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Universidad del País Vasco
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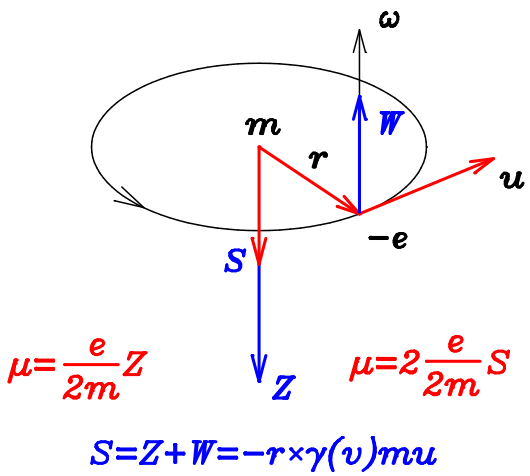
Kinematical Theory of elementary spinning particles



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- If I can't picture it, I can't understand it.
  You know, it would be sufficient to really understand the electron.

  A. Einstein[1]

- Everything should be as simple as possible, but not simpler.

  A. Einstein?, William of Ockham?[2]

- If a spinning particle is not quite a point particle, nor a solid three dimensional top, what can it be? What is the structure which can appear under probing with electromagnetic fields as a point charge, yet as far as spin and wave properties are concerned exhibits a size of the order of the Compton wavelength?

  A.O. Barut [3]

- The picture on the front page represents the circular motion, at the speed of light, of the center of charge of the electron in the center of mass frame. This motion is not modified by any interaction. The center of mass is always a different point that the center of charge. The radius of this motion is $R = \hbar/2mc$, half Compton's wavelength, as is sugggested by Barut. The frequency of this motion, when the center of mass is at rest, is $\omega = 2mc^2/\hbar$. This frecuency, twice the frequency postulated by De Broglie, decreases when the center of mass moves. The local clock is going slower when moving. In this way, elementary matter has an internal periodic motion, and thus a frequency, like waves. We can also associate to matter a wavelength, as the displacement of the center of mass during a complete turn of this internal motion. The spin $\boldsymbol{S}$ has two parts: one $\boldsymbol{Z}$ associated to this relative internal motion and another $\boldsymbol{W}$ in the opposite direction related to the rotation of a local Cartessian frame associated to the center of charge. This frame is not depicted in the figure. The magnetic moment of the electron is produced by the motion of the charge and is related to the orbital part $\boldsymbol{Z}$ of the angular momentum but when expressed in terms of the total spin $\boldsymbol{S}$, which is half the orbital $\boldsymbol{Z}$, is when we obtain the concept of gyromagnetic ratio $g = 2$.

- Classical particle physics, when using so extensively the point particle model to describe experiments, which are always performed with spinning particles, is making a simplification, opposite to the espirit of the above quotations. We have to use spinning particle models to analyze real experiments, because in nature there are no spinless elementary particles.
  In this sense, General Relativity as a theory of gravitation, also makes a simplification when assuming that spacetime has a Riemannian metric structure. This assumption is unnecessary because spacetime has a more general Finslerian metric structure associated to the variational formalism, as we discuss in section **1.6**. To assume that the metric is Riemannian is equivalent to consider a low velocity limit of a more general gravitational theory.

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[1] H. Dehmelt, Proc. Natl. Acad. Sci. USA, **86**, 8618–19 (1989).

[2] See the discussion in http://quoteinvestigator.com/2011/05/13/einstein-simple/#more-2363, about the authorship of this sentence.

[3] A.O. Barut, *Brief History and recent developments in electron theory and Quantumelectrodynamics*, in The electron, New Theory and Experiment, D. Hestenes and A. Weingartshofer (ed.), Kluwer Academic Publishers, Dordrecht (1991).

Contents

Preface

The present notes contain some basic materials, physical and mathematical, of the general formalism for analyzing elementary particles, which under the general name of **Kinematical Theory of Elementary Spinning Particles**, I have been working during the last years. The term *kinematical* makes reference to its close relationship with the kinematical group of space-time transformations associated to the Restricted Relativity Principle which a theoretical framework must necessarily satisfy.

In a certain sense it is a revision of the basic fundamentals of the Lagrangian formalism which leads to Euler-Lagrange equations, Noether's theorem, etc., but looking for solutions which go through the postulated initial and final states of the variational formalism. This produces a classical formalism which is going to be expressed in terms of the set of the two end point variables of the dynamical evolution. This distinguishes this approach from the usual non-orthodox variational approach which expresses the solution in terms of the boundary variables at the initial time. This formalism is, therefore, closer to the quantum mechanical dynamical theory and it is through Feynman's path integral approach that we can find the bridge between them.

These end point variables of the variational formalism, which I propose to call them **kinematical variables**, in the case of elementary particles will necessarilly span a homogeneous space of the kinematical group. In this way, the kinematical group not only reflects the space-time symmetries of the system. It also supplies the necessary variables to describe elementary matter. It is crucial for the description of matter to improve in our knowledge of this kinematical group. In the present notes we shall deal mainly with the Galilei and Poincaré groups, but the formalism is so general that it can be accomodated to any further group we consider as the basic symmetry group of matter.

Another advantage of expressing the orthodox variational formalism in terms of the boundary kinematical variables is that the formalism is equivalent to a geodesic formalism on the kinematical space. This manifold for any arbitrary Lagrangian system is always a **metric Finsler space**. In this sense when we consider the interaction of any mechanical system what produces, from the mathematical point of view, is a change of the Finsler metric of the kinematical space. When we consider the relativistic point particle, the kinematical space is the spacetime manifold with a constant Minkowski metric. This metric is considered Riemannian but it is in fact a constant Finslerian metric which is modified by any interaction. The postulate of General Relativity that gravity produces a pseudo-Riemannian modification of Minkowski metric is an unnecessary restriction.

The formalism is very general, but at the same time is very restrictive, because once this kinematical group is fixed the kind of classical variables which define the initial and final states of an elementary particle in a variational approach, are restricted to belong to homogeneous spaces of the group. This kinematical group is the fundamental object of the formalism and must be defined as a preliminary statement.

For the Galilei and Poincaré groups, a general spinning elementary particle is just a localized and orientable mechanical system. By **localized** we mean that to analyse its evolution in space

we have just to describe the evolution of a single point $\boldsymbol{r}$, where the charge is located and in terms of which the possible interactions are determined. This point $\boldsymbol{r}$ also represents the centre of mass of the particle for spinless particles, while for spinning ones must necessarily be a different point than $\boldsymbol{q}$, the centre of mass, very well defined classically and where we can locate the mass of the particle. It is the motion of the charge around the centre of mass which gives rise to a classical interpretation of the **zitterbewegung**, or trembling motion in Schroedinger's words, and also to the dipole structure of the particle. By **orientable** we mean that in addition to the description of the evolution of the center of charge we also need to describe the change of orientation of the system by analyzing the evolution of a local comoving and rotating frame attached to that point. This local frame has no physical reality such that we can select it in an arbitrary way at any time, thus supplying an additional symmetry group.

If we consider that the kinematical group is Weyl group $\mathcal{W}$, then an elementary particle in addition of being a localizable and orientable system, it is also reescalable. It contains an additional degree of freedom which represents a phase or a change of scale. This means that the most general spacetime symmetry group of the dynamics must contain additional transformations, like local rotations and scale changes. It is possible to find a Lagrangian invariant under the group $\mathcal{W} \otimes SU(2) \otimes U(1)$.

The notes pretend to be selfcontained and in this way we have included at the end of the chapters some mathematical appendices which contain not very well spread materials. The lecture notes are organised as follows. We begin with a Preamble, which could have been written as late as the end of the XIX-th century, and which suggests that the center of charge of an elementary particle moves in a helical motion at the speed of light, so that this point will satisfy, in general, fourth order differential equations. This implies that in a Lagrangian approach we shall have a Lagrangian depending up to the acceleration of this point. We are in the framework of generalized Lagrangian systems.

Instead of postulating models of elementary particles with two separate centers we shall analyze what are the basic fundamental principles that a theory of matter should satisfy. Among these fundamental principles we consider the variational formalism and that is the reason we shall study in the first chapter the formalism of generalized Lagrangian systems, mainly to enhance the role of the kinematical variables in defining a concept of elementary particle. Chapter two will be devoted to the analysis of several relativistic and nonrelativistic models, to show how the standard methods of analyzing symmetries leads to the definition of the relevant observables. In particular, we shall pay attention to the definition of the spin. The spin, as any other observable, will be defined in the classical case in terms of the degrees of freedom and their derivatives, and we shall analyze its mathematical structure.

The next two chapters will cover the quantization of the formalism and the analysis of some relativistic and nonrelativistic examples. The separate fourth chapter is devoted to the model which satisfies Dirac's equation. Special attention is paid to the analysis of Dirac's algebra and its relationship with the classical observables and to show a geometrical interpretation of the difference in chirality between matter and antimatter. This chapter ends with the analysis and enlargement of the spacetime symmetry group of the Dirac particle, going from the Poincaré group to the eleven parameter Weyl group. We shall find a plausible Weyl-invariant interaction Lagrangian which describes a short and long range interaction between two Dirac particles, which has a Coulomb-like behaviour when the spin of the particles is supressed. It also shows that equal charged spinning particles can form metastable bound states provided some boundary conditions are fulfilled. The strength of this interaction is independent of the mass of the particles and is determined by the value of the fine structure constant, when the analysis of this material system is performed in terms of dimensionless variables.

The electromagnetic structure of the model which satisfies Dirac's equation when quantized, is analyzed in a separate fifth chapter. It is not a static electromagnetic field for the center of

mass observer but its time average value has a Coulomb-like behavior in any direction for the electric field and the time average magnetic field is the field of a static magnetic dipole at the origin. The main difference of these fields when compared with the point particle field is that the fields do not diverge at the origin.

Finally, some physical features which are related to the spin of the elementary particles, are described. The electron, because it has an internal frequency it can be considered as a clock. Can we measure this internal frequency? We shall propose to enlarge the energy range of an experiment to determine indirectly the value of this frequency. We shall analyze the gyromagnetic ratio and the dipole structure of the electron, which in the quantum case has a relationship with the Darwin term of Dirac's Hamiltonian. We shall also see how the spin structure allows us to justify in a classical framework the tunnel effect, which will be responsible of the gyant magnetoresistence of several materials. We are entering what in technological terms is called spintronics. Compton's effect analyzes the sccatering of photons by free electrons by using only the energy and linear momentum conservation laws. But the electromagnetic interaction also conserves the total angular momentum. This additional conservation implies to consider both spins of either electron and photon. If we are able to control the spin orientation of the free electrons we can determine the frequency of the sccatered photon. To end this section we shall consider the possibility from the classical point of view that under certain conditions two electrons with their spins parallel to each other can form a metastable bound state of spin 1 and charge $2e$, and therefore the justification of the formation of a Bose-Einstein condensate at finite temperature. A consequence of this pairing is the analysis of the quantum Hall effect. As a final example we shall consider the spin structure of the proton, considered as a bound system of three Dirac particles, the quarks. We shall see that the proton spin crisis can be related to the lacking of a term in the spin of the proton, because Dirac spin operator represents the spin of a Dirac particle with respect to its center of charge and not with respect to the center of mass.

