tag{8}$$

From now on, we shall use the hypothesis of a very small disk thickness h to keep only the lowest order in an expansion in powers of h. In consequence, we will have

$$V(r,\phi,z) \approx V(r,\phi) \equiv V(r,\phi,0), \tag{9}$$

$$B_z(r,z) \approx B(r) \equiv B_z(r,0), \tag{10}$$

$$B_r(r,z) \approx B_r(r,0) = 0. \tag{11}$$

In the last expression we have assumed[5] that the plane $z=0$ has been chosen to have $B_r(r,0)=0$. In this approximation the dependence on the z variable may be ignored[4] and we get a two-dimensional equivalent problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V(r,\phi)}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 V(r,\phi)}{\partial \phi^2}$$
$$= \omega R \frac{dB(r)}{dr}\cos\phi + \omega \frac{1}{r}\frac{d}{dr}[r^2 B(r)]. \tag{12}$$

The structure of the right-hand side (rhs) in (12) suggests using the ansatz $V(r,\phi) = \alpha(r)\cos\phi + \beta(r)$, which effectively separates this equation into two linear ordinary differential equations:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\alpha(r)}{dr}\right] - \frac{1}{r^2}\alpha(r) = \omega R \frac{dB(r)}{dr}, \tag{13}$$

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\beta(r)}{dr}\right] = \omega \frac{1}{r}\frac{d}{dr}[r^2 B(r)]. \tag{14}$$

Since both equations are of the Cauchy–Euler type, they can be solved:

$$\alpha(r) = C_1 r + \frac{C_2}{r} + \frac{\omega R}{r}\int_0^r uB(u)du, \tag{15}$$

$$\beta(r) = C_3 + C_4 \ln r + \omega \int_0^r uB(u)du. \tag{16}$$

The integration constants C_1, C_2, C_3, and C_4 are readily computed by using the conditions that the potential is regular at $r=0$ and vanishes for $r \to \infty$. The final result for the potential is

$$V(r,\phi) = \omega \frac{R\cos\phi + r}{r}\int_0^r uB(u)du - \omega\int_0^a uB(u)du. \tag{17}$$

The uniqueness theorem guarantees that this is the only regular solution of the two-dimensional Poisson's equation (12) that goes to zero as r approaches infinity.

To write the solution in a more convenient way, we will use the vector potential in the Coulomb gauge. By using $\mathbf{B} = \text{curl } \mathbf{A}$ and the field symmetry given by (10) and (11), it is easy to see that $\mathbf{A} = A(r)\hat{\boldsymbol{\phi}}$ with

$$A(r) = \frac{1}{r}\int_0^r uB(u)du. \tag{18}$$

In consequence, the potential (17) takes the form

$$V(r,\phi) = \omega(R\cos\phi + r)A(r) - \omega a A(a). \tag{19}$$

Notice that for $r > a$ the magnetic field vanishes, the vector potential is

$$A(r) = \frac{aA(a)}{r}, \tag{20}$$

and the potential (19) can also be written in the following form, which shows explicitly its dipolar nature for $r > a$:

$$V(r,\phi) = \omega R a A(a)\frac{\cos\phi}{r}. \tag{21}$$

Since the total magnetic flux through the disk surface is $2\pi a A(a)$, the quantity $aA(a)$ and the expressions (19) and (21) are independent of a, as far as it is large enough to have $B(r)=0$ for $r > a$.

The components of the current density (4) are easily computed by using Eqs. (2) and (19):

$$j_r = \sigma\omega R\frac{A(r)}{r}\cos\phi, \quad j_\phi = -\sigma\omega R\frac{dA(r)}{dr}\sin\phi. \tag{22}$$

Since Eq. (18) may be written as

$$B(r)=\frac{dA(r)}{dr}+\frac{A(r)}{r}, \tag{23}$$

we see from (20) and (22) that for $r>a$, where there is no magnetic field, the current density decreases as $1/r^2$.

