# A lacking term in the proton spin

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**Abstract.** The spin structure of an elementary particle shows that the center of mass and center of charge of the particle are two different points. Dirac's spin operator is the angular momentum of the particle with respect to the center of charge. This implies that the addition of the three Dirac spin operators of the three quarks cannot produce the spin of the proton. It is necessary to add at least, the angular momenta of the three quarks with respect to their corresponding center of mass. This lacking term, which is related to the separation between both points has a clear relationship with Dirac's electric dipole term.

## 1. Two centers

Let us think that the following classical analysis was performed before 1920, i.e., before the emergence of quantum mechanics. The assumption is that the center of mass $\boldsymbol{q}$ and the center of charge $\boldsymbol{r}$, of a charged elementary spinning particle are two different points. If this is the case we can define the angular momentum of the particle with respect to both points. Let us call $\boldsymbol{S}$ the angular momentum w.r.t. the center of charge (CC for short) and $\boldsymbol{S}_{CM}$ the corresponding angular momentum w.r.t. the center of mass (CM for short). They are not independent, because if $\boldsymbol{p}$ is the linear momentum of the particle, then $\boldsymbol{S}_{CM} = (\boldsymbol{r}-\boldsymbol{q})\times\boldsymbol{p}+\boldsymbol{S}$. But both spins satisfy two different dynamical equations in the free case and under some external electromagnetic interaction.

For any arbitrary inertial observer, the total angular momentum of the particle w.r.t. the origin of observer's frame can be written either as

$$\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{p}+\boldsymbol{S}, \quad \text{or} \quad \boldsymbol{J} = \boldsymbol{q}\times\boldsymbol{p}+\boldsymbol{S}_{CM}.$$

If the particle is free, $\boldsymbol{J}$ is conserved and thus

$$\frac{d\boldsymbol{J}}{dt} = 0 = \frac{d\boldsymbol{r}}{dt}\times\boldsymbol{p}+\frac{d\boldsymbol{S}}{dt}, \quad \text{or} \quad \frac{d\boldsymbol{J}}{dt} = 0 = \frac{d\boldsymbol{q}}{dt}\times\boldsymbol{p}+\frac{d\boldsymbol{S}_{CM}}{dt},$$

so that

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{p}\times\boldsymbol{u}, \quad \text{or} \quad \frac{d\boldsymbol{S}_{CM}}{dt} = 0,$$

because the conserved $\boldsymbol{p}$ is along the CM velocity $\boldsymbol{v} = d\boldsymbol{q}/dt$, but not along the CC velocity $\boldsymbol{u} = d\boldsymbol{r}/dt$. The CM spin is a conserved observable for a free particle while the CC spin is not. It is moving in an orthogonal direction to the linear momentum, and only its projection on $\boldsymbol{p}$, the helicity $\boldsymbol{S}\cdot\boldsymbol{p}$, is conserved.

Let us assume now that the particle is under some external electromagnetic force $\boldsymbol{F}$ defined at the CC position. In this case, $\boldsymbol{J}$ and $\boldsymbol{p}$ are no longer conserved and thus $d\boldsymbol{J}/dt = \boldsymbol{r}\times\boldsymbol{F}$ and $d\boldsymbol{p}/dt = \boldsymbol{F}$.

$$\frac{d\boldsymbol{J}}{dt} = \boldsymbol{r}\times\boldsymbol{F} = \frac{d\boldsymbol{r}}{dt}\times\boldsymbol{p} + \boldsymbol{r}\times\frac{d\boldsymbol{p}}{dt} + \frac{d\boldsymbol{S}}{dt}, \quad \text{or} \quad \frac{d\boldsymbol{J}}{dt} = \boldsymbol{r}\times\boldsymbol{F} = \frac{d\boldsymbol{q}}{dt}\times\boldsymbol{p} + \boldsymbol{q}\times\frac{d\boldsymbol{p}}{dt} + \frac{d\boldsymbol{S}_{CM}}{dt},$$

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{p}\times\boldsymbol{u}, \quad \text{or} \quad \frac{d\boldsymbol{S}_{CM}}{dt} = (\boldsymbol{r}-\boldsymbol{q})\times\boldsymbol{F}.$$

The CC spin satisfies the same dynamical equation as in the free case, it moves in an orthogonal direction to the linear momentum, although now $\boldsymbol{p}$ is not conserved. The CM spin satisfies the usual torque equation: the torque of the external force w.r.t. the CM is the time variation of this spin.

Both spins can be found in the literature. The Bargmann-Michel-Telegdi spin [1] is the covariant generalization of the CM spin. The CC spin satisfies the same dynamical equation as Dirac's spin operator in the quantum case.

