δ-function converging sequences

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(Received 18 July 2001; accepted 12 October 2001)

We discuss the usefulness and physical interpretation of a simple and general way of constructing sequences of functions that converge to the Dirac delta function. The main result, which seems to have been largely overlooked, includes most of the δ-function converging sequences found in textbooks, is easily extended, and can be used to introduce many useful generalized functions to physics students with little mathematical background. We show that some interesting delta-function identities are simple consequences of the one discussed here. An illustrative example in electrodynamics is also analyzed, with the surprising result that the formalism allows as a limit an uncharged massless particle which creates no electromagnetic field, but has a nonzero electromagnetic energy–momentum tensor. © *2002 American Association of Physics Teachers.*
[DOI: 10.1119/1.1427087]

I. INTRODUCTION

Generalized functions are useful in electrodynamics, quantum mechanics, and other branches of physics.[1] Apart from their usefulness for solving differential equations, they provide a natural way to describe a distribution of point charges. We can understand a point charge as a continuous charge distribution whose density has a known finite integral, but that to a good approximation vanishes except in a very small neighborhood. This intuitive picture of point charges naturally leads to the consideration of sequences of continuous functions with finite integrals that converge to the Dirac delta function.

Particular examples of such sequences of functions are discussed heuristically in some textbooks on mathematical methods.[2] More advanced textbooks discuss theorems that prove the convergence.[3,4] However, determining if these theorems are satisfied is often a daunting task for undergraduate students. In Sec. II we discuss a simplified version of a known convergence theorem.[4] This version appears under more stringent conditions as an exercise in Ref. 5 and for a restricted case and with a different proof in Ref. 6. It is not as general as some theorems given in advanced textbooks, but it is easily proven, can be readily extended (see Sec. III), includes almost all the examples discussed in introductory textbooks, and has an intuitive physical interpretation.

We will see in Sec. IV that we can use the simplified theorem as the starting point for introducing the delta function in a quick and intuitive way that requires little mathematical background, but is still general enough to be of real use in practical applications. In Sec. V we show how it can be used to quickly prove some interesting differential identities involving the delta function. An application in electrodynamics is discussed in Sec. VI, where we will also see that the aforementioned theorem helps one to understand why the product of generalized functions is not always well defined.

II. THE BASIC RESULT

Consider a function g such that the integral

$$\mathcal{G}\equiv\int_{-\infty}^{\infty}g(x)\,dx \tag{1}$$

exists and is finite. We also consider a family of functions that depends on the parameter $\gamma>0$ defined as follows:

$$g_\gamma(x)\equiv\gamma g(\gamma x). \tag{2}$$

Increasing γ amounts to stretching a plot of the function in the vertical direction while compressing it in the horizontal direction in such a way that the integral does not change, as can be seen by defining $y=\gamma x$:

$$\int_{-\infty}^{\infty}g_\gamma(x)\,dx=\int_{-\infty}^{\infty}\gamma g(\gamma x)\,dx=\int_{-\infty}^{\infty}g(y)\,dy=\mathcal{G}. \tag{3}$$

Our main result is that the sequence $g_\gamma(x)$ is proportional to the Dirac delta function as $\gamma\to\infty$:

$$\lim_{\gamma\to\infty}g_\gamma(x)=\mathcal{G}\delta(x). \tag{4}$$

That is,

$$\lim_{\gamma\to\infty}\gamma g(\gamma x)=\delta(x)\int_{-\infty}^{\infty}g(u)\,du. \tag{5}$$

