V. CONCLUSIONS

It was shown that adding an insulating layer to an object that is to be cooled in a much colder fluid causes the cooling to be more rapid. This effect was demonstrated with a simple experiment, suitable for an undergraduate experimental course, and is due to the fact that the insulation causes the surface temperature to be lower. This, in turn, causes an anticipated change of boiling heat-transfer mode, from film to transition or nucleate regime, and the subsequent increase in heat flux which shortens the cooling period.

A critical thickness of insulation was defined which approximately gives the thickness of minimum cooling period. It was found to be in good agreement with the experimental data.

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The Liénard–Wiechert potential and the retarded shape of a moving sphere

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The subtleties in the derivation of the retarded Liénard–Wiechert potential for a point charge are stressed by explicitly computing and drawing the retarded shape of a moving sphere. This shape is the effective integration region for the charge density and it is computed, with the aid of the "information collecting sphere," in the limit of vanishing radius (or, equivalently, from the point of view of a remote observer).

I. INTRODUCTION

The retarded scalar potential $\phi(P,T)$, created at time T and position P by a charge density distribution $\rho(\mathbf{r},t)$, is given by[1]

$$\phi(P,T)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r},T-R/c)}{R}\,dV. \tag{1}$$

Here, R is the retarded distance from P to the point $\mathbf{r}$ at which the source was located at the retarded time $t=T-R/c$.

In the case of a point charge moving with constant velocity v, for given values of P and T the retarded distance R has a single value, say R_0, over the whole charge and the corresponding potential can be written as

$$\phi(P,T)=\frac{1}{4\pi\epsilon_0 R_0}\int\rho\left(\mathbf{r},T-\frac{R}{c}\right)dV. \tag{2}$$

One is then tempted to substitute the total charge q for the integral appearing in the last expression. This, however, would give us an incorrect result. By using the correct value for that integral, namely

$$\int\rho\left(\mathbf{r},T-\frac{R}{c}\right)dV=\frac{q}{(1-\beta\cos\theta)}, \tag{3}$$

one gets the Liénard–Wiechert potential:

$$\phi(P,T)=\frac{1}{4\pi\epsilon_0}\frac{q}{R_0(1-\beta\cos\theta)}. \qquad (4)$$

As usual $\beta = v/c$, and θ is the angle between the particle velocity and the vector joining the charge retarded position with the observation point.

Most undergraduate students have major difficulties understanding where the additional factor $(1-\beta\cos\theta)^{-1}$ comes from. In fact, one might think that it would be enough to substitute R_0 for R in the integrand to compute the integral (3). But this would give the wrong result because, unlike R, in the case of a point charge ρ is not a smooth function and it must be modeled by a Dirac delta function. The latter provides the fastest method to compute the right value of the integral in (3).[2] But the proof is too advanced for most students, who see the magic of generalized functions as a marvelous but mysterious trick.

From a pedagogical point of view, it may be better to start with a finite charge distribution whose size will ultimately vanish. Then, it is often hard to convince the students that, even when the charge distribution shrinks to a point, the left-hand side of (3) does not represent the total particle charge because the charge density in the integrand is taken at different retarded times and, thus, (3) is not a simultaneous integral. Even if the charge density is assumed to be constant over the particle, the effective integration region [i.e., the region where $\rho(\mathbf{r},T-R/c)$ is different from 0] is not the spatial region occupied by the charge at a single instant of time.

It is worth stressing that in this approach taking R out of integral (1) amounts to neglecting terms of the order of a/R_0, where a is the characteristic length of the considered charge distribution. Of course, these terms vanish in the limit $a\to 0$. But the lack of simultaneity in the integral gives rise to a zeroth-order contribution that survives to the process of taking the aforementioned limit.

