IL NUOVO CIMENTO VOL. 72 B, N. 1 11 Novembre 1982

A Quantitative Analysis of the Trouton-Noble Experiment.

A. HERNÁNDEZ, M. RIVAS and J. M. AGUIRREGABIRIA

Departamento de Física, Facultad de Ciencias, Universidad del País Vasco
Apdo. 644, Bilbao, Spain

(ricevuto il 9 Giugno 1982)

Summary. — We analyse quantitatively, in the frame of the usual relativistic theories of continuum media and electrodynamics, a simplified version of the Trouton-Noble experiment. It is checked at first, in the v^2/c^2 approximation as well as in the exact form, that the negative result of the experiment is a consequence of the application of the angular-momentum theorem to the electromagnetic subsystem. An analogous result is obtained when analysing the mechanical subsystem. The apparent paradox can be also explained by considering the angular-momentum conservation law of the closed total system. Finally, energy considerations of the closed total system lead, obviously, to the same conclusion.

1. – Introduction.

It is well known that one of the experiments to verify the Earth motion through the ether was that proposed by TROUTON [1] in 1902, and its negative result has been considered as a proof in favour of the special theory of relativity.

The original idea is the following: when a charged plane-plate capacitor is moving with speed $\boldsymbol{v}$ with respect to the ether, if $\boldsymbol{v}$ is parallel to the plates, there exists a magnetic field $\boldsymbol{B}$, besides the electric field $\boldsymbol{E}$ between them, such that the electromagnetic energy

$$U_{\text{I}} = \frac{1}{2}\int\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \mathrm{d}v = U_{\text{E}}(1+\beta^2)\,,$$

(1) F. T. TROUTON: *Sci. Trans. R. Dublin Soc.*, **7**, 379 (1902).

where U_E is the energy of the electric field alone, $\beta = v/c$ and c the speed of light. If $\boldsymbol{v}$ is orthogonal to the plates, the electromagnetic energy is $U_\perp = U_E$, leading to Trouton's conclusion that, when a capacitor is hang up, moving freely near the Earth's surface (and consequently being dragged with it in its motion through the ether), it will rotate until the energy attains a minimum, *i.e.* the capacitor plates become orthogonal to $\boldsymbol{v}$.

An alternative analysis was proposed by SEARLE (2) by considering the torques and can be found in the Panofsky and Phillips (3) book. It is as follows. If we consider a pointlike charge $+q$ in one of the plates and its image $-q$ in the other, each one moves in the electromagnetic field of the other, being acted, respectively, by equal but opposite forces with different lines of action, producing consequently a torque that will rotate the capacitor until its plates become orthogonal to the speed.

If one considers the usual relativistic electrodynamics, a torque still appears, giving rise to an apparent paradoxical situation, since the experiment shows that the capacitor does not rotate.

The problem has been analysed from different points of view. BUTLER (4) has proposed a new definition of the electromagnetic-energy density, which implies that the electromagnetic energy of the capacitor is independent of its orientation with respect to the velocity; STRNAD (5) gives an interpretation in terms of the virtual-work principle; GRØN (6) considers the problem as an example to which the asynchronous formulation of relativistic statics can be applied, and FURRY (7) thinks that it is a problem in which a hidden momentum, required by the relativity theory, explains the null result of the experiment in the way of reasoning of Pauli (8).

The aim of the present work is to show by explicit calculation how the usual relativistic theories of electromagnetism and continuum media allow one to completely explain the apparent paradox. In sect. **2**, a simplified version of the problem will be stated, by using instead of a capacitor a system consisting of two opposite electric charges joined by a nonconducting rod. In sect. **3** a solution in the β^2 approximation will be given. In sect. **4** the exact expression of the angular momentum associated to the electromagnetic field will be obtained by direct calculation, checking that its time derivative matches with the external torque that acts on the electromagnetic subsystem. An analogous answer to the paradox, but concerning this time the mechanical subsystem, will

(2) G. F. C. SEARLE: *Phylos. Trans. R. Soc. London Ser. A*, **187**, 708 (1896).
(3) W. K. H. PANOFSKY and M. PHILLIPS: *Classical Electricity and Magnetism*, 2nd edition (Reading, Mass., 1962), p. 274, 349.
(4) J. W. BUTLER: *Am. J. Phys.*, **36**, 936 (1968); **37**, 1258 (1969).
(5) J. STRNAD: *Contemp. Phys.*, **11**, 59 (1970).
(6) Ø. GRØN: *Nuovo Cimento B*, **17**, 141 (1973).
(7) W. H. FURRY: *Am. J. Phys.*, **37**, 621 (1969).
(8) W. PAULI: *Theory of Relativity* (New York, N. Y., 1958), p. 127.

be given in sect. **5**, remarking how it can be explained in terms of the total closed system. We conclude in sect. **6** with another explanation based upon the energetic analysis of both subsystems.