The proof, which can be used as a template for related results, starts by considering the integrals of the functions in the sequence:

$$G_\gamma(x)\equiv\int_{-\infty}^{x}g_\gamma(u)\,du=\gamma\int_{-\infty}^{x}g(\gamma u)\,du=\int_{-\infty}^{\gamma x}g(v)\,dv, \tag{6}$$

where we have made the change of variables $v=\gamma u$. Because $\gamma>0$, the limit of this sequence is easily calculated by using the last expression in Eq. (6) and the definition of the Heaviside unit step function $\theta(x)$:

$$\lim_{\gamma\to\infty}G_\gamma(x)=\left\{\begin{array}{ll}\int_{-\infty}^{\infty}g(v)\,dv & \text{if } x>0\\ \int_{-\infty}^{-\infty}g(v)\,dv & \text{if } x<0\end{array}\right\}=\mathcal{G}\times\left\{\begin{array}{ll}1 & \text{if } x>0\\ 0 & \text{if } x<0\end{array}\right\}=\mathcal{G}\theta(x). \tag{7}$$

We have to take the derivative of the result in Eq. (7), but limits and derivatives of ordinary functions rarely commute

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unless further mathematical assumptions are fulfilled. However, one of the most desirable properties of generalized functions[4] is that their derivatives and limits always commute, so that $[\lim_{\gamma\to\infty} G_\gamma]' = \lim_{\gamma\to\infty} G'_\gamma$, because for any regular function $f(x)$ that decreases fast enough at infinity we have

$$\begin{aligned}
\int_{-\infty}^{\infty} [\lim_{\gamma\to\infty} G'_\gamma(x)] f(x)\,dx
&= \lim_{\gamma\to\infty} \int_{-\infty}^{\infty} G'_\gamma(x) f(x)\,dx \\
&= -\lim_{\gamma\to\infty} \int_{-\infty}^{\infty} G_\gamma(x) f'(x)\,dx \\
&= -\int_{-\infty}^{\infty} [\lim_{\gamma\to\infty} G_\gamma(x)] f'(x)\,dx \\
&= \int_{-\infty}^{\infty} [\lim_{\gamma\to\infty} G_\gamma(x)]' f(x)\,dx. \qquad (8)
\end{aligned}$$

Here we have used the fact that the limit of any sequence $h_\gamma(x)$ of generalized functions is defined by

$$\int_{-\infty}^{\infty} [\lim_{\gamma\to\infty} h_\gamma(x)] f(x)\,dx \equiv \lim_{\gamma\to\infty} \int_{-\infty}^{\infty} h_\gamma(x) f(x)\,dx, \qquad (9)$$

and the derivative of the generalized function $h(x)$ is given by

$$\int_{-\infty}^{\infty} h'(x) f(x)\,dx = -\int_{-\infty}^{\infty} h(x) f'(x)\,dx. \qquad (10)$$

By taking the derivative of the result in Eq. (7) as a generalized function and remembering that $\theta'(x) = \delta(x)$ and $G'_\gamma(x) = g_\gamma(x)$, we recover Eq. (5).

Although theorem (5) or related results must have been known by many people for a long time, it seems that its generality and usefulness have been overlooked in textbooks on mathematical methods in physics and in other physics textbooks that use the delta function, for example, in electrodynamics. As an illustration, let us consider the Gaussian function

$$g(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}, \qquad (11)$$

which leads to the sequence of integrals

$$\int_{-\infty}^{\gamma x} g(v)\,dv = \frac{1+\mathrm{erf}(\gamma x)}{2} \xrightarrow{\gamma\to\infty} \theta(x), \qquad (12)$$

as displayed in Fig. 1 for $\gamma = 1,\ldots,10$. The derivatives are shown in Fig. 2 and converge to the delta function:

$$\gamma g(\gamma x) = \frac{\gamma}{\sqrt{\pi}} e^{-\gamma^2 x^2} \xrightarrow{\gamma\to\infty} \delta(x). \qquad (13)$$

An example that provides one of the most useful representations of the delta function is the Fourier transform. If we define $g(x) = \sin x/\pi x$, then