In order to compute (3), one can change variables to have a simultaneous integral. This gives the factor $(1-\beta\cos\theta)^{-1}$ as the Jacobian of the transformation, but the actual computation is a bit cumbersome. The most elegant method we know of is the use of the "information collecting sphere" in the textbook by Panofsky and Phillips.[3] Related but far more restrictive methods are used by Feynman, Leighton, and Sands[4] and Griffiths[5] in the case of rather odd charge distributions (a cubic charge moving directly toward the observations point, and a charged rod, respectively).

The aim of this note is to provide an elementary and more visual computation of the integral (3) by using the "information collecting sphere," but applying it (for an arbitrary direction of motion) to the most natural starting charge distribution. Surprisingly, this happens to be a rather easy task.

II. THE RETARDED SHAPE OF A MOVING SPHERE

Let us consider a charge which when at rest is a sphere of radius a much smaller than any other distance in the problem and has an uniform charge density $\rho^* = 3q/4\pi a^3$. For an inertial observer who sees the charge moving with constant velocity v, the simultaneous shape of the sphere is that of an ellipsoid of revolution with two principal axes of length a and the third one lying along the direction of motion and having a length $\gamma^{-1}a$, with $\gamma=(1-\beta^2)^{-1/2}$. The charge density measured by the observer is $\gamma\rho^*$. Without loss of generality we can select the coordinates in such a way that the charge center moves along the positive direction of the x axis and the observation point P lies in the x-y plane. We can also choose $t=0$ to coincide with the moment at which the center of the charge passes through the origin of coordinates attached to the observer. The situation is depicted in Fig. 1. The simultaneous shape of the ellipsoid is then given by

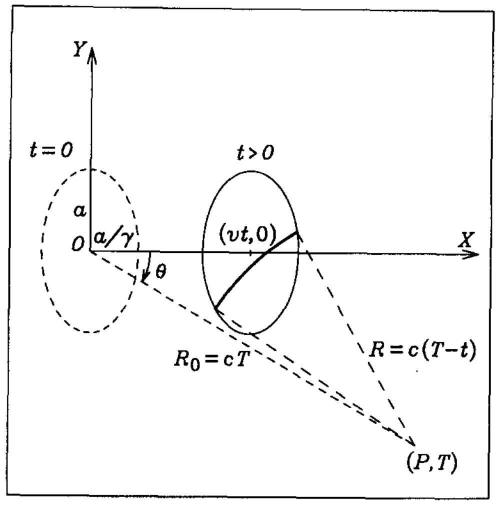

Fig. 1. The intersection of the simultaneous shape of the moving charge and the information collecting sphere collapsing towards the observation point P (at which it arrives at time T).

$$\gamma^2(x-vt)^2+y^2+z^2=a^2. \qquad (5)$$

The contribution to integral (3) corresponding to a fixed retarded time $t=T-R/c$ is due to those points $\mathbf{r}$ for which the integrand $\rho(\mathbf{r},t)$ is different from zero. These are the points at the intersection of the ellipsoid (5) and the information collecting sphere given by $R=c(T-t)$ or, equivalently, by

$$(x-R_0\cos\theta)^2+(y+R_0\sin\theta)^2+z^2=(R_0-ct)^2, \qquad (6)$$

with $R_0=cT$. The trace of this intersection in the plane x-y appears as an arc in Fig. 1.

Now if we keep in mind that we are interested in the limit of a point particle, we have $|x|, |y|, |z|\leqslant a$, $c|t|\lesssim a$ and $a\ll R_0$. Thus, we only need the lowest-order contribution in a/R_0 to (6):

$$x\cos\theta - y\sin\theta = ct. \qquad (7)$$

Subsequently, the information collecting sphere collapsing with the speed of light toward a center at the observation point P becomes in this limit rather an "information collecting plane" moving with the same speed in the observation direction, as depicted in Fig. 2.