## 2. Classical model of a Dirac particle

If an elementary spinning particle has two separate centers, the free motion implies that the CM is moving at a constant velocity $\boldsymbol{v}$. But, what about the CC motion? If the motion is free it means that we are not able to distinguish, at two different instants, a different dynamical behaviour. But if the trajectory of the CC is a regular curve (i.e. a continuous and differentiable trajectory) it means that the velocity of the CC has to be of a constant modulus, the same at any time, and the trajectory of a constant curvature and torsion. The CC travels along a helix at a constant velocity, and this description must be valid for any inertial observer.

This implies that the CC velocity has to be unreachable for any inertial observer. Otherwise, if some inertial observer is at rest w.r.t. the CC at a certain instant $t$, because the CC motion is accelerated, it will have for that observer, a velocity different from zero at a subsequent time, and thus contradictory with the assumption that the velocity is of constant absolute value for any inertial observer. The only possibility is that the CC velocity is the speed of light and only a relativistic treatment is allowed.

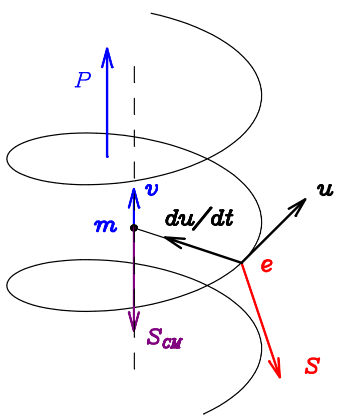


**Figure 1.** Model of a free Dirac particle, with two separate centers, showing the precession of the CC spin $\boldsymbol{S}$ and the conserved CM spin $\boldsymbol{S}_{CM}$. The CC moves along a helix at the speed of light. The CC spin is always orthogonal to the velocity and acceleration of the charge and precesses around $\boldsymbol{p}$. The separation between CC and CM is $\hbar/2mc$, half Compton's wavelength, and the frequency of this internal motion, in the CM frame, is $2mc^2/h$. It is described in [2].

This is precisely the main feature of a classical model of an elementary particle, which satisfies Dirac's equation when quantized, we have developped [2]. The free motion of this model is depicted in Figure 1, where we see the straight motion of the CM and the helical motion at the speed of light of the CC. We also depict the two above mentioned spins, $\boldsymbol{S}$ and $\boldsymbol{S}_{CM}$.

## 3. Dirac's analysis of the electron

In his original 1928 papers [3, 4] Dirac describes an electron in terms of a four-component spinor $\psi(t, \boldsymbol{r})$, defined at point $\boldsymbol{r}$, and a Hamiltonian

$$H = c(\boldsymbol{p} - e\boldsymbol{A}(t, \boldsymbol{r})) \cdot \boldsymbol{\alpha} + \beta mc^2 + e\phi(t, \boldsymbol{r})$$

where $\beta$ and $\boldsymbol{\alpha}$ are Dirac's matrices and $\phi$ and $\boldsymbol{A}$ the scalar and vector external potentials, also defined at the point $\boldsymbol{r}$.

When computing the velocity of point $\boldsymbol{r}$, Dirac arrives at: $\boldsymbol{u} = i/\hbar[H, \boldsymbol{r}] = c\boldsymbol{\alpha}$, which is expressed in terms of $\alpha$ matrices and writes, '*... a measurement of a component of the velocity of a free electron is certain to lead to the result $\pm c$. This conclusion is easily seen to hold also when there is a field present*', because it holds even if the external potentials are not vanishing.

The point $\boldsymbol{r}$ oscillates in a region of order of Compton's wavelength: '*The oscillatory part of $x_1$ is small, ... , which is of order of magnitude $\hbar/mc$, ...*'. This is the amplitude of the motion of the CC around the CM in our model.

The linear momentum does not have the direction of the velocity $\boldsymbol{u}$, but must be related to some average value of it: ... '*the $x_1$ component of the velocity, $c\alpha_1$, consists of two parts, a constant part $c^2p_1H^{-1}$, connected with the momentum by the classical relativistic formula, and an oscillatory part, whose frequency is at least $2mc^2/h$, ...*', the same as in the above classical model.

The total angular momentum w.r.t. the origin of observer's frame, takes the form

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} + \frac{\hbar}{2}\boldsymbol{\sigma} = \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{S}$$

where the orbital part $\boldsymbol{r} \times \boldsymbol{p}$ and the spin part $\boldsymbol{S} = \hbar\boldsymbol{\sigma}/2$, are not separately conserved for a free electron but the spin satisfies,

$$\frac{d\boldsymbol{S}}{dt} = \frac{i}{\hbar}[H, \boldsymbol{S}] = \boldsymbol{p} \times c\boldsymbol{\alpha} = \boldsymbol{p} \times \boldsymbol{u}.$$

even under some external interaction. This is the dynamical equation of the CC spin.

