The interaction Lagrangian of two spin $1/2$ elementary Dirac particles

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Abstract. It has been recently shown that the spacetime symmetry group of a Dirac particle is larger than the Poincaré group. It also contains spacetime dilations and local rotations. In this work we obtain the interaction Lagrangian for two Dirac particles, which is invariant under this enlarged spacetime group. We analyze the interaction between two particles, and show that it is possible the existence of metastable bound states for two particles of the same charge, provided some initial conditions are fulfilled.

1 Introduction

The variational formalism for describing elementary spinning particles [1] determines that the kinematical variables which define the initial and final states of the evolution necessarily span a homogeneous space of the kinematical group of spacetime symmetries of the theory. The Lagrangian for describing these systems will depend on these kinematical variables and their next order time derivative. If the evolution is described in terms of some invariant evolution parameter τ, then the Lagrangian is a homogeneous function of first degree of the $\tau-$derivatives of all kinematical variables.

For a relativistic particle the spacetime symmetry group is the Poincaré group. In a recent work [2] we have shown that this spacetime symmetry group can be enlarged to include also spacetime dilations and local rotations of the body frame of the particle. This group for the classical particle is $\mathscr{S} = \mathscr{W} \otimes SO(3)_L$, where $\mathscr{W}$ is the Weyl group and $SO(3)_L$ is the group of local rotations of the body frame, which commutes with $\mathscr{W}$. The Lagrangian for a free Dirac particle is invariant under this group $\mathscr{S}$.

The kinematical variables of a Dirac particle are time t, position of a point $\mathbf{r}$, which is the location of the charge, its velocity $\mathbf{u}$ with the constraint $u = c$, the orientation α which can be interpreted as the orientation of a local frame with origin at $\mathbf{r}$ and, finally, the phase β of the internal motion of the

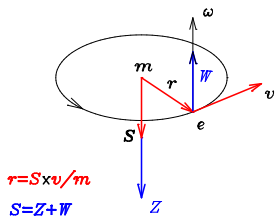

Figure 1. Motion of the charge of the particle, in the centre of mass frame. The total spin S is the sum of the orbital part Z and the rotational part of the body frame W. It is not depicted the local body frame, with origin at point r, which rotates with angular velocity ω. The motion of the charge, with respect to the fixed spin direction, is left-handed. The phase β corresponds to the phase of the circular motion.

charge around the center of mass. The spinning particle has a center of mass **q**, which is always different than the point **r**, such that, for the center of mass observer, the point **r** is moving in circles at the speed of light, around the point **q** in a flat trajectory contained in a plane orthogonal to the spin **S**. The kinematical variable β describes the phase of this internal or *zitterbewegung* motion. The rotation subgroup of $\mathscr{P}$ transforms the kinematical variables **r**, **u** and α among the inertial observers, and the local rotation subgroup $SO(3)_L$ only affects to the local change of particle frame, i.e., to the variables α. Because the Weyl group has no central extensions [3], in the quantum representation the symmetry group becomes $\widetilde{\mathscr{S}} = \mathscr{W} \otimes SU(2)_L$.

2 The interaction Lagrangian. Synchronous description

The Casimir operators of the group $\mathscr{S}$ are the Casimir operator of the Weyl group $\mathscr{W}$ which is the spin of the particle S and the Casimir of the $SO(3)_L$ part, I which corresponds to the spin projection on the body frame. We make the assumption that the structure of an elementary particle is not modified by any interaction so that its intrinsic properties, the spin S and the spin projection on the body frame I cannot be altered by the interaction with an external field or by the presence in its neigbourhood of any other particle.

Let us consider a compound system formed by two spinning particles with the same kind of kinematical variables. Then the kinematical space of

the compound system is spanned by the variables $(t_a, \mathbf{r}_a, \mathbf{u}_a, \alpha_a, \beta_a), a = 1,2$. The Lagrangian of the system will be $L = L_1 + L_2 + L_I$, where the L_a, $a = 1,2$, are the free Lagrangians of each particle and L_I is the interaction Lagrangian we are looking for. Both L_a are invariant under the enlarged group $\mathscr{S}$ and we are going to find an interaction Lagrangian L_I also invariant under $\mathscr{S}$. The general structure of the free Lagrangian L_a of each particle, which only depends on the corresponding kinematical variables, is

