Is General Relativity a $v/c \to 0$ limit of a Finsler geometry?

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Abstract Gravity is understood as a geometrization of spacetime. But spacetime is also the manifold of the boundary values of the spinless point particle in a variational approach. The manifold of the boundary variables for any mechanical system, instead of being a Riemannian space it is a Finsler metric space such that the variational formalism can always be interpreted as a geodesic problem on this manifold. This manifold is just the flat Minkowski spacetime for the free relativistic point particle. Any interaction modifies its flat Finsler metric. In the spirit of unification of all forces, gravity cannot produce, in principle, a different and simpler geometrization than any other interaction. This implies that that the basic assumption that what gravity produces is a Riemannian metric instead of a Finslerian one is a strong restriction so that general relativity can be considered as a low velocity limit of a more general gravitational theory.

1 The geodesic interpretation of the variational formalism

Let us consider any mechanical system of n degrees of freedom described by a Lagrangian, $L(t,q_i,q_i^{(1)})$. The variational approach means that the path followed by the system makes stationary the action functional

$$\mathscr{A}[q(t)] = \int_{t_1}^{t_2} L(t,q_i,q_i^{(1)})dt,$$

between the initial state $x_1 \equiv (t_1,q_i(t_1))$ and final state $x_2 \equiv (t_2,q_i(t_2))$. If the evolution is described in parametric form $t(\tau)$, $q_i(\tau)$ in terms of some arbitrary parameter τ, then $q_i^{(1)}(\tau) = \dot{q}_i/\dot{t}$, the variational approach will be written as [1]

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$$\int_{\tau_1}^{\tau_2} L(t,q_i,\dot{q}_i/\dot{t})\dot{t}d\tau = \int_{\tau_1}^{\tau_2} \widetilde{L}(x,\dot{x})d\tau, \quad \widetilde{L} = L\dot{t},$$

$\widetilde{L}$ is independent of τ and is a homogeneous function of first degree of the derivatives $\dot{x}$. $\widetilde{L}^2$ is a positive definite homogeneous function of second degree of $\dot{x}$. Therefore $\widetilde{L}^2 = g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j$, and the definite positive metric g_{ij} are computed as [2, 3]

$$g_{ij}(x,\dot{x}) = \frac{1}{2}\frac{\partial^2\widetilde{L}^2}{\partial\dot{x}^i\partial\dot{x}^j} = g_{ji}. \tag{1}$$

The variational formalism looks now

$$\int_{\tau_1}^{\tau_2} \widetilde{L}(x,\dot{x})d\tau = \int_{\tau_1}^{\tau_2} \sqrt{\widetilde{L}^2(x,\dot{x})}d\tau = \int_{\tau_1}^{\tau_2} \sqrt{g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j}d\tau = \int_{x_1}^{x_2} ds,$$

where ds is the arc length on the X manifold w.r.t. the metric g_{ij}. The variational statement has been transformed into a geodesic problem with a Finsler metric.

The relativistic point particle of mass m has a kinematical space spanned by time t and the position of the point $\mathbf{r}$, so that the free Lagrangian $\widetilde{L}_0 = \pm mc\sqrt{c^2\dot{t}^2 - \dot{\mathbf{r}}^2}$, is a homogeneous function of first degree of the derivatives $\dot{t}$ and $\dot{\mathbf{r}}$.

2 Examples of Finsler spaces

In the case of a uniform gravitational field $\mathbf{g}$, the dynamical equations $d\mathbf{p}/dt = m\mathbf{g}$, come from the Lagrangian

$$\widetilde{L}_g = \widetilde{L}_0 + m\mathbf{g}\cdot\mathbf{r}\dot{t}. \tag{2}$$

It corresponds from (1) to an evolution in a spacetime with the Finsler metric:

$$g_{00} = m^2c^2 + m^2(\mathbf{g}\cdot\mathbf{r})^2/c^2 - \frac{m^2c(\mathbf{g}\cdot\mathbf{r})}{(c^2-u^2)^{3/2}}(2c^2-3u^2),$$

$$g_{11} = -m^2c^2 + \frac{m^2c(\mathbf{g}\cdot\mathbf{r})}{(c^2-u^2)^{3/2}}(c^2-u_y^2-u_z^2), \; g_{22} = -m^2c^2 + \frac{m^2c(\mathbf{g}\cdot\mathbf{r})}{(c^2-u^2)^{3/2}}(c^2-u_x^2-u_z^2),$$

$$g_{33} = -m^2c^2 + \frac{m^2c(\mathbf{g}\cdot\mathbf{r})}{(c^2-u^2)^{3/2}}(c^2-u_x^2-u_y^2),$$

$$g_{01} = -\frac{m^2u^2(\mathbf{g}\cdot\mathbf{r})}{(c^2-u^2)^{3/2}}u_x, \quad g_{02} = -\frac{m^2u^2(\mathbf{g}\cdot\mathbf{r})}{(c^2-u^2)^{3/2}}u_y, \quad g_{03} = -\frac{m^2u^2(\mathbf{g}\cdot\mathbf{r})}{(c^2-u^2)^{3/2}}u_z,$$

$$g_{12} = \frac{m^2c(\mathbf{g}\cdot\mathbf{r})}{(c^2-u^2)^{3/2}}u_xu_y, \quad g_{23} = \frac{m^2c(\mathbf{g}\cdot\mathbf{r})}{(c^2-u^2)^{3/2}}u_yu_z, \quad g_{13} = \frac{m^2c(\mathbf{g}\cdot\mathbf{r})}{(c^2-u^2)^{3/2}}u_xu_z.$$

If the velocity is negligible with respect to c, the nonvanishing coefficients are

$$g_{00} = m^2c^2\left(1 - \frac{\mathbf{g}\cdot\mathbf{r}}{c^2}\right)^2, \quad g_{ii} = -m^2c^2\left(1 - \frac{\mathbf{g}\cdot\mathbf{r}}{c^2}\right), \quad i = 1,2,3,$$

where g_{00} is the same as the component of the Rindler metric.