The electron, '*... behaves as though it has a magnetic moment given by*

$$\mu = g\frac{e}{2m}\boldsymbol{S} = \frac{e\hbar}{2m}\boldsymbol{\sigma}, \quad g = 2,$$

*an also an instantaneous electric dipole*'

$$\boldsymbol{d} = \frac{ie\hbar}{2mc}\boldsymbol{\alpha}.$$

If the previous classical analysis of an elementary particle with two separate centers is taken into account, it is not difficult to conclude that Dirac's electron is an object with two centers, described by a spinor $\psi(t, \boldsymbol{r})$ which is a function of the CC position $\boldsymbol{r}$. The linear momentum is not lying along the velocity of point $\boldsymbol{r}$, but around some average value of it. Dirac spin operator is not the angular momentum w.r.t. the CM, but it represents the angular momentum w.r.t. the CC, even under some external interaction. The magnetic moment is produced by the motion of the charge, and the separation between these two points defines an electric dipole moment $\boldsymbol{d} = e(\boldsymbol{r} - \boldsymbol{q})$.

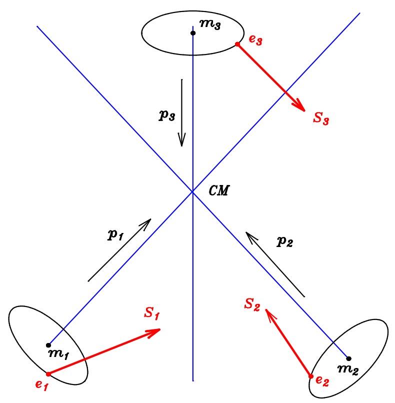


**Figure 2.** Model of a proton as a bound system of three Dirac particles in a $L = 0$ angular momentum state, and in the CM frame. It is shown the motion of the CM of each quark, which has to be a straight trajectory passing through the CM of the proton. The Dirac spin operator of each quark is defined with respect to the corresponding CC, so that the addition of the three $\hbar\boldsymbol{\sigma}/2$ cannot produce the angular momentum w.r.t. the CM of the proton.

## 4. The spin of the proton

Let us assume as usual that the proton is a bound system of three quarks which are in a zero orbital angular momentum $L = 0$, w.r.t. the CM of the proton. We also assume, as usual, that quarks are Dirac particles, i.e., charged particles of spin 1/2 and gyromagnetic ratio $g = 2$, so that we can apply to them the same classical model as the above for the electron.

If they move in a $L = 0$ state, this means literally, when making the analysis in the CM frame of the proton, that the CM of each quark is moving in a straight trajectory passing through the common CM of the proton, and therefore all three trajectories are lying on a plane, such that the total linear momentum is zero. Let us consider that the spin of the proton is the angular momentum of this system of three quarks w.r.t. the common CM, at rest. As we see in figure 2, the three Dirac spin operators represent the angular momenta w.r.t. the CC of each quark, so that the addition of the three Dirac spin operators cannot give us the angular momentum of the proton. We need to add for each quark the corresponding angular momentum $(\boldsymbol{r}_i - \boldsymbol{q}_i) \times \boldsymbol{p}_i$, $i = 1, 2, 3$. Taking into account Dirac's electric dipole moment $\boldsymbol{d} = e(\boldsymbol{r} - \boldsymbol{q})$ we see that the lacking term in the proton spin is the addition of the three operators, one for each quark,

$$\frac{i\hbar}{2mc}\boldsymbol{\alpha} \times \boldsymbol{p} = \frac{i\hbar}{2mc}\boldsymbol{\alpha} \times \frac{\hbar}{i}\nabla = \frac{\hbar^2}{2mc}\boldsymbol{\alpha} \times \nabla.$$

This term for each quark is not negligible because when the CM of each quark reaches the CM of the proton, the average value of $p$ is around 325 MeV/$c$. If $q$ represents the quark spinor field, the angular momentum of the proton must contain at least the terms:

$$\sum_{i=1}^{3} q_i^\dagger \left(\frac{\hbar}{2}\boldsymbol{\sigma}_i\right) q_i + \sum_{i=1}^{3} q_i^\dagger \left(\frac{\hbar^2}{2mc}\boldsymbol{\alpha}_i \times \nabla_i\right) q_i$$

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## References

[1] Bargmann V, Michel L and Telegdi VL 1959 Precession of the Polarization of particles moving in a homogeneous electromagnetic field *Phys. Rev. Lett.* **2** 435.
[2] Rivas M 2001 *Kinematical theory of spinning particles* (Dordrecht: Kluwer)
[3] Dirac PAM 1928 The Quantum Theory of the Electron *Proc. Roy. Soc. Lon.* **A117**, 610.
[4] Dirac PAM 1928 The quantum theory of the electron. Part II *Proc. Roy. Soc. Lon.* **A118**, 351.