The torque exerted by the external magnetic field on this current density,

$$\mathbf{M}=\int \mathbf{O'P}\times(\mathbf{j}\times\mathbf{B})d^3x, \tag{24}$$

is perpendicular to the disk, $\mathbf{M}=M\hat{\mathbf{z}}$, and this component is given by

$$M=\int [R(j_\phi \sin\phi - j_r \cos\phi)-rj_r]B(r)d^3x, \tag{25}$$

where $d^3x=r\ dr\ d\phi\ dz$ is the volume element of the disk.

After performing the integrals in z and ϕ and making use of (23), we get

$$M=-\pi\sigma h\omega R^2\int_0^a rB^2(r)dr, \tag{26}$$

In a similar way, it is easy to see that the force exerted by the external magnetic field on this current density,

$$\mathbf{F}=\int \mathbf{j}\times\mathbf{B}\ d^3x, \tag{27}$$

is perpendicular to the radius R, i.e., $\mathbf{F}=F\hat{\mathbf{y}}$, with

$$F=-\pi\sigma h\omega R\int_0^a rB^2(r)dr. \tag{28}$$

We see that, as a consequence of the symmetry of the magnetic field, we have simply $\mathbf{M}=\mathbf{R}\times\mathbf{F}$, as if the total force was applied at point O, instead of throughout the cylindrical region $r<a$.

III. BORDER EFFECTS

Until now to neglect the edge effects, we have considered a disk of infinite radius. In what follows, we shall assume that the radius is finite and satisfies the condition $R_d\geqslant R+a$, which guarantees that the applied magnetic field vanishes along the disk edge where the potential will satisfy Laplace's equation. Under these assumptions, the charge density is still given by Eq. (5) and, as a consequence of the superposition principle, the potential will be the sum of the solution (19) of Poisson's equation and a solution of Laplace's equation. The physical condition that will fix this unique solution to the problem, is that the component of the current density in the radial direction of the disk (or, equivalently, the radial derivative of the potential) vanishes at the edge.

To take advantage of the symmetry of the new problem we will use cylindrical coordinates (r',θ) around the disk center O' (see Fig. 1). They are related to the coordinates (r,ϕ) by the equations

$$r\cos\phi=r'\cos\theta-R,\quad r\sin\phi=r'\sin\theta, \tag{29}$$

and the potential (21) appears for $r>a$, and in particular near the border, as follows:

$$V(r',\theta)=\omega RaA(a)\ \frac{r'\cos\theta-R}{r'^2+R^2-2Rr'\cos\theta}. \tag{30}$$

This potential satisfies Laplace's equation. To make its radial derivative vanish when $r'=R_d$, for all values of θ, it seems natural to add a term with the same (r',θ) structure but with different constants:

$$\tilde{V}(r',\theta)=B\ \frac{r'\cos\theta-C}{r'^2+C^2-2Cr'\cos\theta}. \tag{31}$$

This term will automatically satisfy Laplace's equation in the region $r>a$, and the condition $\partial(V+\tilde{V})/\partial r'=0$, for $r'=R_d$ is fulfilled if and only if we take $B=-\omega CaA(a)$ and $C=R_d^2/R$. An analogous ansatz was made by Smythe[3] to compute the magnetic scalar potential.

The result we have just found and the uniqueness theorem show that to take into account the edge effects the potential (19) must be corrected by an additional term that after using transformation (29) appears in the form:

$$\tilde{V}(r,\phi)=-\omega R_d^2 aA(a)\times\frac{Rr\cos\phi-(R_d^2-R^2)}{R^2r^2+(R_d^2-R^2)^2-2R(R_d^2-R^2)r\cos\phi}. \tag{32}$$

For the same reason, one has to correct the braking torque (26) with the following additional term:

$$\tilde{M}=-\sigma\int\left[R\left(\frac{\sin\phi}{r}\frac{\partial\tilde{V}}{\partial\phi}-\cos\phi\ \frac{\partial\tilde{V}}{\partial r}\right)-r\ \frac{\partial\tilde{V}}{\partial r}\right]B(r)d^3x. \tag{33}$$