In some places, the lectures will be complemented with numerical simulations whenever the theoretical solution is not available or very difficult to interpret because of the mathematical complexity. A numerical computer program, appropriate for the anaylis of dynamical systems, is *Dynamics Solver*, created by Juan María Aguirregabiria [4], which has been very fruitful for many of the numerical analysis contained in these notes. I am very glad for his kindness to show me the way to manage it. In the whole text, mathematical expressions which contain greek or latin characters in bold face, like $\boldsymbol{a}$ or $\boldsymbol{\alpha}$, they must be undertood as three-dimensional vector magnitudes, while letters like a or α represent, in general, real numbers.

Things should be done simply but not simpler. Simplification has to be done when analyzing some problems and according to the values of the physical variables, and not in the preliminary steps of the formalism. That is why assumptions about that the dynamical equations of physics are second order differential equations, have to be justified on physical grounds. It works with spinless point particles but in Nature do not exist spinless elementary particles. We have to reject the point particle model as a fundamental model for elementary particles and its use to understand some mechanical effects. We shall see that the center of charge of the electron is a different point than its center of mass and satisfies fourth order differential equations.

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Bilbao, March 2022.

[4] J.M. Aguirregabiria, *Dynamics Solver*, available through the web page of his autore, at the sever of the Theoretical Physics Department of the University of the Basque Country, <http://tp.lc.ehu.eus/jma.html>

Preamble: Helical motion of the center of charge

In this preliminary chapter we shall give three different kinds of arguments suggesting that the center of charge and the center of mass of an elementary particle are two different points. The center of charge moves in a helical motion at the speed of light, and it thus satisfies, in general, a fourth order differential equation. This analysis selects the relativistic formalism instead of the nonrelativistic one, and the fact that the dynamical equations of a point are fourth order differential equations, as differential geometry shows, opposite to the usual suggestion of second order differential equations of many classical mechanics books.

This means that a Lagrangian formalism for describing elementary particles has to depend, at least, up to the acceleration of the position of the charge, to properly obtain fourth order dynamical equations. By this reason, we shall start our formalism by describing in chapter 1, the way the generalized Lagrangian formalism produces the general results of Euler-Lagrange equations, the conserved quantities through Noether theorem, and the generalized canonical formalism.

We shall begin with a physical, and therefore restricted, concept of center of charge of an elementary particle.

The center of charge

The concept of center of mass of any distribution of matter is well known. If we have n point particles of masses m_i located at the corresponding points $\boldsymbol{r}_i$ the center of mass location of the system is

$$\boldsymbol{R}_{CM} = \frac{\sum m_i \boldsymbol{r}_i}{\sum m_i}.$$

If we also assume Newton's third law, this point describes a trajectory such that the time variation of the linear momentum is the sum of the external forces.

From the electromagnetic point of view, if we have an arbitrary distribution of charges and currents, the electromagnetic field they generate can be expressed as the field produced from a single point where we locate there the total charge and the different electric and magnetic multipoles defined with respect to this point. If we consider a different point the total charge is the same but the multipoles are different. If we try to define a center of charge R_{CC} like the above definition of the center of mass we have the problem that $\sum_i q_i = 0$. We can alternatively define the center of charge of either the positive and negative charges $R^{\pm}_{CC}$, and the separate fields they generate with the corresponding multipoles, because Maxwell's equations are linear in the sources.

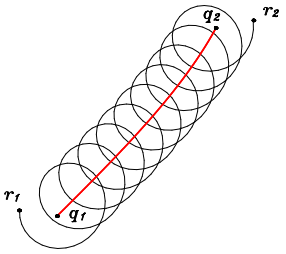
Another question is to calculate the external force produced on a system of charges and currents. Is it possible to write this external force in terms of the total charge and the different multipoles located at a single point or at least in two points? In general this will not be possible for an arbitrary system. But to fix ideas let us consider a simple system of a static and spherical

positive charge distribution in an inertial reference frame. The field it produces is the Coulomb field from the symmetry center of the distribution. If now an external field is acting on this system, and we consider it behaves like a conductor, this will produce in general a modification of the charge distribution and therefore the appearence of dipole momenta with respect to the symmetry center. If it behaves like an insulator some electric polarization will arise.

We do not know if an elementary particle behaves like a conductor or like an insulator, if it is a rigid body or it is not. But in the section devoted to fundamental principles we shall make the hypothesis that an elementary particle is an undeformable mechanical system (Atomic Principle). If its charge and current distribution have a spherical symmetry with respect to some point, such that the electric and magnetic field it produces will be expressed in terms of the location and velocity of this point and no further multipoles, we shall call this point the **center of charge**. If the elementary particle cannot be deformed by any interaction leads us to postulate that the external force acting on it is just the Lorentz force defined at the center of charge. We are making the physical hypothesis that, from the electromagnetic point of view, it behaves like a unique charge located at the center of charge and no other multipoles.

Rigid body arguments

Let us consider that an elementary particle were described as a rigid body. A rigid body is a mechanical system of six degrees of freedom. Three represent the position of a point and the other three the orientation of a body frame attached to that point. Usually, it is described by the location of the center of mass, which is represented by the point $\boldsymbol{q}$, and the orientation by the principal axis of inertia located around $\boldsymbol{q}$. The center of mass satisfies second order dynamical equations and moves like a point of mass m, the total mass of the system, under the total external force. In this way a rigid body moves and rotates.



If instead of considering the description of the center of mass we take a different point $\boldsymbol{r}$, it will follow a helical trajectory around the center of mass, like the one depicted in the figure.

If an elementary particle is a charged rigid body, it is clear that we also need to know its electromagnetic structure, which can be reduced to the knowledge of the center of charge and the different multipoles. If assumed a spherical symmetry for the electric field produced by the particle we are left with the location of the center of charge to compute the actions of the external fields. In general, depending how the mass and charge are distributed, these two points will be different points as we shall assume here. Therefore, if we try to describe the evolution of the center of mass we have to determine also at any time the location of the center of charge to compute the external forces. Newton's dynamical equations for the center of mass will be written as

$$m\frac{d^2\boldsymbol{q}}{dt^2} = e\left(\boldsymbol{E}(t,\boldsymbol{r}) + \frac{d\boldsymbol{r}}{dt}\times\boldsymbol{B}(t,\boldsymbol{r})\right) = \boldsymbol{F}(t,\boldsymbol{r},d\boldsymbol{r}/dt). \tag{1}$$

The electromagnetic force $\boldsymbol{F}$ depends, in general, on the electric and magnetic external fields defined at the charge position $\boldsymbol{r}$ and on the velocity of the charge $d\boldsymbol{r}/dt$ which appears in the magnetic term.

For the relative motion of the center of charge around the center of mass we have that if this relative motion between $\boldsymbol{r}$ and $\boldsymbol{q}$ is a kind of circular motion, in particular in the free case, we can define a unit vector $\boldsymbol{n}$ in the direction of the normal acceleration $d^2\boldsymbol{r}/dt^2$ of point $\boldsymbol{r}$, and thus

$$\boldsymbol{n} = \frac{1}{\omega^2 R}\frac{d^2\boldsymbol{r}}{dt^2},$$

where R is the radius of the circular motion and ω its angular velocity. Then the center of mass position can be written as

$$\boldsymbol{q}(t) = \boldsymbol{r}(t) + \frac{1}{\omega^2}\frac{d^2\boldsymbol{r}}{dt^2}. \tag{2}$$

Then, it will be simpler, from a theoretical point of view, just to describe the evolution of a single point, the center of charge $\boldsymbol{r}$, instead of the center of mass $\boldsymbol{q}$, which will be in some average position of the other, and obtained from (2) once the trajectory of $\boldsymbol{r}$ is computed. The elimination of the $d^2\boldsymbol{q}/dt^2$ among equations (1) and (2) will give us, in general, a fourth order differential equation for the variable $\boldsymbol{r}$. Because the angular velocity is also orthogonal to the plane subtended by the velocity and acceleration of point $\boldsymbol{r}$,

$$\boldsymbol{\omega} = \frac{1}{u^2}\frac{d\boldsymbol{r}}{dt} \times \frac{d^2\boldsymbol{r}}{dt^2}, \tag{3}$$

we have also solved the problem of the rotation of the charged rigid body by analyzing the evolution of just the center of charge.

The second order differential equations for the center of mass position and the orientation of the principal axes of inertia $\boldsymbol{\alpha}$, of the free rigid body become

$$\ddot{\boldsymbol{q}} = 0, \quad \dot{\boldsymbol{\omega}} = 0,$$

and they have been replaced by the fourth-order dynamical equations of the center of charge $\boldsymbol{r}$,

$$\frac{d^4\boldsymbol{r}}{dt^4} + \omega^2\frac{d^2\boldsymbol{r}}{dt^2} = 0.$$

In this way a rigid body can be interpreted as a system of three degrees of freedom, the center of charge $\boldsymbol{r}$, which satisfies fourth order differential equations and therefore in a variational description, the Lagrangian will depend on the acceleration of the center of charge.

The dynamical equations under interaction are:

$$\frac{m}{\omega^2}\frac{d^4\boldsymbol{r}}{dt^4} + m\frac{d^2\boldsymbol{r}}{dt^2} = e\left(\boldsymbol{E}(t,\boldsymbol{r}) + \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{B}(t,\boldsymbol{r})\right), \tag{4}$$

in terms of the three degrees of reedom $\boldsymbol{r}$, where the external fields are defined.

A plausible nonrelativistic Lagrangian depending on the acceleration of the point $\boldsymbol{r}$, like this

$$L = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 - \frac{m}{2\omega^2}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)^2 - e\phi(t,\boldsymbol{r}) + e\boldsymbol{A}(t,\boldsymbol{r}) \cdot \frac{d\boldsymbol{r}}{dt}$$

will reproduce the above dynamical equations (4), where the rigid body will rotate with a constant angular velocity ω, which in this example represents a constant and unmodified intrinsic property.