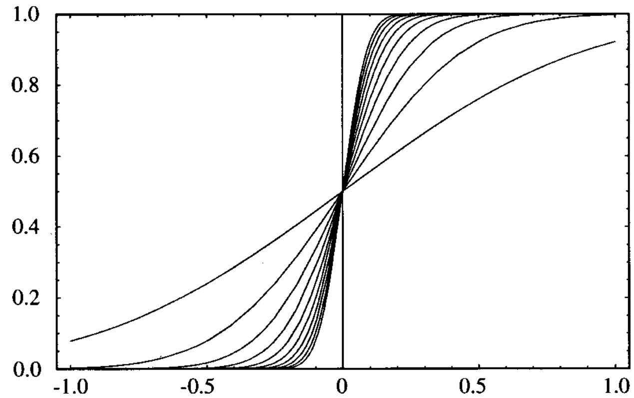

Fig. 1. Plot of the sequence of functions $(1+\mathrm{erf}(\gamma x))/2$ for $\gamma = 1,\ldots,10$. They approach the unit step function $\theta(x)$.

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk &= \frac{1}{2\pi}\lim_{\gamma\to\infty}\int_{-\gamma}^{\gamma} e^{ikx}\,dk \\
&= \lim_{\gamma\to\infty} \gamma \frac{\sin \gamma x}{\pi \gamma x} \\
&= \delta(x) \int_{-\infty}^{\infty} \frac{\sin u}{\pi u}\,du = \delta(x). \qquad (14)
\end{aligned}$$

We stress that to obtain $\gamma g(\gamma x) \to \delta(x)$, the function g need not be positive (as required in the theorem proved in Ref. 6) or even symmetric about the origin; we can choose any function with an unit integral. Although the corresponding plots would not look as esthetically pleasing as those in Fig. 2, the limit would be the delta function, which is symmetric, $\delta(-x) = \delta(x)$, and is intuitively understood as non-negative. Notice also that, contrary to the intuitive idea that $\delta(x)$ is infinite at the origin, a sequence converging to $\delta(x)$, that is, corresponding to $\int_{\mathbf{R}} g(u)\,du = 1$ with $\mathbf{R} = (-\infty,\infty)$, will behave at the origin as $\gamma g(0)$, which may converge to $+\infty$, but also to 0, or even to $-\infty$.

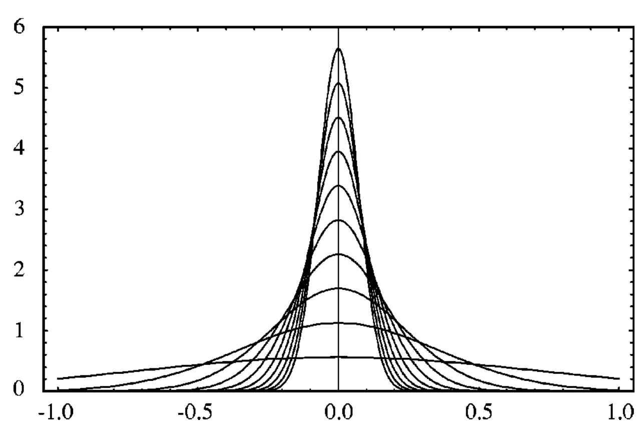

Fig. 2. Plot of the sequence of functions $\gamma g(\gamma x)$ defined in Eq. (13) for $\gamma = 1,\ldots,10$. Their limit is the Dirac delta $\delta(x)$.

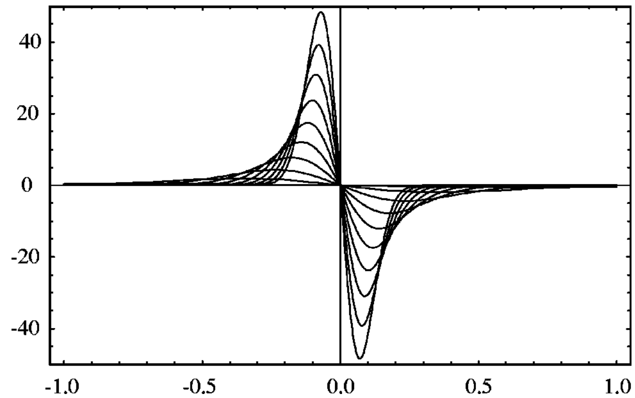

Fig. 3. Plot of the derivatives of the functions plotted in Fig. 2 for $\gamma = 1,...,10$ showing the convergence to $\delta'(x)$.