2. – The problem.

Let us consider, instead of a plane-plate charged capacitor, a device formed by two pointlike electric charges of value $q_1 = +q$ and $q_2 = -q$, respectively, bound together by means of a nonconducting rod (9) and at rest in a reference frame K^*. Let us assume that the charge q_1 is at the point of co-ordinates $(a^*, b^*, 0)$ and q_2 at the point $(-a^*, -b^*, 0)$, $l^* = 2(a^{*2} + b^{*2})^{\frac{1}{2}}$ being the rod length as measured in K^*. We shall assume the rod thickness small enough to be considered negligible.

Let us assume that the frame K^* moves along the OX-axis with speed $\boldsymbol{v}$ with respect to another inertial frame K, such that space-time measurements in both K and K^* are related by means of the special Lorentz transformation

$$x = \gamma(x^* + vt^*)\,, \quad y = y^*\,, \quad z = z^*\,, \quad t = \gamma\left(t^* + \frac{vx^*}{c^2}\right), \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}. \tag{1}$$

For the observer in K^*, the interaction between the charges is Coulomb-like and consequently the total torque on the rod due to the electromagnetic forces is zero and the rod (and charges) remain at rest. However, for the observer in K, each charge is subject to the action of the electric and magnetic field created by the other, such that the forces acting on each other, although equal but opposite, give rise to a nonvanishing torque (fig. 1).

In fact, according to the relativistic transformation of the fields, the electric fields created at a point given by the position vector $\boldsymbol{r}$ are (10)

$$\boldsymbol{E}_i(\boldsymbol{r}) = \frac{\gamma q_i}{4\pi\varepsilon_0}\frac{\boldsymbol{s}_i}{[s_i^2 + (\gamma^2/c^2)(\boldsymbol{v}\cdot\boldsymbol{s}_i)^2]^{\frac{3}{2}}} \qquad (i = 1, 2) \tag{2a}$$

and the magnetic fields

$$\boldsymbol{B}_i(\boldsymbol{r}) = \frac{\boldsymbol{v}}{c^2}\wedge\boldsymbol{E}_i(\boldsymbol{r}) \qquad (i = 1, 2)\,, \tag{2b}$$

(9) This device is that used by PANOFSKY and PHILLIPS (ref. (3), p. 274) and STRNAD (ref. (5)).
(10) W. K. H. PANOFSKY and M. PHILLIPS: *Classical Electricity and Magnetism*, 2nd edition (Reading, Mass., 1962), p. 346.

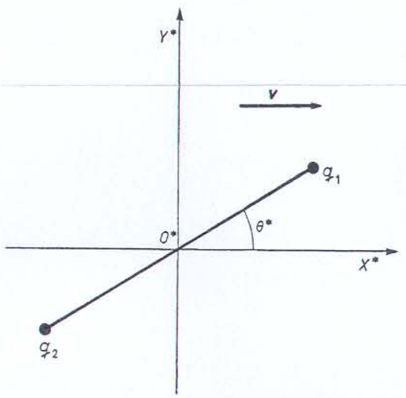

Fig. 1. – The rod at rest in K^* moves along OX with speed $\boldsymbol{v}$ with respect to the frame K.

where $\boldsymbol{s}_i = \boldsymbol{r} - \boldsymbol{r}_i$, $\boldsymbol{r}_i$ being the position vector of the charge q_i. The force acting on the charge q_1 has two components

(3) $$F_{1x} = -\frac{1}{2\pi\varepsilon_0}\frac{q^2 a^*}{l^{*3}}, \qquad F_{1y} = -\frac{1}{2\pi\varepsilon_0}\frac{q^2 b^*}{\gamma l^{*3}},$$

and that on q_2, $\boldsymbol{F}_2 = -\boldsymbol{F}_1$, where the asterisk refers to magnitudes measured in K^*, giving rise to a total torque with respect to the origin of the frame K, $\boldsymbol{M} = M_z \boldsymbol{k}$, $\boldsymbol{k}$ being the unit vector along the OZ-axis, and

(4) $$M_z = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{l^*}\beta^2 \sin 2\theta^*,$$

θ^* is the angle formed by the rod and the OX^*-axis (see fig. 1).