If one substitutes expression (32), the integrand is a rational expression in $\cos\phi$ and $\sin\phi$. After performing the integrals in z and ϕ one gets

$$\tilde{M}=2\pi\sigma h\omega aA(a)\ \frac{R^2R_d^2}{(R_d^2-R^2)^2}\int_0^a rB(r)dr, \tag{34}$$

and by using Eqs. (18), (20), and (26) we finally get the general expression for the total braking torque if the external magnetic field has axial symmetry:

$$M+\tilde{M}=-\pi\sigma h\omega R^2\int_0^a rB^2(r)dr\left[1-\Lambda\ \frac{a^2R_d^2}{(R_d^2-R^2)^2}\right], \tag{35}$$

where Λ is a dimensionless coefficient that measures the profile of the external magnetic field:

$$\Lambda=\frac{2[\int_0^a rB(r)dr]^2}{a^2\int_0^a rB^2(r)dr}. \tag{36}$$

This coefficient is a functional of $B(r)$ and takes values in the range between 0 and 1. For instance, its maximum value is 1 and is obtained for a uniform magnetic field profile, while it equals 2/3 for a linearly decreasing field and its value is 3/4 for a magnetic field of parabolic shape. Its value is 0 if the total flux of the magnetic field through the region $r\leqslant a$ is zero. In consequence, for any magnetic field with axial symmetry, the torque increases as R^2 for small values of R but near the border this parabolic increasing trend is modified by the last term in (35). This behavior is shown in Fig. 2, for $a/R_d=0.05$ and $\Lambda=1$.

In this case of a finite disk, the total braking force is also perpendicular to the radius R and given by $F+\tilde{F}=(M+\tilde{M})/R$, as can be seen after a straightforward calculation.

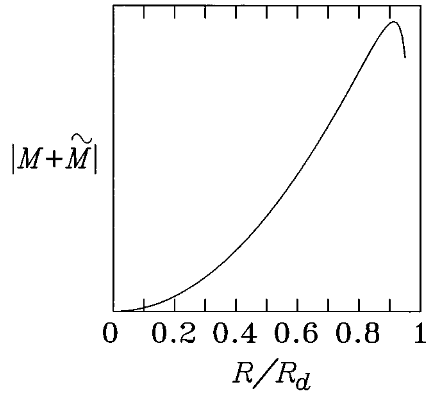

Fig. 2. Total braking torque for $a=0.05R_d$, $\Lambda=1$ and different values of R. The vertical scale is arbitrary, because there is a proportionality coefficient depending on the disc and magnetic field properties.

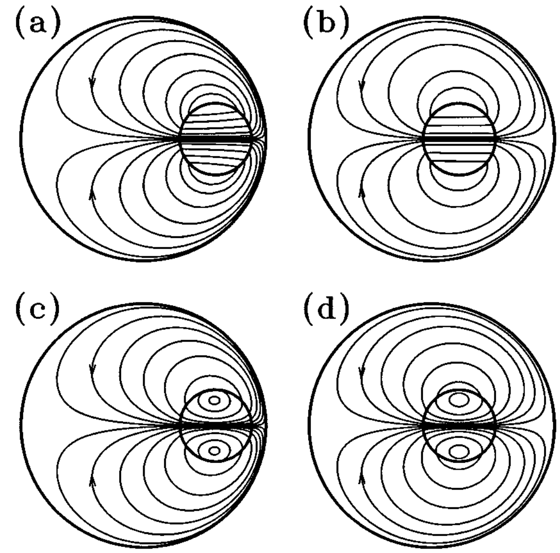

Fig. 3. Some current lines inside the disk for (a) $B=B_0$, $a=0.3R_d$, and $R=0.6R_d$; (b) $B=B_0$, $a=0.3R_d$, and $R=0.23R_d$; (c) $B=6B_0(1-r/a)$, $a=0.3R_d$, and $R=0.6R_d$; (d) $B=6B_0(1-r/a)$, $a=0.3R_d$, and $R=0.23R_d$.

In the experimental setup of Marcuso *et al.*,[5,6] $a/R_d=0.3$ or larger, which prevents us from using our calculations to explain quantitatively the decreasing trend of the torque near the border, because then the footprint will encompass the border and the cylindrical symmetry inside the disk will be broken. However, the tail of the displayed curve may explain qualitatively the decreasing of the torque near the border that they observed experimentally but could not explain, even qualitatively, with the analytical expression they obtained.