$$L_a = T_a \dot{t}_a + \mathbf{R}_a \cdot \dot{\mathbf{r}}_a + \mathbf{U}_a \cdot \dot{\mathbf{u}}_a + \mathbf{W}_a \cdot \omega_a + B_a \dot{\beta}_a$$

where $T_a = \partial L_a / \partial \dot{t}_a$, $\mathbf{R}_a = \partial L_a / \partial \dot{\mathbf{r}}_a$, $\mathbf{U}_a = \partial L_a / \partial \dot{\mathbf{u}}_a$, $\mathbf{W}_a = \partial L_a / \partial \omega_a$ and $B_a = \partial L_a / \partial \dot{\beta}_a$, because of the homogeneity of L_a in terms of the $\tau-$derivatives of the kinematical variables. The spin and the spin projection on the body frame for each particle, are

$$\mathbf{S}_a = \mathbf{u}_a \times \mathbf{U}_a + \mathbf{W}_a, \quad I_{ai} = \mathbf{e}_{\mathbf{a}i} \cdot \mathbf{W}_a$$

where $\mathbf{e}_{\mathbf{a}i}$, $i = 1,2,3$ are three orthogonal unit vectors with origin at point $\mathbf{r}_a$.

The interaction Lagrangian between these two particles L_I will be in general a function of the kinematical variables of both particles and of their $\tau-$derivatives. If both intrinsic properties S_a and I_a of each particle are not modified by any interaction then the interaction Lagrangian cannot be a function of the derivatives of the kinematical variables $\dot{\mathbf{u}}_a$ and ω_a, $a = 1,2$. In this case the functions $\mathbf{U}_a$ and $\mathbf{W}_a$, which give rise to the definition of the spin, are obtained only from the corresponding free Lagrangian L_a.

Then, as far as the $\tau-$derivatives of the kinematical variables are concerned, the interaction Lagrangian L_I will only depend on the variables $\dot{t}_a$, $\dot{\mathbf{r}}_a$ and $\dot{\beta}_a$, $a = 1,2$. In addition to this, it will also be a function of the kinematical variables t_a, $\mathbf{r}_a$, $\mathbf{u}_a$ and β_a, but not of α_a because the general dependence of the Lagrangian on the orientation variables is only through the dependence on the corresponding angular velocity, which is absent in the interaction Lagrangian. The phase β_a is the phase of the orbital motion and of the oriented body frame, then the interaction Lagrangian is also independent of β_a. The interaction Lagrangian will thus be a function of

$$L_I = L_I(t_a, \mathbf{r}_a, \dot{t}_a, \dot{\mathbf{r}}_a),$$

and a homogeneous function of first degree of the derivatives $\dot{t}_a, \dot{\mathbf{r}}_a$, $a = 1,2$. If it is going to be invariant under $\mathscr{W} \otimes SU(2)$, if we call $x_a^\mu \equiv (t_a, \mathbf{r}_a)$, then we get

$$L_I = g \sqrt{\frac{\eta_{\mu\nu} \dot{x}_1^\mu \dot{x}_2^\nu}{\eta_{\mu\nu}(x_1^\mu - x_2^\mu)(x_2^\nu - x_1^\nu)}} = g \sqrt{\frac{\dot{t}_1 \dot{t}_2 - \dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2}{(\mathbf{r}_2 - \mathbf{r}_1)^2 - (t_2 - t_1)^2}}$$

where g is a coupling constant with dimensions of action and $\eta_{\mu\nu}$ is Minkowski's metric tensor. Incidentaly we can also see that the Lagrangian is also invariant under the interchange $1 \leftrightarrow 2$.

Once an inertial observer is fixed it can make a synchronous time description, i.e. to use as evolution parameter the own observer's time t which is the same as the two time variables t_1 and t_2. In this case

$$L_I = g\sqrt{\frac{1-\mathbf{u}_1\cdot\mathbf{u}_2}{(\mathbf{r}_2-\mathbf{r}_1)^2}} = g\frac{\sqrt{1-\mathbf{u}_1\cdot\mathbf{u}_2}}{r} \tag{1}$$

where $r = |\mathbf{r}_1 - \mathbf{r}_2|$ is the instantaneous separation between the corresponding charges and $\mathbf{u}_a = \dot{\mathbf{r}}_a$ the velocity of the charge of particle a.