Invariance arguments

Let us consider the trajectory $\boldsymbol{r}(t)$, $t \in [t_1, t_2]$ followed by a point of a mechanical system for an arbitrary inertial observer O. Any other inertial observer O' is related to the previous one by a transformation of the kinematical group such that their relative space-time measurements of any space-time event are given by

$$t' = T(t, \boldsymbol{r}; g_1, \ldots, g_\alpha), \quad \boldsymbol{r}' = \boldsymbol{R}(t, \boldsymbol{r}; g_1, \ldots, g_\alpha),$$

where the functions T and $\boldsymbol{R}$ define the corresponding transformation of the kinematical group G, of parameters $(g_1, \ldots, g_\alpha)$, among any two observers. Then the description of the trajectory of that point for observer O' is obtained from

$$t'(t) = T(t, \boldsymbol{r}(t); g_1, \ldots, g_\alpha), \quad \boldsymbol{r}'(t) = \boldsymbol{R}(t, \boldsymbol{r}(t); g_1, \ldots, g_\alpha), \quad \forall t \in [t_1, t_2].$$

If we eliminate t as a function of t' from the first equation and substitute into the second we shall get

$$\boldsymbol{r}'(t') = \boldsymbol{r}'(t'; g_1, \ldots, g_\alpha). \tag{5}$$

Since observer O' is arbitrary, equation (5) represents the complete set of trajectories of the point for all inertial observers. Elimination of the $g_1, \ldots, g_\alpha$ group parameters among the function $\boldsymbol{r}'(t')$ and their time derivatives will give us the differential equation satisfied by all the trajectories of the point. Let us assume that the trajectory is unrestricted in such a way that the above group parameters are essential in the sense that no smaller number of them gives the same family of trajectories. This differential equation is invariant under the transformations of the kinematical group by construction because it is independent of the group parameters and therefore independent of any inertial observer. In fact, because (5) is a three-vector expression, each time we take a time derivative we obtain three equations to eliminate the group parameters. When we reach the third order derivative we have up to nine equations. If G is either the Galilei or Poincaré group, it is a ten-parameter group so that we have to work out in general up to the fourth derivative to obtain sufficient equations to eliminate the group parameters. Therefore the order of the invariant differential equation is dictated by the number of parameters and the structure of the kinematical group. If the point $\boldsymbol{r}$ represents the position of the center of charge of an elementary particle we get again that it satisfies, in general, a fourth order differential equation.

But at the same time it is telling us that to obtain the invariant differential equation satisfied by the center of charge of an elementary particle, it is sufficient to obtain its trajectory in an arbitrary reference frame, for instance in the center of mass frame, and to follow the above procedure of elimination of the group parameters. We shall use this method to obtain the invariant differential equation of a spinning electron in section **2.6**.

Geometrical arguments

As is well known in differential geometry, a continuous and differentiable curve in three-dimensional space, $\boldsymbol{r}(s)$, has associated three orthogonal unit vectors, $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$, called respectively the tangent, normal and binormal. If using the arc length s as the curve parameter, they satisfy the Frenet-Serret (1847) equations

$$\dot{\boldsymbol{t}} = \kappa \boldsymbol{n}, \quad \dot{\boldsymbol{n}} = -\kappa \boldsymbol{t} + \tau \boldsymbol{b}, \quad \dot{\boldsymbol{b}} = -\tau \boldsymbol{n},$$

where κ is the curvature and τ the torsion and the overdot means $\dot{} \equiv d/ds$. The knowledge of the functions of s, the curvature $\kappa(s)$ and torsion $\tau(s)$, together the boundary values $\boldsymbol{r}(0)$, $\boldsymbol{t}(0)$,

$\boldsymbol{n}(0)$ and $\boldsymbol{b}(0)$, completely determine the curve, because the above equations are integrable. If we define the vector $\boldsymbol{\omega} = \tau\boldsymbol{t} + \kappa\boldsymbol{b}$, known as Darboux vector, the Frenet-Serret equations can be rewritten as

$$\dot{\boldsymbol{t}} = \boldsymbol{\omega} \times \boldsymbol{t}, \quad \dot{\boldsymbol{n}} = \boldsymbol{\omega} \times \boldsymbol{n}, \quad \dot{\boldsymbol{b}} = \boldsymbol{\omega} \times \boldsymbol{b},$$

so that, in units of arc length, Darboux vector represents the instantaneous angular velocity of the local frame of the three orthogonal unit vectors.

If we call $\boldsymbol{r}^{(k)}(s) \equiv d^k\boldsymbol{r}/ds^k$, and, in particular

$$\boldsymbol{r}^{(1)} = \boldsymbol{t}, \quad \boldsymbol{r}^{(2)} = \kappa\boldsymbol{n}, \quad \boldsymbol{r}^{(3)} = \dot{\kappa}\boldsymbol{n} + \kappa(-\kappa\boldsymbol{t} + \tau\boldsymbol{b})$$

and eliminate the three unit vectors $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$, in terms of the derivatives $\boldsymbol{r}^{(k)}$, $k = 1, 2, 3$, we get

$$\boldsymbol{t} = \boldsymbol{r}^{(1)}, \quad \boldsymbol{n} = \frac{1}{\kappa}\boldsymbol{r}^{(2)}, \quad \boldsymbol{b} = \frac{\kappa^2}{\tau}\boldsymbol{r}^{(1)} - \frac{\dot{\kappa}}{\kappa\tau}\boldsymbol{r}^{(2)} + \frac{1}{\kappa\tau}\boldsymbol{r}^{(3)}$$

and thus

$$\kappa = |\boldsymbol{r}^{(2)}|, \qquad \tau = \frac{1}{\kappa^2}(\boldsymbol{r}^{(1)} \times \boldsymbol{r}^{(2)}) \cdot \boldsymbol{r}^{(3)}$$

are expressed in terms of the derivatives up to the third order. If we replace the three Frenet-Serret unit vectors in the next order derivative, one obtains that the most general differential equation satisfied by the point $\boldsymbol{r}$, is the fourth order differential system

$$\boldsymbol{r}^{(4)} - \left(\frac{2\dot{\kappa}}{\kappa} + \frac{\dot{\tau}}{\tau}\right)\boldsymbol{r}^{(3)} + \left(\kappa^2 + \tau^2 + \frac{\dot{\kappa}\dot{\tau}}{\kappa\tau} + \frac{2\dot{\kappa}^2 - \kappa\ddot{\kappa}}{\kappa^2}\right)\boldsymbol{r}^{(2)} + \kappa^2\left(\frac{\dot{\kappa}}{\kappa} - \frac{\dot{\tau}}{\tau}\right)\boldsymbol{r}^{(1)} = 0, \qquad (6)$$

where the coefficients are only functions of the derivatives of $\boldsymbol{r}$ up to fourth order.

This conclussion is easily obtained if we realize that the three-dimensional space is also a **vector space**. Any curve in three-space is called regular if at any point it has a tangent vector $\boldsymbol{r}^{(1)}$. If it is also differentiable, they will be also defined the subsequent derivatives $\boldsymbol{r}^{(2)}$ and $\boldsymbol{r}^{(3)}$, which, in general, will be no collinear. But the next derivative $\boldsymbol{r}^{(4)}$, will be necessarily a linear combination of the other three. Every regular curve in three dimensional space satisfies a fourth order differential equation. This is what equation (6) represents.

Let us consider that an elementary particle, instead of being a rigid body, is just a localized mechanical system. By localized we mean that, at least, it is described by the evolution of a single point $\boldsymbol{r}$. This point could be the center of mass, but, as mentioned before, in order to determine the external forces to obtain the center of mass evolution, we also need to know the location of the center of charge to compute the actions of the external fields. Let us assume that the elementary particle is charged. By the previous arguments, if assumed spherical symmetry of its electric field, we are reduced to know the evolution just of the center of charge. The particle will have a center of mass but we make the assumption that the center of mass and the center of charge are not necessarily the same point.

Then, the center of charge of an elementary particle will satisfy, in general, a fourth order differential equation of the form (6) where $\kappa(s)$ and $\tau(s)$ will depend on the external forces and torques.

Free motion

Let us assume now that the motion of the particle is free. This means that we cannot distinguish one instant of the evolution from another, so that the above equations (6) must be explicitely independent of the parameter s. The Frenet-Serret triad moves and rotates. It must be desplaced at a velocity of constant absolute value and the Darboux vector has to be a constant vector in the comoving frame. The velocity $ds/dt = u$ and the value of Darboux vector $\omega^2 = \kappa^2 + \tau^2$ must be constant.

The curvature and torsion are necessarily constants of the motion. Thus $\dot{\kappa} = \dot{\tau} = 0$, and, in the free case, these equations are simplified and reduced to

$$\boldsymbol{r}^{(4)} + \left(\kappa^2 + \tau^2\right) \boldsymbol{r}^{(2)} = \frac{d^2}{ds^2}\left(\boldsymbol{r}^{(2)} + \omega^2 \boldsymbol{r}\right) = 0.$$

If the curvature and torsion are constant the curve is a helix, which can be factorized in terms of a central point

$$\boldsymbol{q} = \boldsymbol{r} + \frac{1}{\kappa^2 + \tau^2}\boldsymbol{r}^{(2)}, \quad \frac{d^2\boldsymbol{q}}{ds^2} = 0,$$

which is moving along a straight trajectory, while the point $\boldsymbol{r}$ satisfies

$$\boldsymbol{r}^{(2)} + \omega^2 \left(\boldsymbol{r} - \boldsymbol{q}\right) = 0,$$

an isotropic harmonic motion of frequency $\omega = \sqrt{\kappa^2 + \tau^2}$, around point $\boldsymbol{q}$. The point $\boldsymbol{q}$ clearly represents the centre of mass position of the free particle. Going further, let us assume that the free evolution is analyzed by some inertial observer. Then this observer cannot distinguish one instant from another, so that, the arc length $ds = |\boldsymbol{u}|dt$, where $\boldsymbol{u} = d\boldsymbol{r}/dt$ is the velocity of the charge, must be also independent of the time t. Otherwise, if ds is not the same we can distinguish one instant of the evolution from another, as far as the displacement of the charge is concerned. The center of charge of a free elementary particle is describing a helix at a constant velocity for any inertial observer.

A first conclusion is that the velocity of the center of charge has to be an unreachable velocity for every inertial observer. The helical motion is an accelerated motion in one frame and thus it is accelerated in all inertial frames. If one observer is at rest with respect to the charge at one instant. t, it measures $u = 0$ at this time, but $u \neq 0$ at time $t + dt$, which contradicts that the velocity has to be constant in this frame. This means that the constant velocity cannot be zero in any frame and no inertial observer can reach that velocity.