To test that behavior more accurately, magnets of very small radius a must be used and try to close the magnetic field lines outside the disk in order to make the magnetic flux through the region $r\leqslant a$, and thus the parameter Λ, as large as possible.

By using expressions (5), (19), and (32) we may compute the current density $\mathbf{j}$ inside the disk. In Fig. 3 we have plotted some current lines in the case $a/R_d=0.3$ and for $R/R_d=0.6$, and 0.23, using a computer program.[10] In (a) and (b) we consider a uniform magnetic field ($B=B_0$ for $r<a$) and we see that when the border effects are negligible the current is uniform inside the magnetic field footprint $r<a$. If the magnetic field profile is triangular, $B=6B_0(1-r/a)$, we obtain the plots (c) and (d). Now, inside the circle $r=a$ the current is not even approximately uniform, but outside it has exactly the same value as that in the case of the uniform magnetic field, because for $r>a$ the potential and the current depend only on the total magnetic flux and we have chosen the constants in order to have the same flux in both cases. Notice that field lines in Fig. 3(a) and (b) have a discontinuous normal component at $r=a$ due to the existence of a surface charge density distribution induced by the discontinuity of the magnetic field.

IV. A CYLINDRICAL BATTERY

Saslow[11] analyzed a spherical battery that produced a uniform nonelectromagnetic force per unit charge $\mathbf{F}$, such that in the interior of the battery the current was given by $g(\mathbf{E}+\mathbf{F})$. Since there are induced currents, we may consider the disk analyzed in previous sections as a circuit in which the emf is supplied by a cylindrical battery of height h and radius a located around O. In our case, we may explicitly identify the nonelectromagnetic force $\mathbf{F}$, because its role is played by the vector $\mathbf{v}\times\mathbf{B}$, which happens to be nonuniform in this case.

To simplify the following discussion, let us consider the particular case in which the disk radius is infinite and the external magnetic field is constant and uniform in the region $r<a$ and vanishes outside it:

$$\mathbf{B}=\begin{cases} B_0\hat{\mathbf{z}}, & \text{for } r<a \\ 0, & \text{for } r>a. \end{cases} \tag{37}$$

In this case the potential (19) is

$$V(r,\phi)=\begin{cases} \frac{1}{2}\omega B_0(Rr\cos\phi+r^2-a^2), & \text{for } r\leqslant a, \\ \frac{1}{2}\omega B_0 Ra^2(\cos\phi/r), & \text{for } r\geqslant a. \end{cases} \tag{38}$$

The electric field and current density are, respectively,

$$E_r(r,\phi)=\begin{cases} -\frac{1}{2}\omega B_0(R\cos\phi+2r), & \text{for } r<a, \\ \frac{1}{2}\omega B_0 Ra^2(\cos\phi/r^2), & \text{for } r>a; \end{cases} \tag{39}$$

$$E_\phi(r,\phi)=\begin{cases} \frac{1}{2}\omega B_0 R\sin\phi, & \text{for } r\leqslant a, \\ \frac{1}{2}\omega B_0 R(a^2/r^2)\sin\phi, & \text{for } r\geqslant a; \end{cases} \tag{40}$$

$$j_r(r,\phi)=\begin{cases} \frac{1}{2}\sigma\omega B_0 R\cos\phi, & \text{for } r\leqslant a, \\ \frac{1}{2}\sigma\omega B_0 Ra^2(\cos\phi/r^2), & \text{for } r\geqslant a; \end{cases} \tag{41}$$

$$j_\phi(r,\phi)=\begin{cases} -\frac{1}{2}\sigma\omega B_0 R\sin\phi, & \text{for } r\leqslant a, \\ \frac{1}{2}\sigma\omega B_0 R(a^2/r^2)\sin\phi, & \text{for } r\geqslant a. \end{cases} \tag{42}$$

Since the magnetic and electric fields are discontinuous at $r=a$, in addition to the volume charge density (5) which reduces in this case to the constant value $\rho(r,\phi)=-2\epsilon_0\omega B_0$, there exists a surface charge density in the cylinder lateral surface $r=a$, given by

$$\sigma_c(a,\phi)=\epsilon_0 \lim_{\eta\to 0}[E_r(a+\eta,\phi)-E_r(a-\eta,\phi)]$$
$$=\epsilon_0\omega B_0(R\cos\phi+a). \quad (43)$$