We are facing now an apparent paradox. In frame K^* the total torque is null and, consequently, according to the angular-momentum theorem, the rod will stay in equilibrium. On the contrary, in the frame K there exists a total torque $\boldsymbol{M}$ that does not modify the mechanical angular momentum.

3. – Solution up to the β^2 approximation.

The paradox disappears when the two sybsystems, in the sense of Møller [11], *i.e.* the electromagnetic subsystem formed by the electric and magnetic fields

[11] C. Møller: *The Theory of Relativity*, 2nd edition (Oxford, 1972), Chapt. 7.

and the mechanical subsystem formed by the rod, that together form the total system, are analysed separately.

Since electromagnetic forces do not obey the principle that the action equals reaction, it is necessary, in order to maintain the linear and angular momentum and energy conservation laws, to assign these magnitudes to the electromagnetic field, giving rise to the Poynting theorem in electrodynamics and, in general, to the energy-momentum tensor formalism.

In this case, the angular-momentum theorem for the electromagnetic subsystem implies that the variation per unit time of the angular momentum associated to the electromagnetic field will be equal to the torque of the external forces that act on this subsystem, *i.e.* equal to $-\boldsymbol{M}_z$.

The angular momentum of the electromagnetic field in empty space, with respect to the origin, $\boldsymbol{L}_{\text{e.m.}}$ is given by

(5) $$\boldsymbol{L}_{\text{e.m.}} = \frac{1}{c^2}\int (\boldsymbol{r}\wedge\boldsymbol{S})\,\mathrm{d}V,$$

where $\boldsymbol{S} = (1/\mu_0)\boldsymbol{E}\wedge\boldsymbol{B}$ is Poynting's vector.

Expression (5) can be written in another form if certain conditions are satisfied. In fact, in stationary or quasi-stationary situations, in order to ensure that the potential vector $\boldsymbol{A}$ satisfies the Coulomb gauge and if either the electric field $\boldsymbol{E}$ decreases with distance faster than R^{-2}, or $\boldsymbol{A}$ faster than R^{-1} (or even both satisfy these conditions), the angular momentum of a system of pointlike charges and currents is given by [12]

(6) $$\boldsymbol{L}_{\text{e.m.}} = \sum_i \boldsymbol{r}_i \wedge q_i \boldsymbol{A}(\boldsymbol{r}_i),$$

where $\boldsymbol{A}(\boldsymbol{r}_i)$ is the potential vector at the location $\boldsymbol{r}_i$ of the charge q_i.

When $v \ll c$, it can be accepted up to the β^2 approximation that the electromagnetic field of the two charges is quasi-stationary [13]; furthermore, the electric field at large distances decreases like that of a pointlike dipole, *i.e.* faster than R^{-3}. Consequently expression (6) holds for the angular momentum of the electromagnetic field created by two charges, in this approximation.

Symmetry considerations allow us not to consider the self-terms in order to compute the angular momentum with respect to the origin. Even if the symmetry considerations would fall, since we are considering the Lorentz force and not the self-forces (there is no acceleration), we must disregard the self-terms anyway. In our system, expression (6) takes the form

(7) $$\boldsymbol{L}_{\text{e.m.}} = \boldsymbol{r}_1 \wedge q_1 \boldsymbol{A}_{12} + \boldsymbol{r}_2 \wedge q_2 \boldsymbol{A}_{21},$$

[12] J. M. Aguirregabiria and A. Hernández: *Eur. J. Phys.*, **2**, 168 (1981).
[13] J. D. Jackson: *Classical Electrodynamics* (New York, N. Y., 1962), p. 411.