III. RELATED RESULTS

The theorem shown in Eq. (5) provides a template for similar results that can be proven in much the same way. For instance, the delta function need not be centered at the origin; one obviously has

$$\lim_{\gamma\to\infty} \gamma g(\gamma(x-a)) = \delta(x-a)\int_{-\infty}^{\infty} g(u)\,du, \tag{15}$$

and the equivalent formula obtained by making $\epsilon = 1/\gamma$:

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon} g\left(\frac{x-a}{\epsilon}\right) = \delta(x-a)\int_{-\infty}^{\infty} g(u)\,du. \tag{16}$$

Moreover, we can suggest to students that they prove, in the same spirit as the derivation shown in Sec. II, that for any function $h(\gamma)$ with finite limit $a = \lim_{\gamma\to\infty} h(\gamma)$, we have

$$\lim_{\gamma\to\infty} \gamma g[\gamma(x-h(\gamma))] = \delta(x-a)\int_{-\infty}^{\infty} g(u)\,du. \tag{17}$$

We obtain another useful result by taking the derivative of Eq. (15) as a generalized function:

$$\lim_{\gamma\to\infty} \gamma^2 g'(\gamma(x-a)) = \delta'(x-a)\int_{-\infty}^{\infty} g(u)\,du. \tag{18}$$

Further derivatives are straightforward. An example of such a sequence of derivatives, which is useful for representing dipoles, is displayed in Fig. 3 for the derivatives of $\gamma g(\gamma x)$ defined in Eq. (13).

The results are readily extensible to higher dimensions. For example, in three dimensions, the equivalent of Eq. (16) is

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^3} g\left(\frac{\mathbf{r}-\mathbf{r}'}{\epsilon}\right) = \delta^{(3)}(\mathbf{r}-\mathbf{r}')\int_{\mathbf{R}^3} g(\mathbf{u})\,d^3\mathbf{u}, \tag{19}$$

where the three-dimensional delta function is defined as usual:

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}') \equiv \delta(x-x')\,\delta(y-y')\,\delta(z-z'). \tag{20}$$

The proof of this result can be proposed as a problem.

IV. INTRODUCING THE DELTA FUNCTION

Although the result in Eq. (5) seems to have been largely overlooked, it contains the sequences that are usually discussed in other ways in textbooks on mathematical methods of physics.[2] In fact, Eq. (5) contains an infinite number of such sequences; their common limit defines (up to a constant) the delta function in the spirit of generalized functions as a limit of sequences of ordinary functions (see, for example, Ref. 5) rather than from Schwartz's point of view,[7] where the delta function is more formally defined as an object (a "distribution") that associates with any test function f its value at the origin:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0). \tag{21}$$

On the other hand, rather than as a theorem derived from the theory of generalized functions, results (5) and (15) can be used as a quick and dirty way to introduce generalized functions to students of physics with little mathematical background. Students will not object to the following calculation of a limit by means of the change of variables $v = \gamma(u-a)$, if they are told that it is correct provided that the function f satisfies the appropriate mathematical conditions of regularity and decrease at infinity:

$$\lim_{\gamma\to\infty}\int_{-\infty}^{\infty} \gamma g(\gamma(u-a)) f(u)\,du$$
$$= \lim_{\gamma\to\infty}\int_{-\infty}^{\infty} g(v) f\left(\frac{v}{\gamma}+a\right)dv = f(a)\int_{-\infty}^{\infty} g(v)\,dv. \tag{22}$$

Because this result does not depend on the details of f and g, we can then tell students that we can summarize all the limits of type shown in Eq. (22) by means of the single expression (15), where the δ "function" only makes sense when introduced inside an integral with another (suitable) function:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a). \tag{23}$$

The remaining elementary properties of the delta function and its derivatives can be easily established as usual. One easy student problem would be to interpret the result in Eq. (19) in a way similar to Eq. (22).