The braking force and torque acting on the disk are according to (26) and (28):

$$F=-\tfrac{1}{2}\pi\sigma h\omega Ra^2B_0^2,\quad M=RF. \quad (44)$$

The power of the electric force inside the battery $(r<a)$ and outside it $(r>a)$ is

$$P_{\substack{<\\>}}=\int_{\substack{r<a\\r>a}}\mathbf{E}\cdot\mathbf{j}d^3x=\mp\frac{1}{4}\pi\sigma h\omega^2R^2a^2B_0^2. \quad (45)$$

Of course, the total power of the conservative electric field is zero. On the other hand, the power supplied by the battery is

$$P_m=\int_{r<a}(\mathbf{v}\times\mathbf{B})\cdot\mathbf{j}d^3x=\frac{1}{2}\pi\sigma h\omega^2R^2a^2B_0^2, \quad (46)$$

which is the loss of kinetic energy per unit time of the disk or, alternatively, the mechanical power $-M\omega$ produced by some external agent to maintain the disk rotating with constant angular velocity.

The total current supplied by the battery is

$$I=\int_S\mathbf{j}\cdot ds=\sigma h\omega RaB_0, \quad (47)$$

where the integration surface S is, for instance, the cylinder cross section $(r<a,\ \phi=\pm\pi/2)$, or the upper half of its lateral surface $(r=a,\ -\pi/2\leqslant\phi\leqslant\pi/2)$, the outer part of the battery $(r>a,\ \phi=\pm\pi/2)$, or any other convenient surface. The result is, of course, independent of S because of the conservation law (3).

Since the current is spread through the volume of the disk, we cannot use the usual definition of resistance as an integral along the conducting wire. Furthermore, we cannot use the expression $\mathscr{R}_e=V/I$ to define the external resistance of the circuit, because our battery does not have two clear poles to define V. Nevertheless, it seems reasonable to use the expression $P=I^2\mathscr{R}$ to define the external resistance of the ''circuit'' as follows:

$$\mathscr{R}_>=P_>/I^2=\pi/4\sigma h. \quad (48)$$

In a similar way, we may define the internal resistance of the battery by the relation

$$\mathscr{R}_<=(P_<+P_m)/I^2=\pi/4\sigma h, \quad (49)$$

which has the same value as the external resistance.

The energy balance may also be performed in terms of the Poynting vector. Because the current density (41) and (42) has no j_z component and j_r and j_ϕ are functions independent of z, the corresponding vector potential has the form $\mathbf{A}'=A_r'(r,\phi)\hat{\mathbf{r}}+A_\phi'(r,\phi)\hat{\boldsymbol{\phi}}$, so that these induced currents generate inside the disk a magnetic field $\mathbf{B}'=\nabla\times\mathbf{A}'=B'(r,\phi)\hat{\mathbf{z}}$, with a single nonvanishing component along OZ given by

$$B'(r,\phi)=\begin{cases}\tfrac{1}{2}\mu_0\sigma\omega RB_0r\sin\phi, & \text{for } r\leqslant a,\\ \tfrac{1}{2}\mu_0\sigma\omega RB_0a^2\dfrac{\sin\phi}{r}, & \text{for } r\geqslant a.\end{cases} \quad (50)$$

Let us call $\mathbf{E}'$ the electric field induced by $\mathbf{B}'$, which satisfies $\nabla\times\mathbf{E}'=-\partial\mathbf{B}'/\partial t=0$. The lowest order terms in the Poynting vector for $r<a$ will be

$$\mathbf{N}=\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}+\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}'+\frac{1}{\mu_0}\mathbf{E}'\times\mathbf{B}. \quad (51)$$