where $\boldsymbol{A}_{ij}$ is the potential vector created by the charge q_j at the location of the charge q_i and has the value [13]

$$A_{12} = -\frac{\mu_0}{8\pi}\frac{q}{l}\left[\boldsymbol{v} + \frac{(\boldsymbol{v}\cdot\boldsymbol{l})\,\boldsymbol{l}}{l^2}\right], \tag{8}$$

since $\boldsymbol{l} = 2((a^*/\gamma)\,\boldsymbol{i} + b^*\boldsymbol{j}) = \boldsymbol{r}_2 - \boldsymbol{r}_1$ and $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors along OX and OY, respectively. Since $\boldsymbol{A}_{12} = -\boldsymbol{A}_{21}$ and

$$\boldsymbol{r}_1 = \boldsymbol{v}t + \boldsymbol{l}/2\,, \qquad \boldsymbol{r}_2 = \boldsymbol{v}t - \boldsymbol{l}/2\,, \tag{9}$$

expressing (8) in magnitudes measured in K^* and disregarding terms of order greater than β^2, we get that the variation of the electromagnetic angular momentum per unit time is just $-M_z$.

The angular momentum could also be calculated from the Darwin Lagrangian for two interacting particles [13], which includes relativistic effects up to order β^2 and is given by

$$\mathscr{L}_{\text{int}} = \frac{q_1 q_2}{4\pi\varepsilon_0 l}\left\{-1 + \frac{1}{2c^2}\left[\boldsymbol{v}_1\cdot\boldsymbol{v}_2 + \frac{(\boldsymbol{v}_1\cdot\boldsymbol{l})(\boldsymbol{v}_2\cdot\boldsymbol{l})}{l^2}\right]\right\} \tag{10}$$

and leads to the same expression for $\boldsymbol{L}_{\text{e.m.}}$ as (7).

4. – Angular momentum of the electromagnetic field.

Since in the problem at hand the particles move at constant speed, no radiation effects have to be considered and consequently the force between the particles is the Lorentz force instead of the Lorentz-Dirac force, which is equivalent to withdrawing the self-terms in expression (5), which, by writing $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$ and $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$, becomes

$$\boldsymbol{L}_{\text{e.m.}} = \int_{R^3} \varepsilon_0 \boldsymbol{r}\wedge[(\boldsymbol{E}_1\wedge\boldsymbol{B}_2) + (\boldsymbol{E}_2\wedge\boldsymbol{B}_1)]\,\mathrm{d}V\,. \tag{11}$$

Symmetry considerations also lead to expression (11) for the electromagnetic angular momentum.

By using the Cartesian co-ordinates

$$\begin{cases} \boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}\,, & \boldsymbol{s}_1 = \boldsymbol{r} - \boldsymbol{R} - \dfrac{\boldsymbol{l}}{2}\,,\\ \boldsymbol{R} = \boldsymbol{v}t\,, & \boldsymbol{s}_2 = \boldsymbol{r} - \boldsymbol{R} + \dfrac{\boldsymbol{l}}{2}\,,\\ \dfrac{\boldsymbol{l}}{2} = a\boldsymbol{i} + b\boldsymbol{j}\,, & a = a^*/\gamma\,, \quad b = b^*\,, \end{cases} \tag{12}$$

the angular momentum $\boldsymbol{L}_{\text{e.m.}}$ can be written in the form

$$\boldsymbol{L}_{\text{e.m.}} = N\int_{R^3} f_1\cdot f_2\cdot\boldsymbol{h}\,\mathrm{d}V\,, \tag{13}$$

where

$$\begin{cases} N = \dfrac{\gamma q_1 q_2}{(4\pi)^2 c^2 \varepsilon_0}\,, \quad \boldsymbol{h} = h_x\boldsymbol{i} + h_y\boldsymbol{j} + h_z\boldsymbol{k}\,,\\ f_1 = [\gamma^2(x - vt - a)^2 + (y - b)^2 + z^2]^{-\frac{3}{2}}\,,\\ f_2 = [\gamma^2(x - vt + a)^2 + (y + b)^2 + z^2]^{-\frac{3}{2}}\,,\\ h_x = -2abvz\,, \quad h_y = 2vz[x^2 + y^2 + z^2 - xvt - b^2]\,,\\ h_z = -2vy[x^2 + y^2 + z^2 - xvt - b^2] + 2abvx \end{cases} \tag{14}$$

and the integral is extended to the whole three-dimensional space, because of the null rod thickness.