An illustrative example is a point charge q located at the origin, by which we mean a charge density $\rho(\mathbf{r})$ about which we know only that it is negligible except very near the origin, but has a finite integral

$$\int_{\mathbf{R}^3}\rho(\mathbf{r})\,d^3\mathbf{r} = q. \tag{24}$$

There are infinite number of forms for ρ; it is enough to pick one integrable function $g(\mathbf{r})$ that decreases as $r = |\mathbf{r}|$ increases, and satisfies

$$\int_{\mathbf{R}^3} g(\mathbf{r})\,d^3\mathbf{r} = q. \tag{25}$$

In fact, for any $\epsilon > 0$, the density $\rho_\epsilon(\mathbf{r}) = \epsilon^{-3} g(\mathbf{r}/\epsilon)$ leads to the same charge q,

$$\int_{\mathbf{R}^3}\rho_\epsilon(\mathbf{r})\,d^3\mathbf{r} = \int_{\mathbf{R}^3} g(\mathbf{r})\,d^3\mathbf{r} = q, \tag{26}$$

and is located about the origin at distances not much larger than ϵ. For small enough ϵ, ρ_ϵ is a good approximation to a point charge whatever the profile $g(\mathbf{r})$ might be.

Most students will agree that it makes sense to write

$$\lim_{\epsilon\to 0}\rho_\epsilon(\mathbf{r})=q\,\delta^{(3)}(\mathbf{r}), \tag{27}$$

to obtain

$$\lim_{\epsilon\to 0}\int_{\mathbf{R}^3}\rho_\epsilon(\mathbf{r})f(\mathbf{r})d^3\mathbf{r}=qf(\mathbf{0}), \tag{28}$$

instead of using a modification of Eq. (22). In this way, for example, the potential at point $\mathbf{r}$ may be calculated as

$$V(\mathbf{r})=\lim_{\epsilon\to 0}\frac{1}{4\pi\varepsilon_0}\int_{\mathbf{R}^3}\frac{\rho_\epsilon(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}, \tag{29}$$

or simply as

$$V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int_{\mathbf{R}^3}\frac{q\,\delta^{(3)}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}. \tag{30}$$

Of course if ϵ is small but nonzero, the profile function $g(\mathbf{r})$ will also appear in the form of a dipole moment, as the following problem shows.

(i) Use a Taylor expansion and the partial derivative of Eq. (19) to prove that

$$\rho_\epsilon(\mathbf{r})=q\,\delta^{(3)}(\mathbf{r})-\mathbf{p}\cdot\boldsymbol{\nabla}\,\delta^{(3)}(\mathbf{r})+O(\epsilon^2), \tag{31}$$

where $\mathbf{p}$ is the electric dipole moment of the charge distribution $\rho_\epsilon(\mathbf{r})$:

$$\mathbf{p}=\int_{\mathbf{R}^3}\mathbf{r}\rho_\epsilon(\mathbf{r})d^3\mathbf{r}=\epsilon\int_{\mathbf{R}^3}\mathbf{r}g(\mathbf{r})d^3\mathbf{r}. \tag{32}$$

(Note that $\mathbf{p}$ vanishes in the limit $\epsilon\to 0$.)

(ii) Assume that $f(\mathbf{r})$ decreases fast enough as $\mathbf{r}\to\infty$, and prove the following result by using integration by parts:

$$\int_{\mathbf{R}^3}\frac{\partial\delta^{(3)}(\mathbf{r})}{\partial x_i}f(\mathbf{r})d^3\mathbf{r}=-\frac{\partial f(\mathbf{0})}{\partial x_i}. \tag{33}$$

(iii) Show that if terms proportional to ϵ^2 are negligible, the potential is given by

$$V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r}+\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right]. \tag{34}$$

In Sec. V we discuss how the complete fields of electric (and magnetic) dipoles can be calculated from the corresponding potential.