If we make a nonrelativistic analysis, the relationship of the velocity measurements among two arbitrary inertial observers O and O', is given by $\boldsymbol{u}' = \boldsymbol{u} + \boldsymbol{v}$, where $\boldsymbol{v}$ is the constant velocity of O as measured by O'. Now,

$$u'^2 = u^2 + v^2 + 2\boldsymbol{v} \cdot \boldsymbol{u}.$$

If u' has to be also constant for observer O', irrespective of $\boldsymbol{v}$, this means that the vector $\boldsymbol{u}$ must be a constant vector. The center of charge necessarily moves along a straight trajectory at a constant velocity, for every inertial observer, and the above general helix degenerates into a straight line and $\boldsymbol{q} = \boldsymbol{r}$. This is the usual description of the spinless or pointlike free elementary charged particle, whose center of charge and center of mass are represented by the same point.

In the relativistic case we get simmilarly

$$\boldsymbol{u}' = \frac{\boldsymbol{u} + \gamma(v)\boldsymbol{v} + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot\boldsymbol{u})\boldsymbol{v}}{\gamma(1 + \boldsymbol{v}\cdot\boldsymbol{u}/c^2)}, \quad u'^2 = \frac{u^2 - c^2}{\gamma^2\left(1 + \boldsymbol{v}\cdot\boldsymbol{u}/c^2\right)^2} + c^2.$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$, and taking the time derivative we also obtain that $\boldsymbol{v}\cdot\dot{\boldsymbol{u}} = 0$, and thus $\boldsymbol{u}$ has to be a constant vector, for any time t, irrespective of the value of $\boldsymbol{v}$.

However, in the relativistic analysis, there is one alternative not included in the nonrelativistic approach. The possibility that the charge of an elementary particle will be moving at the speed of light and, in that case, $u = u' = c$, for any inertial observer. This means that the center of the helix is always moving at a velocity $|d\boldsymbol{q}/dt| < c$, and, if it represents the center of mass, this particle is a massive particle. In a variational description of this system the Lagrangian should depend up to the acceleration of the point $\boldsymbol{r}$ in order to obtain fourth order differential

equations. We will show that this dependence on the acceleration will give a contribution to the spin of the particle and there is also another contribution from the rotation of the system, because the body frame rotates with angular velocity $\boldsymbol{\omega}$. The motion of the charge around the center of mass produces the magnetic moment of the particle.

In summary, there are only two possibilities for a free motion of the center of charge of an elementary particle. One, the charge is moving along a straight line at any constant velocity, and the system has no magnetic moment. In the other, the particle has spin and magnetic moment, and the charge moves along a helix at the speed of light. Because all known elementary particles, quarks and leptons, are spin 1/2 particles, we are left only with the last possibility. This is consistent with Dirac's theory of the electron, because the eigenvalues of the components of Dirac's velocity operator are $\pm c$. This means that Dirac's spinor $\psi(t, \boldsymbol{r})$ is expressed in terms of the position of the charge $\boldsymbol{r}$, because the external fields $A^{\mu}(t, \boldsymbol{r})$ are defined and computed at this point.

This last possiblity is the description of the center of charge of a relativistic spinning elementary particle obtained in the kinematical formalism to be developed in this course, and which satisfies Dirac's equation when quantized.

In this formalism Dirac particles are localized and also orientable mechanical systems. By orientable we mean that we have to attach to the above point $\boldsymbol{r}$, a local cartesian frame to describe its spatial orientation. This frame could be the Frenet-Serret triad. The rotation of the frame will also contribute to the total spin of the particle. When quantizing the system, the spin 1/2 is coming from the presence of the orientation variables. Otherwise, if there are no orientation variables, no spin 1/2 structure is described when quantizing the system. This twofold structure of the classical spin has produced a pure kinematical interpretation of the gyromagnetic ratio [5]. The dependence of the Lagrangian on the acceleration is necessary for the particle to have magnetic moment and for the separation between the center of mass and center of charge.

Two centers, two spins

It is usually called spin to the angular momentum of an elementary particle. But an angular momentum is a mechanical property which is defined with respect to some definite point. If an elementary particle has two characteristic points, we can determine the angular momentum with respect to both of them.

Let us consider an electron which is characterized by the location of its center of mass (CM) $\boldsymbol{q}$, and its center of charge (CC) $\boldsymbol{r}$, and let $\boldsymbol{k}$ be another point of the electron, different from the previous ones, in a certain reference inertial frame with origin at the point O (see figure 1).

Let us call $\boldsymbol{S}$ the angular momentum of the particle with respecto to the centre of charge (CC) $\boldsymbol{r}$. The angular momentum $\boldsymbol{S}_{CM}$ with respect to the centre of mass (CM) $\boldsymbol{q}$, will be

$$\boldsymbol{S}_{CM} = (\boldsymbol{r} - \boldsymbol{q}) \times \boldsymbol{p} + \boldsymbol{S},$$

where $\boldsymbol{p}$ is the linear momentum of the particle in this frame.

Let us call $\boldsymbol{v} = d\boldsymbol{q}/dt$ and $\boldsymbol{u} = d\boldsymbol{r}/dt$, to the velocities of CM and CC, respectively. Let $\boldsymbol{S}_k$ be the angular momentum with respect to the point $\boldsymbol{k}$. The total angular momentum of the particle with respect to the origin of the reference frame of any inertial observer, can be written as

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{S}, \quad \text{or} \quad \boldsymbol{J} = \boldsymbol{q} \times \boldsymbol{p} + \boldsymbol{S}_{CM}, \quad \text{or} \quad \boldsymbol{J} = \boldsymbol{k} \times \boldsymbol{p} + \boldsymbol{S}_k.$$

[5] M. Rivas, J.M. Aguirregabiria and A. Hernández, "A pure kinematical explanation of the gyromagnetic ratio $g = 2$ of leptons and charged bosons", Phys. Lett. **A 257**, 21–25 (1999).

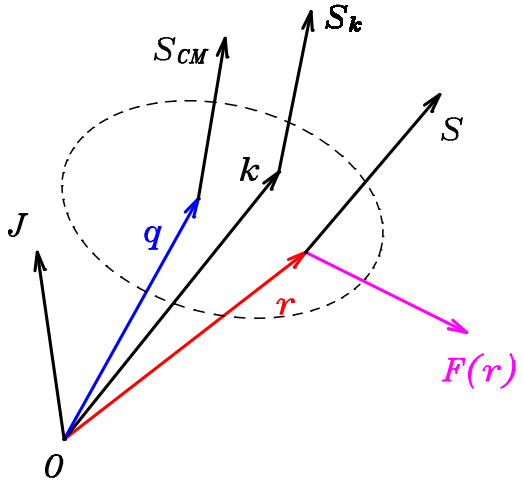


Figure 1: **Different angular momenta S, S_{CM}, S_k and J of the electron with respect to different points in some inertial reference frame, with origin at the point O. It is also depicted the external electromagnetic force F defined at the Center of Charge. The dotted line suggests some arbitrary, but localized, form or shape of the electron.**

If the particle is free, $\boldsymbol{p}$ and also $\boldsymbol{J}$ are conserved. Since $d\boldsymbol{J}/dt = 0$, this leads to

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{p} \times \boldsymbol{u}, \quad \frac{d\boldsymbol{S}_{CM}}{dt} = 0,$$

because $\boldsymbol{p}$ has the direction of $\boldsymbol{v}$, but not of $\boldsymbol{u}$.

The center of mass spin $\boldsymbol{S}_{CM}$ is a conserved magnitude for a free particle, but the center of charge spin $\boldsymbol{S}$ is not. It satisfies a dynamical equation which implies that its time variation is orthogonal to the linear momentum. It is suggesting that $\boldsymbol{S}$ precess or oscillate around the constant vector $\boldsymbol{p}$. Morover, for a free particle $\boldsymbol{u}$ cannot be a constant vector, otherwise the centre of charge spin $\boldsymbol{S}$, will rise continuosly.

Let $\boldsymbol{F}$ be the external electromagnetic force applied at the centre of charge $\boldsymbol{r}$. Now neither $\boldsymbol{J}$ nor $\boldsymbol{p}$ are conserved quantities. The force and the torque with respect to the origin satisfy

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}, \quad \frac{d\boldsymbol{J}}{dt} = \boldsymbol{r} \times \boldsymbol{F},$$

and thus

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{p} \times \boldsymbol{u}, \quad \frac{d\boldsymbol{S}_{CM}}{dt} = (\boldsymbol{r} - \boldsymbol{q}) \times \boldsymbol{F}, \quad \frac{d\boldsymbol{S}_k}{dt} = \boldsymbol{p} \times \frac{d\boldsymbol{k}}{dt} + (\boldsymbol{r} - \boldsymbol{k}) \times \boldsymbol{F}.$$

We can distinguish between these spins by their different dynamical behavior. The spin dynamics not only supplies information about the spin evolution. It also gives us information about what is the point where these spins are defined.

It is clear that if $\boldsymbol{r} = \boldsymbol{q}$, the center of mass spin must always be conserved. Conversely, if $\boldsymbol{S}_{CM}$ is not conserved, this means that $\boldsymbol{r} \neq \boldsymbol{q}$, and therefore the electron has a centre of mass and center of charge which are different points.

We can find in the literature examples of both spins. Bargmann, Michel and Telegdi spin [6] satisfies a dynamical equation which is a covariant generalization of the dynamics of the $\boldsymbol{S}_{CM}$.

[6] V. Bargmann,L. Michel y V.L. Telegdi, *Precession of the polarization of particles moving in a homogeneous electromagnetic field*, Phys. Rev. Lett. **2**, 435 (1959).

It is linear in the external fields and is conserved for a free particle. The center of charge spin $\boldsymbol{S}$, satisfies the same dynamical equation than Dirac's spin operator in the quantum case, as we shall see in this lecture course. The existence of these different dynamical equations for the different spins suggest that the two centers are, necessarily, different points.

In this formalism we are going to find a definition of elementary particle which produces relativistic and nonrelativistic models of spinning particles, such that one of the main features is the separation between the center of mass and the center of charge. Finally, the only model which satisfies Dirac equation when quantized is the model, depicted on the front page, whose center of charge is moving at the speed of light.

Three in one universal constants

The three universal constants $\hbar$, c and e represent basic properties of the electron. The first $\hbar$, is related to the mechanical property, the unique value of the spin $s = \hbar/2$ of this particle. The second c, is the limit velocity that the center of charge of the electron has to be moving, if it is a different point than the center of mass. Finally the third e, is its interacting electromagnetic intensity. It is the value of the electric charge which takes a unique value, independent of their masses, for those particles which only interact in an electromagnetic way. These three universal constants define a dimensionless universal constant α, named by Arnold Sommerfeld **the fine structure constant**, which takes the value

$$\alpha = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c} \approx \frac{1}{137}.$$

It is a characteristic of the charged electromagnetic interacting particles with spin, independent of their masses. If we show that the value of this constant is unique for these particles, these three universal constants are not independent. According to Pauli[7] a theory which is not able to determine this constant, is an incomplete theory.