By integration in the z-variable, the contributions of h_x and h_y vanish since they are odd functions of z. Consequently, the electromagnetic angular momentum is along the OZ-axis. The tedious calculation of the integral is given in the appendix. The final result obtained for $\boldsymbol{L}_{\text{e.m.}} = L_{\text{e.m.}}\boldsymbol{k}$ is

$$L_{\text{e.m.}} = -\frac{\gamma q^2 abv^2 t}{8\pi\varepsilon_0 c^2[\gamma^2 a^2 + b^2]^{\frac{3}{2}}}\,. \tag{15}$$

The time derivative of $L_{\text{e.m.}}$ is just $-M_z$, as can be easily checked when expressing the right-hand side of (15) in terms of magnitudes measured in the frame K^*.

Boyer [14] has proven that the variation per unit time of the electromagnetic angular momentum of two charges moving at constant speed is equal to the torque, changed in sign, of the electromagnetic forces that act on them, without explicitly computing the mentioned angular momentum.

5. – The mechanical subsystem.

We have mentioned in sect. **2** that the rod is, as measured in the frame K, under the action of two equal but opposite forces with different lines of action. According to the usual relativistic theory of continuum media [15], there will

[14] T. H. Boyer: *Am. J. Phys.*, **39**, 257 (1971).
[15] C. Møller: *The Theory of Relativity*, 2nd. edition (Oxford, 1972), p. 203.

be, stored in the rod, an angular momentum, its z-component being

$$L_z = \int_\omega (x g_y - y g_x)\, d\omega\,, \tag{16}$$

where [16]

$$g_x = \gamma^2 \varrho^* v + \frac{\gamma^2 v}{c^2} t^{*xx}\,, \qquad g_y = \frac{\gamma v}{c^2} t^{*yx}\,, \tag{17}$$

ϱ^* and $\overline{\overline{t^*}}$ being, respectively, the rest mass density and the elastic stress tensor in the frame K^* and ω the volume of the rod. Because of the angular momentum associated to the elastic stresses, the rod can stay in equilibrium in the frame K, without vanishing of the torque of the external forces.

At the time t, expression (16) of L_z leads to

$$L_z = \int_{vt-a^*/\gamma}^{vt+a^*/\gamma} (\gamma v/c^2)\, dx \int_{\sigma_x} x t^{*yx}\, d\sigma_x - \int_{vt-a^*/\gamma}^{vt+a^*/\gamma} (\gamma^2 v/c^2)\, dx \int_{\sigma_x} y(t^{*xx} + \varrho^* c^2)\, d\sigma_x\,, \tag{18}$$

where σ_x is the rod cross-section of abscissa x. Since the rod is in equilibrium and of negligible thickness, we can write

$$\int_{\sigma_x^*} t^{*yx}\, d\sigma_x^* = F_y^*\,, \qquad \int_{\sigma_x^*} t^{*xx}\, d\sigma_x^* = F_x^*\,. \tag{19}$$

One can see that the second integral on the right-hand side of (18) is null. The first one leads for L_z to the value of

$$L_z = \frac{1}{8\pi\varepsilon_0} \frac{q^2}{l^*} \beta^2 t \sin 2\theta^* \tag{20}$$

and, consequently, we check that for the mechanical subsystem the time derivative of the angular momentum coincides with the torque of external forces given by (4).

This result that the time derivative of the angular momentum, stored in an elastic solid in equilibrium, equals the torque of external forces has been shown previously for a particular as well as for a general case [17].

An alternative analysis consists in considering the system as a whole, *i.e.* the rod and the electromagnetic field, which is then a closed system, the z-component of its angular momentum being the sum of (15) and (20) which is just

([16]) C. MØLLER: *The Theory of Relativity*, 2nd edition (Oxford, 1972), p. 190.
([17]) A. CHAMORRO and A. HERNÁNDEZ: *Lett. Nuovo Cimento*, **22**, 553 (1978); G. CAVALLERI and G. SPINELLI: *Lett. Nuovo Cimento*, **26**, 261 (1979); G. SPINELLI: *Nuovo Cimento B*, **54**, 145 (1979).

zero at every time, as was expected because of the symmetry of the system and of the conveniently chosen co-ordinate axis. We can arrive alternatively at this conclusion, due to the fact that, in the rod rest frame, the linear momentum, the angular momentum and the centre of mass are null and consequently the angular momentum is also null in whichever inertial frame.

6. – Energy analysis.