The first term is linear in ω, while the other two are of order of ω^2. Since the curls of $\mathbf{E}$, $\mathbf{E}'$ and $\mathbf{B}$ vanish, the only nonvanishing divergence of $\mathbf{N}$ comes from the second term on the rhs of (51), which is explicitly given by

$$\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}'=\frac{1}{4}\sigma\omega^2RB_0^2r[R\sin\phi\hat{\mathbf{r}}+(R\cos\phi$$
$$+2r)\hat{\boldsymbol{\phi}}]\sin\phi, \quad (52)$$

such that the divergence of the Poynting vector is

$$\operatorname{div}\mathbf{N}=\tfrac{1}{4}\sigma\omega^2RB_0^2(R+2r\cos\phi), \quad (53)$$

which coincides with the expression $-\mathbf{E}\cdot\mathbf{j}$, as required by the energy conservation. Outside the battery, $r>a$, the Poynting vector is

$$\mathbf{N}=\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}'=\frac{1}{4}\sigma\omega^2a^4R^2B_0^2\frac{\sin\phi}{r^3}$$
$$\times(\sin\phi\hat{\mathbf{r}}-\cos\phi\hat{\boldsymbol{\phi}}), \quad (54)$$

and we also can check that, in this region

$$\operatorname{div}\mathbf{N}=-\mathbf{E}\cdot\mathbf{j}=-\frac{1}{4}\sigma\omega^2R^2B_0^2\frac{a^4}{r^4}. \quad (55)$$

Finally, it is easy to check that the flux of vector $\mathbf{N}$ through the lateral surface of the cylinder of radius a exactly matches the power $P_>$ dissipated by the Joule effect in the outer region.

V. COMMENTS AND DISCUSSION

Wiederick *et al.*[1] consider the case where the applied magnetic field is uniform and of value B_0 within a rectangular region of sides l and w. The magnet is sufficiently far away from the axis of rotation to consider that the velocity throughout the rectangular region is uniform. Also, this region is sufficiently far from the disk border to permit one to assume that the disk is of infinite radius. With these assumptions, they obtain by an approximate method that the braking force is

$$\mathbf{F}=-\alpha\sigma h\omega RB_0^2lw\hat{\mathbf{y}}, \quad (56)$$

where

$$\alpha=\frac{1}{1+\mathscr{R}_>/\mathscr{R}_<}, \quad (57)$$

and $\mathscr{R}_<$ and $\mathscr{R}_>$ are the resistance of the pieces of disk located, respectively, inside and outside the external magnetic field.

It is to be remarked that this coincides with our expression (44) if we take for $\mathscr{R}_>$ and $\mathscr{R}_<$ the values (48) and (49), respectively. However, our definition of $\mathscr{R}_>$ and $\mathscr{R}_<$ and the geometry of the region where the magnetic field is defined are different from the ones given in Ref. 1.

Marcuso *et al.*[5] consider the action of a nonuniform magnetic field of cylindrical symmetry confined to a circular region of radius a as in our model (1). When the border effects may be considered negligible and the velocity at all points in the magnetic field region is uniform, $v=\omega R$, our expression (28) for the braking force F_y equals their expression (50). If the velocity is not uniform, our expression (26) for the torque differs from their expression (43) in a constant term that Marcuso *et al.* consider negligible in their experimental setup. They give no expression for the border effects.

In the particular case where the external magnetic field is uniform and has value B_0 for $r \leqslant a$, but the influence of the border is not negligible, expression (35) reduces to

$$M=-\frac{1}{2}\,\sigma h \omega R^2 \pi a^2 B_0^2 \left(1-\frac{a^2 R_d^2}{(R_d^2-R^2)^2}\right), \tag{58}$$

which is precisely the value given by Smythe[3] for this case. It should be stressed that our expression (35) may be used not only for this case but also for any external magnetic field with cylindrical symmetry inside the disk provided it vanishes for $r>a$.

In our approach, we have not taken into account the influence of the magnetic field $\mathbf{B}'$ produced by the induced currents in the disk. This is equivalent to consider that the applied field B_0 is such that $B'_{\max} \ll B_0$. From (50), this is equivalent to

$$\omega \ll \frac{2}{\mu_0 \sigma R a}. \tag{59}$$

For a copper disk ($\sigma=5.74\times 10^7\,\Omega^{-1}\ \mathrm{m}^{-1}$) of radius $R=100$ mm and $a=10$ mm we get $\omega \ll 27.75$ rad/s. This requirement was satisfied in the experimental setup of Marcuso *et al.*[6]

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