An average over the charge position and velocity in the center of mass of one of the particles implies that the interaction becomes the instantaneous Coulomb interaction between the center of mass of the first particle (which is also the average position of its charge) and the charge position of the other. The average over the other then corresponds to the interaction of two spinless point particles. This average about the internal motion is equivalent to neglect the spin structure and it suggests that $g \sim \pm e^2$.

3 Analysis of a two-particle system

The dynamical equation of a free Dirac particle is a fourth-order differential equation for the position of the charge which can be separated into a system of coupled second order differential equations for the center of mass $\mathbf{q}$ and center of charge $\mathbf{r}$ in the form:[4]

$$\ddot{\mathbf{q}} = 0, \quad \ddot{\mathbf{r}} = \frac{1-\dot{\mathbf{q}}\cdot\dot{\mathbf{r}}}{(\mathbf{q}-\mathbf{r})^2}(\mathbf{q}-\mathbf{r})$$

In the case of interaction the second equation remains the same because it corresponds to the definition of the center of mass position which is unchanged by the interaction. The first equation for particle a is replaced by $d\mathbf{p}_a/dt = \mathbf{F}_a$ where $\mathbf{p}_a$ is the linear momentum of particle a and the force $\mathbf{F}_a$, computed from the interaction Lagrangian (1) takes the form:

$$\mathbf{F}_1 = -g\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}\sqrt{1-\mathbf{u}_1\cdot\mathbf{u}_2} + \frac{d}{dt}\left(\frac{g\mathbf{u}_2}{2|\mathbf{r}_1-\mathbf{r}_2|\sqrt{1-\mathbf{u}_1\cdot\mathbf{u}_2}}\right).$$

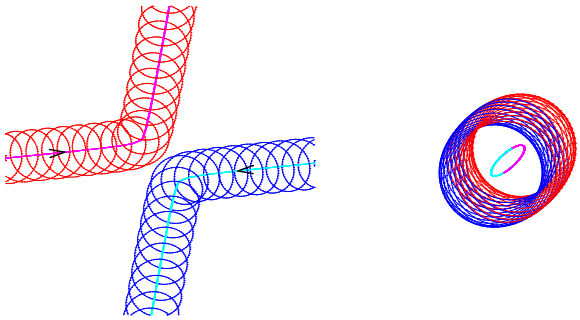

Figure 2. (a) The trajectories of the centers of mass and charge of two spinning particles with initial center of mass velocity $v = 0.1$ and a small impact parameter.(b) Bound motion of the centers of mass and charge of two spinning particles with parallel spins and with a center of mass velocity $v \leq 0.01$, for an initial separation between the centers of masses $0.2\times$Compton's wavelength.

Then the system of differential equations to be solved are

$$\ddot{\mathbf{q}}_a = \frac{\alpha}{\gamma(\dot{\mathbf{q}}_a)}\left(\mathbf{F}_a - \dot{\mathbf{q}}_a(\mathbf{F}_a\cdot\dot{\mathbf{q}}_a)\right) \tag{2}$$

$$\ddot{\mathbf{r}}_a = \frac{1-\dot{\mathbf{q}}_a\cdot\dot{\mathbf{r}}_a}{(\mathbf{q}_a-\mathbf{r}_a)^2}(\mathbf{q}_a-\mathbf{r}_a), \quad a=1,2 \tag{3}$$

where $\alpha = g/m$ is the fine structure constant in the case of electromagnetic interaction and once all the variables are dimensionless. All terms of equation (2) which depend on the acceleration of the charges have to be replaced by the expressions of (3).

See in figure 2(a) the sccatering of two equal charged particles with parallel spins and in figure 2(b) the bound motion of both electrons when their centers of mass are separated below Compton's wavelength.

References

[1] Rivas M 2001 *Kinematical theory of spinning particles*, (Dordrecht: Kluwer). Some other works related to this formalism can be obtained through the web page `http://tp.lc.ehu.es/martin.htm`.

[2] Rivas M 2006 *J. Phys. A: Math. Gen.* **39** 4291 (*Preprint* hep-th/0511244).

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[4] Rivas M 2003 *J. Phys. A: Math. Gen.* **36** 4703 (*Preprint* physics/0112005).