The problem can also be solved under energy considerations. In fact, the energies of the electromagnetic and mechanical subsystems depend on the orientation of the rod with respect to the direction of motion, but the energy of the total closed system is independent of that orientation, as will be seen in the following.

When the self-terms are excluded, the electromagnetic energy of the two particles is given by

$$U_{\text{e.m.}} = \int_{R^3} \left[\varepsilon_0 \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 + \frac{1}{\mu_0} \boldsymbol{B}_1 \cdot \boldsymbol{B}_2\right] dV\,, \tag{21}$$

which, after considering relations (2) and (12), is transformed into

$$U_{\text{e.m.}} = \int_{R^3} \frac{\varepsilon_0 q_1 q_2}{(4\pi\varepsilon_0)^2} f_1 \cdot f_2 \cdot W\, dV\,, \tag{22}$$

where f_1 and f_2 are the same as those of formula (14) and

$$W = \gamma^2[(x - vt)^2 - a^2] + (2\gamma^2 - 1)[y^2 + z^2 - b^2]\,. \tag{23}$$

By a similar procedure as that followed in the appendix for the calculation of $\boldsymbol{L}_{\text{e.m.}}$, we arrive, after a long integration process, to the result for $U_{\text{e.m.}}$

$$U_{\text{e.m.}} = -\frac{q^2}{4\pi\varepsilon_0 l^*} \gamma [1 + \beta^2 \cos^2\theta^*]\,. \tag{24}$$

The energy of the mechanical sybsystem is [16]

$$U_{\text{me}} = \int_\omega (\varrho^* c^2 + \beta^2 t^{*xx}) \gamma^2\, d\omega\,. \tag{25}$$

Expression (25) can be written in the form

$$U_{\text{me}} = \int_\omega \gamma^2 \varrho^* c^2\, d\omega + \int_{vt-a^*/\gamma}^{vt+a^*/\gamma} \gamma^2 \beta^2\, dx \int_{\sigma_x} t^{*xx}\, d\sigma_x\,. \tag{26}$$

The first integral on the right-hand side of (26) has the value γH^*, where $H^* = \int_{\omega^*} \varrho^* c^2 \, d\omega$ is the rest energy of the rod, and the second term gives $(q^2\gamma/4\pi\varepsilon_0 l^*)\beta^2 \cos^2\theta^*$, and thus the energy of the closed system is

$$U = U_{\text{e.m.}} + U_{\text{me}} = \gamma\left(H^* - \frac{q^2}{4\pi\varepsilon_0 l^*}\right), \tag{27}$$

which is independent of the rod orientation, and one cannot expect the rod to rotate. This result was the expected one since the term in brackets in expression (27) is just the total rest energy of the total system and, for closed systems, the total energy transforms like the time component of a 4-vector.

Appendix

To calculate (13), we make a change of co-ordinates such that the two charges are located, respectively, at the points $(\alpha, 0, 0)$ and $(-\alpha, 0\ 0)$ since $\alpha = (\gamma^2 a^2 + b^2)^{\frac{1}{2}}$. This change is

$$X = \frac{\gamma^2 a}{\alpha}(x - vt) + \frac{b}{\alpha} y ,$$

$$Y = -\frac{b}{\alpha}\gamma(x - vt) + \frac{\gamma a}{\alpha} y ,$$

$$Z = z .$$

Since the Jacobian of this transformation is γ^{-1} and some of the terms that appear are odd in X or Y and consequently of vanishing contribution, we arrive at

$$L_z = G \int f_1 \cdot f_2 (\alpha^2 - X^2 + Y^2) \, dX \, dY \, dZ ,$$

where

$$G = \frac{q_1 q_2 a b \gamma v^2 t}{8\pi^2 c^2 \varepsilon_0 \alpha^2} ,$$

$$f_1 = [(X - \alpha)^2 + Y^2 + Z^2]^{-\frac{3}{2}} , \quad f_2 = [(X + \alpha)^2 + Y^2 + Z^2]^{-\frac{3}{2}} .$$

By passing now to spherical co-ordinates,

$$X = r\cos\theta ,$$

$$Y = r\sin\theta\cos\varphi ,$$

$$Z = r\sin\theta\sin\varphi .$$

After integration in φ and making $\cos\theta = u$ and $r = \alpha s$, we get

$$L_z = \frac{G\pi}{\alpha} \int \frac{2s^2 + s^4 - 3s^4 u^2}{[(s^2+1)^2 - 4s^2u^2]^{\frac{3}{2}}} \, du \, ds .$$