Theory of elementary particles

We are going to obtain in these notes a general formalism to describe elementary particles from the classical and quantum mechanical point of view. Classical mechanics is a formalism which describes the dynamical laws of material systems in terms of ordinary differential equations for the variables which represent the different independent degrees of freedom. Today we know that the different material systems are formed by small indivisible objects which are called elementary particles. It is thus necessary, that the formalism contains also the possibility of distinguishing whether a mechanical system is elementary or not. To achive this goal we shall postulate as one fundamental principle, called the atomic principle, which will establish the physical distinction, and also its mathematical translation, between elementary systems and compound systems of elementary particles. This fundamental principle will fix the degrees of freedom which characterize an elementary particle and its distinction from a non-elementary system.

In this Preamble we have analized what would happen if the center of charge of an elementary particle is a different point than its center of mass. We are suggesting that elementary particles are localizable dynamical systems, localization that is determined by the knowledge of three degrees of freedom, the position of a point, the center of charge. The arguments given in the previous sections suggest that the center of charge must satisfy a system of ordinary differential equations of fourth order. This analysis also implies that if the evolution of the center of charge

[7] W. Pauli, Nobel Lectures, vol 13, 1942-1962, Elsevier, Amsterdam (1964)

is known, the evolution of the center of mass is completely determined. Are sufficient these three degrees of freedom to describe an elementary particle? Certainly not. The different material systems, elementary or not, are localized in certain spatial regions and also have orientable properties, such as magnetic moments, angular momenta, etc., i.e., material systems move and rotate. The have, at least, degrees of freedom which describe their localization and orientation in three-dimensional space. Elementary particles also have orientable properties and therefore they also move and rotate, and thus in addition of being localizable objects they are also orientable. We have to use classical variables to describe its orientation. For example, we shall describe its orientation by attaching to the center of charge a Cartesian comoving orthogonal sytem of three unit vectors, which their orientation is changing during the evolution of the particle. This implies that, at least, an elementary particle will have six degrees of freedom. One question arises: Do we need more degrees of freedom to describe an elementary particle than the location of a point and the spatial orientation of a comovil Cartesian frame? The answer will depend on another of the fundamental principles we are going to state in the next chapter, but in this preamble what we want to stress is that, at least, an elementary particle is a mechanical localizable and orientable system.

Predictions

The formalism we are going to introduce in this lecture course is not complete. We have not been able to determine the value of the fine structure constant. Nevertheless it predicts several results and phenomena which are consistent with the standard model description of matter and others which have to be determined experimentally. Most of these predictions do not appear in the standard description of elementary particles considered as point particles. They are analyzed along the quoted sections and chapters, and we just ennumerate them here:

1. For a massive elementary particle, with two different CC and CM centers, the velocity of the CC is unreachable for every inertial observer. Since the CM velocity can never reach the velocity of the CC, there exists for any massive body a maximum limit of the velocity of its center of mass. (**Preamble**)

2. Recently, from an idea of Aníbal Hernández [8], we have obtained the Lorentz transformations with no reference to light or electromagnetic phenomena, with the hypothesis of the existence of a limit velocity for massive bodies.

3. If an elementary particle were a point particle, there will be no limit to the velocity of its center of mass which can have any arbitrary velocity. This simple criterion, contradictory with the experimental fact that material bodies have a limit velocity, rejects the possibility that an elementary particle could be described as a point particle. (**Preamble**)

4. The center of charge and center of mass of a massive elementary particle which satisfies Dirac's equation (and which we shall call from now on a **Dirac particle**) are two different points, separated by a distance $R_0 = \hbar/2mc$, in the center of mass frame. (Sec. **2.5.2**)

5. The separation between these two points is not constant for an arbitrary observer, and depends on the velocity of the center of mass with respect to the observer and the orientation of its spin. (Sec. **2.6.2**)

6. Since a Dirac particle has two different characteristic points, the center of mass (CM) $\boldsymbol{q}$ which moves at the speed $\boldsymbol{v}$ and the center of charge (CC) $\boldsymbol{r}$ moving at velocity $\boldsymbol{u}$, we can

[8] J.M Aguirregabiria, A. Hernández, M. Rivas, *Law of inertia, clock synchronization, speed limit and Lorentz transformations*, Eur. J. Phys. **41**, 045601 (2020).

define two spins with respect to both points, $\boldsymbol{S}_{CM}$ and $\boldsymbol{S}$, respectively, wich are different mechanical properties, which can be expressed in terms of the kinematics of both points in the form:

$$\boldsymbol{S} = \mp\gamma(v)m(\boldsymbol{r}-\boldsymbol{q})\times\boldsymbol{u}, \quad \boldsymbol{S}_{CM} = \mp\gamma(v)m(\boldsymbol{r}-\boldsymbol{q})\times(\boldsymbol{u}-\boldsymbol{v}),$$

where the sign $-$ is for the particle and $+$ for the antiparticle. Since $\boldsymbol{r}-\boldsymbol{q}$ has the direction opposite to the acceleration of the center of charge $\boldsymbol{r}$, which is always orthogonal to the velocity $\boldsymbol{u}$, the spin $\boldsymbol{S}$ has the opposite direction to the binormal of the trayectory of the CC for the particle and in the direction of the binormal for the antiparticle. (Sec. **2.5.4**)

7. If the elementary particle moves along a straight line at the speed of light, it is a massles particle, and if it represents an electromagnetic quantum of energy, then necessarily its spin, which lies along the direction of motion, can take only the values $S = \pm\hbar$. It is a boson from the quantum point of view. (Sec. **3.3**)

8. A photon rotates. The direction of the spin of the photon has the direction of its angular velocity but they are functionally independent. The spin of the photon is the same for all inertial observers while the angular velocity transforms according to the prescriptions of the Doppler effect. (Sec: **2.5.1**)

9. Photons are massless particles which rotate with an angular velocity along the direction of the motion, pointing forward or backwards, and of a frequency which is the same as the frequency of the corresponding electromagnetic radiation of which they represent the energy quanta. (Sec: **2.5.1**)

10. The formalism predicts that antiphotons are different particles than photons. (Sec: **2.5.1**)

11. We have made the design of a telescope for focusing antiphotons (Sec: **2.7.1**). [9]

12. The relativistic formulation forbids the existence of a spinless massive point particle moving at the speed of light. Its Lagrangian will vanish identically. If it moves at the speed of light it has to have more than three degrees of freedom and it has spin. Photons have orientation and rotate along an axis which has the direction of the linear velocity. The spin of the photon has the direction of the angular velocity but it is not related to it. (Sec. **3.3**)

13. For a massive elementary particle the center of charge is moving in circles at the speed of light around the center of mass, with a frequency $\nu_0 = 2mc^2/h$, or period $T_0 = 1/\nu_0 = h/2mc^2$, in the center of mass frame. There exists a natural clock associated to this internal motion. (Sec. **2.5.2**)

14. For another inertial observer who sees the center of mass of the electron moving at the velocity v, the electron clock is going slower, with a greater period $T = \gamma(v)T_0$, where $\gamma(v) = (1 - v^2/c^2)^{-1/2}$. (Sec. **6.2.1**)

15. The mechanical energy and linear momentum of a Dirac particle, can be expressed in terms of the center of mass velocity, like in the case of the point particle:

$$H = \pm\gamma(v)mc^2, \quad \boldsymbol{p} = \pm\gamma(v)m\boldsymbol{v},$$

where the sign $+$ is for the particle and $-$ for the antiparticle. (Sec. **2.5.2**)

[9] M. Rivas, *Considerations about photons and antiphotons*, Indian J. Phys. **96** 583-591 (2022).

16. From the classical point of view, the point where the interacting properties of an elementary particle are localized, is moving with an unreachable velocity for every inertial observer. The universal constant c represents both, the unreachable velocity of the point where the photon is localized as much as the velocity of the center of charge of a massive elementary particle with spin. When we quantize these classical systems implies that the fermionic matter and the spinning bosons which mediate in their interaction, are moving at the speed of light. (Sec. **2.5**)

17. For the center of mass observer, an elementary particle has, in addition to charge, a magnetic moment with respect to the center of mass $\boldsymbol{\mu}$, orthogonal to the trajectory plane of the center of charge and also an electric dipole moment $\boldsymbol{d}$ orthogonal to $\boldsymbol{\mu}$. (Sec. **2.5.6**)

18. The magnetic moment of an elementary particle is produced by this relative motion of the center of charge, which is not modified by any external interaction. (Sec. **2.5.6**)

19. The electron, in addition to the magnetic moment created by the motion of the center of charge, has an electric dipole moment with respect to the center of mass, already predicted by Dirac, although he considered irrelevant, and which rotates with the internal frequency of the electron.(Sec. **2.5.6**) It is also related to the Darwin term of Dirac's Hamiltonian. (Sec. **6.3.1**)

20. The quantum gyromagnetic ratio $g = 2$, is related to the double structure of the spin from the classical and quantum mechanical point of view. The spin has two parts $\boldsymbol{S} = \boldsymbol{W} + \boldsymbol{Z}$, one $\boldsymbol{W}$ related to the rotation of the particle and which does not produce magnetic moment and another $\boldsymbol{Z}$ associated to the relative motion between the center of mass and center of charge (Zitterbewegung). (Sec. **6.1**)

21. If we assume, like in the standard model, that elementary matter are Dirac particles, then from the quantum point of view their spin is necessarily $S = \hbar/2$, independently of its mass and charge. This means that leptons and quarks are fermions of spin $S = \hbar/2$. (Chap.**4**)

22. The formalism is independent of the kinematical group of space-time symmetries which define the relationship among inertial observers. It thus produces models of relativistic and non-relativistic elementary spinning particles. (Sec. **1.5**)

23. In three-dimensional space, if the center of charge of an elementary particle moves at the speed of light, the kinematical group of spacetime symmetries has to be a 11-dimensional group. This extension of the Poincaré group can be the Weyl group $\mathcal{W}$, which in addition to spacetime translations, rotations and boosts also include spacetime scale transformations which conserve the speed of light c. (Sec. **6.10**)

24. If we admit that the spacetime symmetry group of an elementary particle is the Weyl group, then every elementary particle has nonvanishing mass and spin $\hbar/2$. In the standard model, leptons and quarks are massive objects of spin $\hbar/2$. (Sec. **4.4**)