The dynamics of a point particle in a Newtonian potential and its Lagrangian

$$\frac{d\mathbf{p}}{dt} = -\frac{GmM}{r^3}\mathbf{r}, \quad \widetilde{L}_N = \widetilde{L}_0 + \frac{GmM}{cr}c\dot{t}.$$

and from (1) the metric coefficients are

$$g_{00} = m^2c^2 + \frac{G^2m^2M^2}{c^2r^2} - \frac{Gm^2Mc}{r(c^2-u^2)^{3/2}}(2c^2 - 3u^2),$$

$$g_{11} = -m^2c^2 + \frac{Gm^2Mc^3}{r(c^2-u^2)^{3/2}} - \frac{Gm^2Mc(u_y^2+u_z^2)}{r(c^2-u^2)^{3/2}},$$

$$g_{22} = -m^2c^2 + \frac{Gm^2Mc^3}{r(c^2-u^2)^{3/2}} - \frac{Gm^2Mc(u_x^2+u_z^2)}{r(c^2-u^2)^{3/2}},$$

$$g_{33} = -m^2c^2 + \frac{Gm^2Mc^3}{r(c^2-u^2)^{3/2}} - \frac{Gm^2Mc(u_x^2+u_y^2)}{r(c^2-u^2)^{3/2}},$$

$$g_{01} = -\frac{Gm^2Mu^2u_x}{r(c^2-u^2)^{3/2}}, \quad g_{02} = -\frac{Gm^2Mu^2u_y}{r(c^2-u^2)^{3/2}}, \quad g_{03} = -\frac{Gm^2Mu^2u_z}{r(c^2-u^2)^{3/2}},$$

$$g_{12} = \frac{Gm^2Mcu_xu_y}{r(c^2-u^2)^{3/2}}, \quad g_{23} = \frac{Gm^2Mcu_yu_z}{r(c^2-u^2)^{3/2}}, \quad g_{31} = \frac{Gm^2Mcu_zu_x}{r(c^2-u^2)^{3/2}},$$

It is a Finsler metric, which in the case of low velocities it becomes

$$g_{00} = m^2c^2\left(1 - \frac{GM}{c^2r}\right)^2, \quad g_{ii} = -m^2c^2\left(1 - \frac{GM}{c^2r}\right), \quad i = 1,2,3.$$

This corresponds to the static and spherically symmetric Riemannian metric

$$\left(1 - \frac{GM}{c^2r}\right)^2 c^2dt^2 - \left(1 - \frac{GM}{c^2r}\right)(dr^2 + r^2d\Omega^2).$$

This metric is not a vacuum solution of Einstein's equations, so that it cannot be transformed into the Schwarzschild metric in istropic coordinates.

In all the examples, the free Lagrangian $\widetilde{L}_0$ of the spinless particle, has been transformed by the interactions in the Finsler metric

$$\widetilde{L}_0^2 = m^2c^2\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \quad \Rightarrow \quad \widetilde{L}^2 = g_{\mu\nu}(x,\dot{x})\dot{x}^\mu\dot{x}^\nu. \tag{3}$$

The low velocity limit produces a Riemannian approximation which does not give rise to the usual dynamical equations.

However, General Relativity states that gravity modifies the metric of spacetime producing a new (pseudo-)Riemannian metric $g_{\mu\nu}(x)$, which is related through Einstein's equations to the energy momentum distribution $T^{\mu\nu}$. The motion of a point particle is a geodesic on spacetime, and therefore can be treated as a Lagrangian dynamical problem with a Lagrangian

$$\widetilde{L}_g^2 = g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu. \tag{4}$$

In the spirit of unification of all interactions, one is tempted to extend the formulation of gravity (4) to (3) by allowing the metric to be also a function of the derivatives. Otherwise, to assume only a Riemannian metric is to consider that gravity produces a different geometrization than any other interaction. In a region of a uniform gravitational field, the Lagrangian dynamics is equivalent to a geodesic problem where the metric is necessarily a Finsler metric. The elimination of the velocities in the metric coefficients could be interpreted as a low velocity limit of a more general gravitational theory.

3 Conclusions

The manifold of the boundary variables of any Lagrangian system is a Finsler space. Any variational approach is equivalent to a geodesic statement on this manifold. The metric, is a function of the $x \in X$ and $\dot{x}$, depends on the interaction, and to assume that gravitation only produces a modification of the metric which is only a function of the x, is a restriction of a more general formalism.

In all examples we have seen the Finsler structure of spacetime under different gravitational interactions, although the metrics are obtained by pure Lagrangian statements and not by any field equations. The new metrics are true Finsler metrics which in the case of $v/c \to 0$, resemble the metrics obtained in a general relativity formalism but they are not strict vacuum solutions of Einstein's equations. This could suggest some relationship between general relativity and the low velocity limit of the corresponding Finsler structure of the gravitational problems.

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References

1. Rivas M 2001 *Kinematical theory of spinning particles*, (Dordrecht: Kluwer), Chapter 6.
2. Asanov GS 1985 *Finsler geometry, Relativity and Gauge theories*, (Reidel Pub. Co, Dordrecht).
3. Rund H 1973 *The Hamilton-Jacobi theory in the calculus of variations*, (Krieger Pub. Co., N.Y).