Let us first integrate in u and call

$$I_1 = \int_{-1}^{1} \frac{2s^2 + s^4}{[(s^2+1)^2 - 4s^2u^2]^{\frac{3}{2}}} \, du = \frac{2(2s^2 + s^4)}{(s^2+1)^2(s^2-1)} \operatorname{sgn}(s-1) ,$$

$$I_2 = \int_{-1}^{1} \frac{-3s^4u^2}{[(s^2+1)^2 - 4s^2u^2]^{\frac{3}{2}}} \, du = I_3 + \frac{3s}{4} \arcsin\frac{2s}{s^2+1} ,$$

where

$$I_3 = \frac{-3s^2}{2(s^2-1)} \operatorname{sgn}(s-1) ,$$

$\operatorname{sgn}(x)$ being the function sign of x.

By proceeding to integrate by parts in s,

$$\frac{3}{4}\int_0^\infty s \arcsin\frac{2s}{s^2+1} \, ds = \frac{3}{4}\left[\frac{s^2}{2}\arcsin\frac{2s}{s^2+1}\right]_0^\infty + \int_0^\infty I_4 \, ds$$

with

$$I_4 = \frac{3s^2}{4(s^2+1)} \operatorname{sgn}(s-1) .$$

If we call $I = I_1 + I_3 + I_4$, we will have

$$\frac{\alpha L_z}{G\pi} = \int_0^\infty I \, ds + \lim_{s\to\infty} \frac{3}{4}\left(\frac{s^2}{2}\arcsin\frac{2s}{s^2+1}\right)$$

and

$$\int_0^\infty I \, ds = \int_0^1 I \, ds + \int_1^\infty I \, ds ,$$

where

$$\int_0^1 I \, ds = \frac{1}{2} - \frac{\pi}{16} ,$$

$$\int_1^\infty I \, ds = \frac{1}{2} + \frac{\pi}{16} - \lim_{s\to\infty} \frac{3s}{4}$$

and the two limit terms cancel out:

$$\lim_{s\to\infty}\left\{\frac{3s}{4}-\frac{3}{4}\left[\frac{s^2}{2}\arcsin\frac{2s}{s^2+1}\right]\right\}=0\,,$$

thus concluding that $L_z = G\pi/\alpha$, and, in terms of the values of the two charges,

$$L_z = -\frac{\gamma q^2 abv^2 t}{8\pi\varepsilon_0 c^2[\gamma^2 a^2+b^2]^{\frac{3}{2}}}\,.$$

● RIASSUNTO (*)

Si analizza quantitativamente, nel contesto delle usuali teorie relativistiche dei mezzi nel continuo e nell'elettrodinamica, una versione semplificata dell'esperimento di Trouton-Noble. Si è mostrato per prima cosa, nell'approssimazione v^2/c^2 così come nella forma esatta, che il risultato negativo dell'esperimento è una conseguenza dell'applicazione del teorema dell'impulso angolare al sottosistema elettromagnetico. Un risultato analogo è ottenuto quando si analizza un sottosistema meccanico. L'apparente paradosso può essere anche spiegato considerando la legge di conservazione dell'impulso angolare del sistema chiuso totale. Infine le considerazioni di energia del sistema totale chiuso portano ovviamente alla stessa conclusione.

(*) *Traduzione a cura della Redazione.*

Количественный анализ « Trouton-Noble » эксперимента.

Резюме (*). — В рамках обычной релятивистской теории сплошной среды и электродинамики проводится количественный анализ упрощенной модификации « Trouton-Noble » эксперимента. Сначала проверяется, что в приближении v^2/c^2, а также при точном рассмотрении, отрицательный результат эксперимента является следствием применения теоремы углового момента к электродинамической подсистеме. Аналогичный результат получается при анализе механической подсистемы. Этот кажущийся парадокс можно объяснить, рассматривая закон сохранения углового момента для замкнутой полной системы. Кроме того, энергетическое рассмотрение замкнутой полной системы приводит, очевидно к тому же заключению.

(*) *Переведено редакцией.*

A. HERNÁNDEZ, *et al.*
11 Novembre 1982
Il Nuovo Cimento
Serie 11, Vol. **72** B, pag. 1-12