25. If an elementary particle does not interact strongly (leptons), its electric charge is unique and independent of the value of its mass. This value will be the electron electric charge e, but this value is not yet predicted. The three leptons electron, muon and tau have different masses, but the same electric charge and spin. (Sec. **5.1.5**)

26. If an elementary particle interacts also stronlgy (quarks), its electric charge is necessarily smaller than e. The formalism does not predict that this charge will be a fraction $e/3$ or $2e/3$, as is postulated in the standard model. (Sec. **5.1.5**)

27. If it would be possible to describe a quark from the classical point of view, its interacting properties will be associated to two centers: the center of the electric charge and the center of the color charge. The requirement of the atomic principle that the kinematical space should be a homogeneous space of the Poincaré group implies that both centers have to be the same point. In this way a quark will be, from the classical and quantum mechanical point of view a Dirac particle. The same criterion is aplicable to the electron if we consider that the weak interaction is a different interaction than the electromagnetic one, and the location of the weak charge and electric chrage must be the same point. (last example of Sec. **1.5**)

28. The relative orientation between the spin and magnetic moment of electrically charged elementary particles is the same for the particle and the antiparticle. It depends on the sign of the charge of what we consider is the particle. If we consider that the electron, of negative electric charge, is the particle and the positron its antiparticle, then electrons and positrons have their spin and magnetic moment in the same direction. This relative orientation for leptons has never been measured experimentally. (Sec. **4.2.7**)

29. A measurement of the relative orientation between spin and magnetic moment of electrons bound to atoms could be performed by making the hyperfine transition of atoms of Rb^{87} in an external magnetic field, by means of a beam of circularly polarized light. We have no notice that this experiment has ever been performed. (Sec. **4.2.7**)

30. Tunnel effect is not a pure quantum effect. It can also be produced in a classical framework for spinning particles, and it is related to the separation between the center of mass and the center of charge and of the spin orientation. (Sec. **6.5**)

31. By controlling the spin orientation of the electrons we can modify the probability of crossing of a potential barrier. If the spin is oriented in the direction perpendicular to the current, the probability increases, while if its orientation is along the current, decreases. This is called spin polarized tunnel effect. (Sec. **6.5**)

32. Two electrons, from the classical point of view, can form a metastable bound state of charge $2e$ and spin 1, i.e., a boson, provided their spins are parallel and the relative velocity among their center of masses is below to $0.01c$ and the phases of their internal motions are opposite to each other. This bound state is stable under external electric fields but not stable under magnetic fields orthogonal to the spins. (Sec. **6.6**)

33. In a conductor, under an external magnetic field, if the number of conducting electrons is sufficient, and the temperature is not very high, pairings of electrons with parallel spins can be produced and the paired conducting electrons can be in a superconducting phase. This is possible classically up to a certain high temperature. This maximal temperature from the quantum point of view has not been determined yet. (Sec. **6.6**)

34. For magnetic fields of intensity greater than 1 T, the analysis of the integer quantum Hall effect suggests that the longitudinal conductivity is produced by means of bound pairs of electrons in a superconducting phase. An experiment to extract charge carriers of the main current and of the negatively charged region of the transversal Hall potential would show that these charge carriers correspond to bound states of electrons of charge $2e$ and spin 1. We have no notice that this experiment has ever been performed. (Sec. **6.7**)

35. In the same way, if we apply the pointer of a tunnel effect microscope to a superconducting material, when the temperature is below the critic temperature T_c, it will extract pairs of bound electrons, while if the temperature is above T_c we will only obtain unpaired electrons. (Sec.**6.7**)

36. The classical electromagnetic field generated by a spinning electron at rest is not static. The time average value of the retarded electric field, during a turn of the center of charge, is Coulomb like in any direction and does not diverge at the center of mass. The time average value of the retarded magnetic field, during a turn of the center of charge, is the same as the magnetic field produced by a static magnetic dipole located at the center of mass, with a gyromagnetic ratio $g = 2$. If we compute the time average of the corresponding advanced fields, they do not have the above Coulomb behaviour and magnetic dipole structure, respectively. (Chap. **5**)

37. In the ground state of the Hydrogen atom the electron is in a S-state of orbital angular momentum $l = 0$. This implies, from the classical point of view, that the center of mass of the electron is going through the center of mass of the proton. This is impossible for the spinless point particle. Nevertheless this can be justified classically, because the center of mass and the center of charge of a spinning electrons are different points and their separation is greater than the estimated size of the proton. Then in the ground state of the atom the center of mass of the electron describes a straight trajectory passing through the center of mass of the proton.

38. The usual analysis of the Compton effect as an interaction of two pointlike particles, a photon and a point electron, only considers energy and linear momentum conservation. But the electromagnetic interaction also conserves the total angular momentum. If we consider, in addition to energy and linear momentum conservation, the conservation of angular momentum, shows that by controlling the orientation of the spin of the free electron, the energy and direction of the sccatered photon is uniquelly defined. (Sec. **6.4**)

39. From a theoretical point of view, the Lagrangian of an interacting elementary particle is written as $\widetilde{L} = \widetilde{L}_0 + \widetilde{L}_I$, where $\widetilde{L}_0$ is the free Lagrangian of the particle, which describes its mechanical properties, and $\widetilde{L}_I = -e\phi(t, \boldsymbol{r})\dot{t} + e\boldsymbol{A}(t, \boldsymbol{r})\dot{\boldsymbol{r}}$, is the interacting Lagrangian which predicts only an electromagnetic interaction. (Sec. **2.5.2**)

40. The analysis of the interaction of two Dirac particles leads to the conclusion that the interaction Lagrangian can be invariant under a larger group than Poincaré group, which also contains space-time dilations and local rotations. The whole analysis can be done in terms of dimensionless variables. The coupling constant is 2α where α is the fine structure constant. The $\widetilde{L}_I$ goes like $1/r$ where r is the instantaneous separation between the centers of charge of both particles. The free Lagrangian $\widetilde{L}_0$ can describe free particles of arbitrary masses, but the interaction is independent of the masses of the particles. One feature is that to have an effective interaction between both particles, the velocities $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ of the centers of charge must be different. Two electrons, if at any time, their velocities $\boldsymbol{u}_1 = \boldsymbol{u}_2$, do not interact, independently of the separation between them. (Sec. **4.5**)

41. If we call kinematical variables x, the boundary variables of any mechanical system in a variational approach, then the classical Lagrangian of any mechanical system $\widetilde{L}(x, \dot{x})$ is always a homogenous function of degree 1 of the derivatives of the kinematical variables $\dot{x}$, with respect to some arbitrary, dimensionless evolution parameter τ. (Sec. **1.3.4**)

42. If we consider that the Hamiltonian is the conjugate momentum of the time variable, then all kinematical variables $x \equiv (t, q_i, \ldots, q_i^{(k-1)})$ have associated a conjugate momentum. For any Lagrangian system we have as many conjugate pairs of canonical variables as the number of the kinematical variables. Therefore, the kinematical variables with the time excluded, represent the generalized coordinates of the canonical formalism. If some of these variables are the time derivatives of another ones, then the Lagrangian depends

on higher order derivatives, because the Lagrangian always depends on the kinematical variables and their next order time derivative. (Secs. **1.4, 1.3.2**)

43. The kinematical space of any mechanical system X is always a metric Finsler space, and the variational formulation is equivalent to a geodesic problem on the kinematical space X, where the metric depends on the kind of interaction. For an elementary particle, any interaction modifies the metric of its kinematical space. (Sec. **1.6**)

44. The metric of the kinematical space can be obtained by differentiation of the Lagrangian $\widetilde{L}$, by means of (Sec. **1.6**)

$$g_{ij}(x,\dot{x}) = \frac{1}{2}\frac{\partial^2 \widetilde{L}^2}{\partial \dot{x}_i \partial \dot{x}_j} = g_{ji}(x,\dot{x}).$$

45. The point particle is a possible model for an elementary particle in this formalism, but it corresponds to a spinless elementary particle. The extensive use of this model has to be rejected for the analysis of the behavior of the real elementary matter. It seems that there are no spinless elementary particles in nature. All physical properties associated to the spin will be masked with the use of this model. (Sec. **2.1**)

46. The kinematical space of the point particle is Minkowski space time with the constant metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Gravity, considered as another interaction and in the spirit of unification of all interactions, when applied to the point particle, would modify the Minkowski metric and will be rise, in general, to a Finsler metric but not to a Riemannian metric as is postulated in General Relativity. (Sec. **1.6**)

47. Gravity, considered as another interaction, when applied to the spinning elementary particle, would modify the metric of its kinematical space and will give rise, in general, to a Finsler metric of this manifold, and not only of the spacetime submanifold. (Sec. **1.6**)

48. A consequence of postulating a variational formulation of the dynamics, as a geodesic formulation on the kinematical space, is that this formulation contains a restricted version of the Causality Principle. When the squared metric distance between between two points on the kinematical space is definite positive the evolution between these two points is allowed, while if this is not the case, both points are causally disconected and the evolution between them is not allowed. (Sec. **1.7**)

49. A consequence, not a prediction, of postulating a variational principle like in this formalism is that it is not equivalent to a canonical formulation. Euler-Lagrange equations are equivalent to Hamilton's equations if we try to find solutions by giving boundary conditions at the initial point of the evolution t_1. But the variational formalism that leads to Euler-Lagrange equations has been formulated by the requirement that the particular solution will go from an initial state at time t_1 to a fixed final state at time t_2. The canonical formalism with boundary conditions at time t_1 is not equivalent to Euler-Lagrange equations with boundary values at times t_1 and t_2. In fact, the canonical formalism is unneccesary for the quantization of this kinematical formalism and also for the description of particles with spin. (Sec. **1.8**)

50. If the center of charge of a spinning particle is moving in circles at the speed of light, according to the well know theory of radiation of point-like particles, the free spinning particle should be radiating continuously. This is incompatible with the conservation of energy. It is necessary to develop the theory of radiation of spinning particles, which will

produce radiation whenever the center of mass of the spinning particle is accelerating, i.e., when the particle is under the action of an external force. In this case part of the energy obtained by the particle will be trasformed into radiation. This theory of radiation of spinning particles is not yet done. (Chap. **5**)

51. The requirement that the energy of any material system must be a positive definite observable, might be related to the difference between the active and passive interpretation of time translations. Passive time translations are mathematical transformations in which the zero point of the clock is changed forward or backwards, arbitrarily. From the active point of view we can only perform active time translations to the future. The conserved magnitude associated to the invariance of physical laws under active time translations cannot have both signs. The remaining invariance laws under space translations, rotations and boosts, have not definite sign, because the group transformations have both directions, form the active and passive point of view, and the conserved momenta can have both signs. The concept of negative energy in physics is meaningless. ((Sec: **6.10.3**)

52. The above comment implies that the Restricted Relativity Principle is not associated to a complete symmetry group but, as far as active time translations is concerned, would be related to the semigroup of time translations to the future. (Sec: **6.10.3**)

53. The Law of Inertia is ussually stated in the form: In any inertial reference frame a free body is at rest or is moving at a constant velocity. This formulation makes reference to the motion of the center of mass of a free body. However, elementary matter moves and rotates, and if it is free it moves with a constant velocity (constant CM velocity) and with a constant angular momentum S_{CM}. It is necessary to reformulate the Law of Inertia to include the free rotational motion.