V. PROVING DIFFERENTIAL IDENTITIES

Some time ago Frahm[8] introduced a number of novel differential identities that are useful in electrodynamics. We will show how Eq. (19) can be used to shorten a physicist's proof of these identities and make them easier for students. To illustrate the method, it will be enough to prove the following identity:[8]

$$-\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{r}=\frac{\partial}{\partial x_i}\frac{x_j}{r^3}=\frac{r^2\delta_{ij}-3x_ix_j}{r^5}+\frac{4\pi}{3}\delta_{ij}\delta^{(3)}(\mathbf{r}), \tag{35}$$

where $r^2=x^2+y^2+z^2$. The origin of the δ function in Eq. (35) is of course the fact that the function $f_0=x_j/r^3$ to be derived is not regular at the origin. So, we will start by considering a continuous family of regular functions that goes to f_0 in the appropriate limit:

$$f_\epsilon\equiv\frac{x_j}{(r^2+\epsilon^2)^{3/2}}\xrightarrow{\epsilon\to 0}f_0=\frac{x_j}{r^3}. \tag{36}$$

The derivative of f_ϵ is readily calculated:

$$\frac{\partial}{\partial x_i}\frac{x_j}{(r^2+\epsilon^2)^{3/2}}=\frac{r^2\delta_{ij}-3x_ix_j}{(r^2+\epsilon^2)^{5/2}}+\frac{\epsilon^2\delta_{ij}}{(r^2+\epsilon^2)^{5/2}}. \tag{37}$$

Notice that the first term on the right-hand side will go to the first term in (35) as $\epsilon\to 0$. However, the second term does not go to zero, but can be written as

$$\frac{\epsilon^2}{(r^2+\epsilon^2)^{5/2}}=\frac{1}{\epsilon^3}g\left(\frac{\mathbf{r}}{\epsilon}\right), \tag{38}$$

with

$$g(\mathbf{r})\equiv\frac{1}{(r^2+1)^{5/2}}. \tag{39}$$

Because $\int_{\mathbf{R}^3}g(\mathbf{r})d^3\mathbf{r}=4\pi/3$, we recover Eq. (35) if we take the limit $\epsilon\to 0$ in Eq. (37) and use Eq. (19).

The careful reader will notice that the first term on the right-hand side of Eq. (37) may also be written in a similar way:

$$\frac{r^2\delta_{ij}-3x_ix_j}{(r^2+\epsilon^2)^{5/2}}=\frac{1}{\epsilon^3}g_{ij}\left(\frac{\mathbf{r}}{\epsilon}\right), \tag{40}$$

with

$$g_{ij}(\mathbf{r})\equiv\frac{r^2\delta_{ij}-3x_ix_j}{(r^2+1)^{5/2}}. \tag{41}$$

However, these functions do not converge to the delta function because g_{ij} is not integrable. When we say that this limit is obtained simply by dropping ϵ on the left-hand side, we mean that the short-hand notation $(r^2\delta_{ij}-3x_ix_j)/r^5$ should not be understood as an ordinary function (which would not be locally integrable, and thus would not define a generalized function). Instead, this notation denotes the generalized function that is defined by a careful limiting process, which can be implemented in polar coordinates by first integrating the angular variables to avoid the singularity at the origin (the logarithmic singularity at infinity will be compensated by the asymptotic behavior of the test function f):

$$\int_{\mathbf{R}^3}\frac{r^2\delta_{ij}-3x_ix_j}{r^5}f(\mathbf{r})d^3\mathbf{r}$$
$$=\lim_{\epsilon\to 0}\int_0^\infty\left[\int_0^\pi\int_0^{2\pi}\frac{r^2\delta_{ij}-3x_ix_j}{(r^2+\epsilon^2)^{5/2}}f(\mathbf{r})\sin\theta\,d\theta\,d\varphi\right]r^2\,dr. \tag{42}$$