The effective integration region, which has been called in a different context[6] the "apparent shape" of the sphere, is made up of the points satisfying Eqs. (5) and (7) for all values of t. Of course, these points only exist during the time interval in which the information collecting plane is passing through the charge, i.e., when $-t_0\leqslant t\leqslant t_0$, with

$$t_0=(a/c)\sqrt{(1+\beta\cos\theta)/(1-\beta\cos\theta)}.$$

In fact, we do not need this result because the retarded shape can be directly obtained by eliminating t between

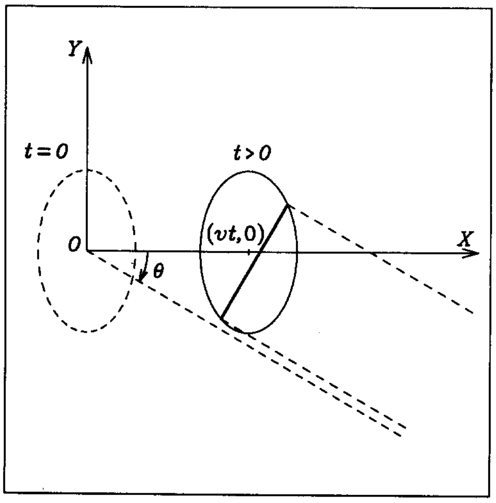

Fig. 2. The intersection of the simultaneous shape of the moving charge and the information collecting plane moving along the observation direction.

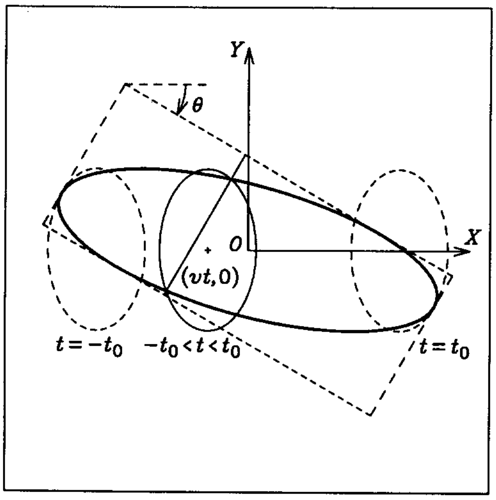

Fig. 3. The retarded shape of the charge approaching the observation point with $\beta = 0.8$ and $\theta = 30°$.

Eqs. (5) and (7). The result is then very simple and reads as follows:

$$\gamma^2(1-\beta\cos\theta)(1-\beta)\left(\cos\frac{\theta}{2}x - \sin\frac{\theta}{2}y\right)^2 + \gamma^2(1-\beta\cos\theta)(1+\beta)\left(\sin\frac{\theta}{2}x + \cos\frac{\theta}{2}y\right)^2 + z^2 = a^2. \quad (8)$$

It is now obvious that the retarded shape is yet another ellipsoid with one principal axis of an unchanged length $a_3 = a$, and direction OZ, and two other axes rotated by an angle $\theta/2$ (independent of v!) around the z axis and having lengths

$$a_1 = \gamma^{-1}a/\sqrt{(1-\beta\cos\theta)(1-\beta)}$$

and

$$a_2 = \gamma^{-1}a/\sqrt{(1-\beta\cos\theta)(1+\beta)},$$

as shown in Fig. 3. Thus the volume of the effective integration region in (3) is $V = 4/3\pi a_1 a_2 a_3 = 4/3\pi a^3\gamma^{-1}/(1-\beta\cos\theta)$, and the integral in (3) can be now readily evaluated:

$$\int \rho\left(\mathbf{r}, T - \frac{R}{c}\right)dV = \gamma\rho^* \int_V dV = \gamma\rho^* V = \frac{q}{(1-\beta\cos\theta)}. \quad (9)$$

This result does not depend on the radius a of the model, as corresponds to the limit case of a point charge.

The Liénard–Wiechert potential (4) is Coulomb-like, but the retarded distance must be used and the effective charge is the one corresponding to the same charge density but extended to a different volume V.

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FEYNMAN ON STUPIDITY

"...we do not know where we are stupid until we stick our necks out."

Richard P. Feynman [Quoted in John S. Bell, "Against 'Measurement,' " CERN-TH-5611/89 (December 1989)].