Appendix: Elementary particles (Standard model)

We list the elementary particles of the standard model, beginning with the intermediate bosons of spin 1, (gluon g, photon γ and massive bosons $W^{\pm}$ and Z), the 6 leptons (electron e, muon μ and tau τ and the corresponding neutrinos) and the 6 quarks, all fermions of spin 1/2. Several quantum numbers, in addition to the mass and charge, are included. The isospin, spin, parity, leptonic number L, barionic number B, strangeness S and colour. We do not include information of the hypothetic graviton, which would be a massless particle of spin 2. We also include information on the recently measured Higgs boson. The leptonic number is characteristic of the three leptons, i.e., they exist three different leptonic numbers L_e, L_μ and L_τ. They exist the antiparticles of all of them, of the same mass and spin, but opposite quantum numbers.

	mass$\cdot c^2$	charge	Isospin	Spin	Par.	L	B	S	Colour	Life
g	0	0	0	1	$-$	0	0			
γ	$< 2\times10^{-16}$eV	0	0 , 1	1	$-$	0	0			
W	80.398 GeV	$\pm e$		1		0	0			
Z	91.187 GeV	0		1		0	0			
e	0.511 MeV	$-e$		1/2		1	0		0	stable
μ	105.65 MeV	$-e$		1/2		1	0		0	10^{-6}s
τ	1.777 GeV	$-e$		1/2		1	0		0	10^{-15}s
ν_e	<0.5 eV	0		1/2		1	0		0	
ν_μ	<0.17 MeV	0		1/2		1	0		0	
ν_τ	<18.2 MeV	0		1/2		1	0		0	
u	1.5 $\sim$ 3.3 MeV	$2e/3$	1/2	1/2	+	0	1/3	0	1	
d	3.5 $\sim$ 6.0 MeV	$-e/3$	1/2	1/2	+	0	1/3	0	1	
c	1.27 GeV	$2e/3$	0	1/2	+	0	1/3	0	1	
s	104 MeV	$-e/3$	0	1/2	+	0	1/3	-1	1	
t	171.2 GeV	$2e/3$	0	1/2	+	0	1/3	0	1	
b	4.2 GeV	$-e/3$	0	1/2	+	0	1/3	0	1	
Higgs	125.3 GeV	0	0	0		0	0	0		

Intensity of the Interactions

Quarks exist in 6 flavours, $u, d, \ldots, b$, have colour charge with three possible values, electric charge and mass and can interact under the four forces: strong, electromagnetic, weak and gravitational. Leptons have no colour and they do not interact strongly. They can interact under the other three forces, except neutrinos which do not interact electromagnetically. The interchange of gluons between quarks implies the change of the colour charge. Ordinary matter, made of aggregates of quarks and leptons, has no colour and therefore quarks and antiquarks can only form bound states of neutral colour. This is called **confinement**.

If the intensity of the strong interaction between quarks is 1 and of a range of order 10^{-15}m, the electromagnetic interaction, by interchange of photons γ, is of the order of the fine structure constant $\alpha = 1/137$ and of infinite range. The weak force is of very short range, around 10^{-18}m with the interchange of massive bosons $W^{\pm}$ and Z ($m>$ 80 GeV) and of intensity 10^{-6} while the gravitational force, of infinite range, is of intensity of 6×10^{-39}. Nevertheless, this intensity depends on the energy of the interacting particles. What it seems to happen is that with increasing energy all three interactions (gravity excluded) have the same intensity and the behavior is like if the particle were free. This is called asymptotic freedom. At very high energy quarks behave like free particles.

Several observables for the electron for different velocities

v/c	v^2/c^2	$\gamma(v)$	p (MeV/c)	E (MeV)	T (K)
0	0	1	0	0.511003	0
0.0001	10^{-8}	1.	0.0000511003	0.511003	8.475
0.001	10^{-6}	1.	0.000511004	0.511004	847.54
0.01	**10^{-4}**	**1.00005**	**0.00511029**	**0.511029**	**$8.47{\cdot}10^4$**
0.1	0.01	1.00504	0.0513578	0.513578	$8.47{\cdot}10^6$
0.5	0.25	1.1547	0.295028	0.590056	$2.11{\cdot}10^8$
0.86603	0.750	2.00003	0.885103	1.02202	$\cdots$
0.9	0.81	2.29416	1.05509	1.17232	$\cdots$
0.99	0.9801	7.08881	3.58618	3.62241	$\cdots$
0.999	0.99800	22.3663	11.4178	11.4292	$\cdots$
0.9999	0.99980	70.7124	36.1307	36.1343	$\cdots$
0.99999	0.99998	223.607	114.263	114.264	$\cdots$
0.999995	**0.999990**	**316.532**	**161.748**	**161.749**	$\cdots$
0.999999	0.999998	707.107	361.334	361.334	$\cdots$
0.9999999	0.9999998	2236.07	1142.64	1142.64	$\cdots$

The observables of the table are

$$\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad p = \gamma(v)mv, \quad E = \gamma(v)mc^2.$$

$$\gamma(v) \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots$$

the factor γ takes the value 2 for $v/c \approx 0.86603$. Please remark that it is necessary to reach the velocity $v/c = 0.99999999999987$ (twelve nines) with a factor $\gamma = 2 \cdot 10^6$, in order that the electron energy will be 1.00213 TeV. Today's accelerators (Tevatrón (FermiLab), LHC (Cern)) reach energies of order from 4 to 8 TeV.

The last column corresponds to the temperature in Kelvin of a nonrelativistic electron gas whose mean velocity is the indicated, and considered a system of seven degrees of freedom. The dots of some sections imply that for those velocitites the nonrelativistic analysis of statistical mechanics does not apply.

$$\frac{7}{2}\kappa T = \frac{1}{2}mv^2, \quad \kappa = 1.38 \cdot 10^{-23}\ \text{J/K (Boltzmann's Constant)}.$$

The first coloured boldface line, corresponding to $v/c = 0.01$, represents the maximal velocity of the center of masses of two electrons with parallel spins, to form a bound state, as we shall analyze in section **6.6**.

The second coloured boldface line corresponds to the experiment, not of very high energy, we shall analize in section **6.2** to measure the electron clock.

Chapter 1

Fundamental Principles

1.1 Newtonian formulation

To our knowledge, the first important approach for a theory of matter where all objects are bound systems of smaller particles is due to Newton. By definition, the simplest material particle is the point. For Newton, matter is composed of aggregates of points of mass m, of arbitrary but fixed value. Each elementary point particle satisfies a dynamical equation

$$m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}$$

where $\boldsymbol{r}$ is the location of the point and $\boldsymbol{F}$ is the total external force acting on it. If we also admit that forces satisfy Newton's third law, we arrive to the conclusion that any aggregate of matter has a characteristic point, its center of mass $\boldsymbol{q}$, defined as

$$\boldsymbol{q} = \frac{\sum m_i \boldsymbol{r}_i}{\sum m_i}, \quad m = \sum m_i$$

which satisfies

$$\sum \boldsymbol{F}_{\text{ext}} = m\frac{d^2\boldsymbol{q}}{dt^2}.$$

This is known as the center of mass theorem: *The center of mass of any material system behaves like a point particle of mass the total mass of the system, under the sum of only the external forces acting on the particles.*

Newton postulates that matter atracts each other with the universal gravitation law, which satisfies Newton's third law. If we try to sepparate a sheet of paper into two parts, assuming two pieces of around 1 g each and separated 10 cm, the gravitational force between them is

$$F = G\frac{m^2}{d^2} = 6.672 \times 10^{-11} \times 0.001^2/0.1^2 = 6.672 \times 10^{-15}\ \text{N},$$

much much smaller than the actual force we have to do to separate the sheet into two parts. Cohesion forces of matter are not of gravitational nature. Among material systems another kind of force should exist to form bound objects. Newtonian theory does not restrict the kind of forces we can have in Nature. If the point particle has a property called charge, this will be located at the same point $\boldsymbol{r}$. Then all matter will be built from arbitrary material points of arbitrary masses and charges, which in addition to the gravitational interaction they attract, and sometimes repel, each other with another kind of force of higher intensity.

If we can make a time travel, come back to Newton's time in Cambridge, and ask him: Sir, we are coming from the future and we know that matter, in addition of having mass, has another unmodified property called spin. It is possible that Sir Isaac, would think about and

would modify his second law to take into account the dynamics of the angular momentum in terms of the external torques. The important aspect is that when around 1920 quantum mechanics enters into the scene, it would produce a different quantization scenario.

Newtonian formalism is not restrictive and for the forces $\boldsymbol{F}$ among particles many kind of interactions are allowed. It is the gauge theory in the quantum case, and the atomic principle in our formalism, which will establish a limit to the allowed interactions. In another context charges, masses, angular momenta of elementary particles are not resticted and can take any value. It is quantum theory which should predict these values. Nevertheless, up to now, quantum theory has only been able to predict the values of the spin, with a total freedom for the remaining properties, like masses and charges.

Newton was already aware of this possibility of internal forces of short and long range, as he writes in his dissertation in the book III of Opticks: [1]

Now the smallest particles of matter may cohere by strongest attractions, and compose bigger particles of weaker virtue; and many of these may cohere and compose bigger particles whose virtue is still weaker, and so on for diverse successions, until the progression ends in the biggest particles on which the operations in chemistry, and the colors of natural bodies depend, and which by cohering compose bodies of a sensible magnitude.

For we must learn from the phenomena of nature what bodies attract one another, and what are the laws and properties of the attraction, before we inquire the cause by which the attraction is perform'd. The attractions of gravity, magnetism, and electricity, reach to very sensible distances, and so have been observed by vulgar eyes, and there may be others which reach to so small distances as to escape observation.