When we say that the sequence $1/\epsilon^3g(\mathbf{r}/\epsilon)$ in Eq. (38) will converge to the delta function, we mean that we need not repeat the calculation of $\lim_{\epsilon\to\infty}1/\epsilon^3\int_{\mathbf{R}^3}g(\mathbf{r}/\epsilon)f(\mathbf{r})d^3\mathbf{r}$ for each choice of $f(\mathbf{r})$, because we have proved that the result will always be proportional to $f(0)$.

By contracting the indices in Eq. (35) and changing the sign, we can recover the elementary solution of the Poisson equation:

$$\nabla^2\frac{1}{r}=-4\pi\delta^{(3)}(\mathbf{r}). \tag{43}$$

Of course, if Eq. (35) has not been discussed, we can prove Eq. (43) directly by considering the regularized function $1/\sqrt{r^2+\epsilon^2}$. Another easy and interesting consequence of Eq. (35) is the appearance of delta-function terms in the fields of electric and magnetic dipoles.[8]

A similar calculation can be used by students to prove the following result:[8]

$$\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\frac{1}{r} = \frac{3r^2(\delta_{ij}x_k+\delta_{jk}x_i+\delta_{ki}x_j)-15x_ix_jx_k}{r^7} - \frac{4\pi}{5}\left[\delta_{ij}\frac{\partial\delta^{(3)}(\mathbf{r})}{\partial x_k}+\delta_{jk}\frac{\partial\delta^{(3)}(\mathbf{r})}{\partial x_i}+\delta_{ki}\frac{\partial\delta^{(3)}(\mathbf{r})}{\partial x_j}\right]. \tag{44}$$

VI. AN APPLICATION IN ELECTRODYNAMICS

We can use δ-function converging sequences to readily solve problems such as "Calculate the field and Poynting's vector $\mathcal{S}$ for a charge moving in a straight line at the speed of light." [9] To solve this problem, consider a point charge e of mass m moving with constant velocity v along the x axis. If the particle is at the origin (0,0,0) at $t=0$, the electric and magnetic fields at any other point and time are

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\frac{\gamma e}{r^3}[(x-vt)\mathbf{i}+y\mathbf{j}+z\mathbf{k}], \tag{45}$$

$$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{\gamma e v}{r^3}(-z\mathbf{j}+y\mathbf{k}), \tag{46}$$

where $\gamma=(1-v^2/c^2)^{-1/2}$, $v=c(1-\gamma^{-2})^{1/2}$, $r^2=\gamma^2(x-vt)^2+y^2+z^2$, and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the usual unit vectors along the coordinate axis.

We are interested in the limit $v\to c$, or equivalently $\gamma\to\infty$. If we assume m as given, the energy $E=\gamma mc^2$ and linear momentum $\mathbf{p}=\gamma m\mathbf{v}$ will diverge. But we may instead consider a family of particles with rest masses that scale as

$$m=\gamma^{-1}\frac{E}{c^2} \tag{47}$$

for a fixed value of E, so that the energy and momentum are finite in the limit $\gamma\to\infty$.

Because

$$\int_{-\infty}^{\infty}\frac{\gamma}{r^3}dx=\int_{-\infty}^{\infty}\frac{1}{(x^2+y^2+z^2)^{3/2}}dx=\frac{2}{\rho^2}, \tag{48}$$

with $\rho^2=y^2+z^2$, we obtain from Eq. (17)

$$\lim_{v\to c}\frac{\gamma}{r^3}=\frac{2}{\rho^2}\delta(x-ct), \tag{49}$$

so that the fields are confined to the plane passing through the charge in the direction perpendicular to the motion:

$$\mathbf{E}=\frac{1}{2\pi\varepsilon_0}\frac{e}{\rho^2}\delta(x-ct)(y\mathbf{j}+z\mathbf{k}), \tag{50}$$

$$\mathbf{B}=\frac{\mu_0}{2\pi}\frac{ec}{\rho^2}\delta(x-ct)(-z\mathbf{j}+y\mathbf{k}). \tag{51}$$

We have made use of the identity $(x-ct)\delta(x-ct)=0$.

However, the problem requires more careful analysis. Because the electromagnetic energy–momentum tensor[10]

$$T=\varepsilon_0\begin{pmatrix}\frac{1}{2}(E^2+c^2B^2) & c(\mathbf{E}\times\mathbf{B})\\ c(\mathbf{E}\times\mathbf{B}) & -[E_iE_j+c^2B_iB_j-\frac{1}{2}\delta_{ij}(E^2+c^2B^2)]\end{pmatrix} \tag{52}$$

is quadratic in the fields, it contains squares of delta functions, which are notoriously ill-defined objects, as can be seen by using δ-function converging series. If $\gamma g(\gamma x)\to\delta(x)\int_{\mathbf{R}}g(u)du$, its square would be expected to diverge,

$$\gamma^2g^2(\gamma x)\to\gamma\delta(x)\int_{\mathbf{R}}g^2(u)du\to\infty, \tag{53}$$

even if g^2 is integrable. In our example the energy–momentum tensor contains diverging terms of the form γ^2/r^6, and thus in the limit $v\to c$, we will have an infinite density of energy, momentum, and stress. This limit casts serious doubts on the physical existence of the limit $v\to c$, which is consistent with the lack of massless charged particles.

Nevertheless, a more interesting limit is obtained[11] if we further assume that the charge also scales as

$$e^2=\gamma^{-1}q^2 \tag{54}$$

for some constant q. In the limit $\gamma\gg1$, the massless particle has no charge and carries no electromagnetic field, but surprisingly has a relic electromagnetic energy–momentum tensor:

$$T=\frac{1}{4\pi\varepsilon_0}\frac{3q^2}{32\rho^3}\delta(x-ct)\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}. \tag{55}$$

In fact, because

$$\int_{-\infty}^{\infty}\frac{\gamma}{r^6}dx=\int_{-\infty}^{\infty}\frac{1}{(x^2+y^2+z^2)^3}dx=\frac{3\pi}{8\rho^5}, \tag{56}$$

we obtain as a consequence of Eq. (17),

$$\lim_{v\to c}\frac{\gamma}{r^6}=\frac{3\pi}{8\rho^5}\delta(x-ct). \tag{57}$$

According to Eq. (57), there is room in the formalism of classical electrodynamics for a family of massless uncharged particles that depends on two parameters: their mechanical energy E and the value q, which essentially measures the finite electromagnetic energy that remains in the limit of vanishing charge and fields.[11]

To calculate the gravitational field of a massless particle in general relativity, Aichelburg and Sexl[12] considered a family of point masses that scales according to Eq. (47) and are represented by a metric obtained from the Schwarzschild solution by a Lorentz boost corresponding to $v \to c$. As in the electromagnetic case, the gravitational field is contained in the plane orthogonal to the motion of the particle. They did a careful calculation to show that

$$\lim_{v\to c}\left[\frac{\gamma}{\sqrt{\gamma^2(x-vt)^2+\rho^2}}-\frac{\gamma}{\sqrt{\gamma^2(x-vt)^2+1}}\right] = -2\ln\rho\,\delta(x-ct), \tag{58}$$

but we see that Eq. (58) is a straightforward consequence of Eq. (17) and

$$\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{x^2+\rho^2}}-\frac{1}{\sqrt{x^2+1}}\right]dx = -2\ln\rho. \tag{59}$$

ACKNOWLEDGMENT

This work was supported by The University of the Basque Country (Research Project No. UPV172.310-EB150/98 and General Research Grant No. UPV172.310-G02/99).

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