1.1.1 Chronology of Mechanics and Lagrangian Mechanics

- **1687**. The Principia Mathematica are published by Newton. $\boldsymbol{F} = m\boldsymbol{a}$
- **1733**. Euler discovers the Variational Formulation.
- **1755**. Lagrange states the necessary conditions of the Variational Formulation: Euler-Lagrange equations.
- **1788**. Lagrange writes the Mécanique Analytique: Lagrangian Formulation.
- **1833**. Hamilton establishes the canonical formulation: Hamilton's Equations.
- **1854**. Riemann formulates the structure of the metric spaces $g_{ij}(x)$.
- **1915**. Einstein postulates gravity as a modification of the Riemannian metric of space-time.
- **1915**. Noether relates continuous symmetries with conservation laws. Noether's Theorem.
- **1918**. Finsler presents a PhD thesis about general metric spaces $g_{ij}(x, \dot{x})$.
- **1934**. Cartan publishes: Les Espaces de Finsler.

The playground of the Lagrangian systems is always a metric Finsler space

[1] I. Newton, *Opticks, A treatise of the Reflections, Refractions, Inflections and Colours of Light*, Dover, NY 1952, p.394.

1.2 Fundamental Principles

Because all known elementary particles, the quarks and leptons, are spinning particles and it seems that there are no spinless elementary particles in nature, we take the challenge of obtaining a classical formalism for describing spin. The interest of a classical description of spinning matter is not important in itself, because matter, at this level, behaves according to the laws of quantum mechanics. But finer a classical description of elementary matter a deeper quantum mechanical formalism, because we will have at hand, when quantizing the system, more classical variables to deal with, and therefore with a more clear physical and/or geometrical interpretation. A second feature is that a classical formalism supplies models. Both goals, in my opinion, have been succesfully achieved.

Feynman, in the first chapter of his *Lectures on Physics* [2], states that *"If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the atomic hypothesis (or the atomic fact or whatever you wish to call it) that all things are made of atoms-little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another."*

If the atomic hypothesis is such an important principle, physics has to take advantage of this fact, and, properly formulated, should be included as a preliminary fundamental principle of elementary particle physics, as we shall do in what follows. The books of Physics, when dealing with the subject of atomism, they just mention Leuccipus and Democritus of Abdera, as the first scientists who proposed the idea that matter is finally a set of discrete undivisible objects (atoms). Democritus adds that these objects are also immutable. It is difficult to understand what Democritus would mean around 2500 years ago, about immutability. But this idea what perhaps means is that a compound system can be modified but an elementary particle cannot. We can excite a molecule, rotate it with some angular velocity, even deform and modify its mass, but this is not possible for an electron. We cannot change the electron mass and charge and we cannot rotate an electron around itself with an arbitrary angular velocity. The most we can do is to modify its orientation in space. The mass and absolute value of its spin are immutable. The atomic principle is going to restrict the number and the kind of classical variables we have to use to describe an elementary particle. These variables are not restricted for arbitrary material systems, but they are restricted for elementary particles. It is a very restrictive principle which will suggest a kind of minimal coupling interaction when analyzing compound systems of elementary particles.

The kinematical formalism for describing elementary spinning particles, previously aimed for the classical spin description of matter, has proven to be a general framework for the description of elementary particles, because it supplies a very precise definition of a classical elementary particle which has, as a quantum counterpart, Wigner's definition. All elementary systems described within this formalism have the feature that, when quantized, their Hilbert space of pure states carries a projective unitary irreducible representation of the kinematical group. It is through Feynman's path integral approach that both formalisms complement each other.

The formalism we propose is based upon the four fundamental principles:

- Restricted Relativity Principle,
- Variational Principle,
- Atomic Principle,
- Quantization Principle.

[2] Feynman RP, Leighton RB and Sands M 1968 *The Feynman Lectures on Physics*, (NY: Addison Wesley) Vol 1, Sec 1-2.

1.2.1 Restricted Relativity Principle

Restricted Relativity Principle: In absence of gravitation, there exists a class of equivalent observers, to whom the laws of physics must be the same.

When using the same kind of variables the fundamental physical laws have to be written in the same form in the different equivalent reference frames. One fundamental law is the Law of inertia which states that a free body in a reference frame can be at rest or moving with constant velocity. If this law of inertia holds for the class of equivalent observers, then the equivalent observers are at rest or moving at a constant velocity with respect to each other. We call them **inertial reference frames**.

This class of inertial observers is defined by the way they relate the measurement of any space-time event. The set of inertial observers is endowed with an equivalence law $\equiv$, i.e., it satisfies the properties of an equivalence law: reflexive $O \equiv O$, symmetric $O \equiv O'$ implies $O' \equiv O$, and transitive, if $O \equiv O'$ and $O' \equiv O''$ then $O \equiv O''$. In the language of composition of transformations this means that there exists the unit transformation, the inverse of any transformation and that the composition of transformations is associative and produces another transformation. The set of transformations among the inertial observers form a group, the **kinematical group of the formalism**.

The equivalent observers are defined with respect to each other by a spacetime transformation group. They are moving with a constant velocity and thus two possibilities arise: (a) the relative velocity has no upper limit, or (b) there exists an upper limit velocity unreachable for all of them. If this velocity exists, we represent it by c, and according to this relativity principle must be the same for all inertial observers.

If we accept that the relative situation among inertial observers contains space-time translations, static rotations and relative displacements at a constant velocity (boosts), the possibility (a) implies that if the observer O measures a space-time event given by the values of time and position t and $\boldsymbol{r}$, respectively, and observer O' measures t' and $\boldsymbol{r}'$ for the same event, these values are related by means of the transformation

$$t' = t + b, \quad \boldsymbol{r}' = R(\boldsymbol{\alpha})\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a},$$

where the ten real numbers $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ are fixed for these two observers and where by $\boldsymbol{\alpha}$ we want to represent the three parameters which define the relative orientation between the corresponding Cartesian reference frames of both observers. These equations represent the action of the Galilei group of transformations on the space-time, which is the kinematical group in the nonrelativistic framework. If instead of these transformations we use those of the Poincaré group,

$$t' = \gamma(v)\left(t + \frac{\boldsymbol{v}\cdot R(\boldsymbol{\alpha})\boldsymbol{r}}{c^2}\right) + b, \quad \boldsymbol{r}' = R(\boldsymbol{\alpha})\boldsymbol{r} + \gamma(v)\boldsymbol{v}t + \frac{\gamma(v)^2}{(1+\gamma(v))c^2}\left(\boldsymbol{v}\cdot R(\boldsymbol{\alpha})\boldsymbol{r}\right)\boldsymbol{v} + \boldsymbol{a},$$

which depend on the same 10 parameters $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ and on the universal constant c, and where $\gamma(v) = (1 - v^2/c^2)^{-1/2}$. We are in the case (b), but now with the restriction that $v < c$, and this is called a relativistic formalism.

The kinematical group associated to this fundamental principle has to be fixed once for ever. This principle is not only a statement about the *restricted universality* of the physical laws, but it is also a statement that the relative measurements between inertial observers of any other observable depends only on this group, i.e., how two inertial observers relate their relative measurements of space-time events. By restricted universality what we mean is that the physical laws are not the same for all possible observers, but only for a restricted class of them, the so called inertial observers, to whom the formalism is restricted.

If some observer is describing an electromagnetic phenomenon and we change to another accelerated observer, in this frame in addition to the corresponding electromagnetic phenomenon we shall also describe the presence of an inertial field, which is undistinguishable from a gravitational field. These two observers do not describe the same kind of phenomena. They are not equivalent observers. We are going to restrict the formalism for observers who describe the same phenomena. It is the General Relativity Principle which admits the invariance of physical laws under any change of arbitrary observers or the use of any system of coordinates, but if we include gravity between the phenomena to be described. The reason is that it is not possible locally to distinguish between a change to an accelerated frame form the presence of a gravitational field. If we admit this restricted relativity principle we have to exclude in its framework the possibility of description of gravitational phenomena.

1.2.2 Variational Principle

The **Variational Principle** states that a property called the action of any mechanical system during its evolution between some initial and final states must be stationary. The action is described in terms of a Lagrangian function which is an explicit function of the time t, the independent degrees of freedom and their subsequent time derivatives up to a finite order, which is what we are going to consider in this formalism. Usually, most mechanical textbooks restrict the Lagrangian to depend up to the first order time derivative of the independent degrees of freedom. This is the case for bound systems of spinless or point particles, for instance in the Newtonian formalism. This implies that dynamical equations for the degrees of freedom are at most second order differential equations. However, differential geometry shows that, in general, a point in a three-dimensional vector space, satisfies a fourth order differential equation. In another context we do not know yet what are the variables we need to describe spinning matter. Are we able to restrict to these unknown variables to satisfy only second order differential equations? This is a mathematical restriction which is not justified physically. Think in the discussion in the Preamble about the motion of the admisible center of charge. We are not going to restrict Lagrangians to depend only on the first order time derivatives of the independent degrees of freedom. The atomic principle will only restrict the Lagrangian to depend on a finite number of degrees of freedom and also of a finite maximum order in their derivatives.

According to this variational principle, there will be a Lagrangian function L, which will be an explicit function of the time, of a finite number of degrees of freedom and their time derivatives up to a finite order, for any mechanical system formed from a finite number of elementary particles. It is the atomic principle which will limit the maximum number of variables to describe an elementary particle.

This variational principle is so strong that when we apply it to material systems which satisfy the atomic principle, we shall arrive to the conclusion that the only allowed interaction for classical elementary particles is the electromagnetic interaction, either for spinless or spinning particles. The dynamical equation of an elementary particle of charge e, in the variational formulation, will be

$$\frac{d\boldsymbol{p}}{dt} = e\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right),$$

where $\boldsymbol{p}$ is the linear momentum of the particle, $\boldsymbol{u}$ is the velocity of the center of charge and $\boldsymbol{E}$ and $\boldsymbol{B}$ the external electromagnetic field. The expression of the linear momentum depends on the framework, either relativistic or not relativistic, i.e., of the kinematical group and in terms of the different degrees of freedom and their derivatives. In the classical variational framework, and with these three fundamental principles, we have not been able to describe other interactions. Weak and strong interactions are described in a quantum context under the assumption of local gauge invariance.

In this way we shall start in section **1.3** with the generalized Lagrangian formalism to obtain the main results in general form.

1.2.3 Atomic Principle

The **Atomic Principle** admits that matter cannot be divided indefinitely. Matter does not satisfy the hypothesis of the continuum. After a finite number of steps in the division of matter we can reach an ultimate and indivisible object, an *elementary particle*. If a theoretical framework pretends to describe real matter, it must contain in the formalism some statement or declaration about the existence of these primordial objects and the possibility to distinguish theoretically between an elementary system and another one which is not elementary.

If we take a piece of matter and we try to break it, the result is that it is first deformed and if our strength is enough it breaks into two or more pieces. The distinction between an elementary particle and any other finite mechanical system is that an elementary particle, in addition of being indivisible, if not destroyed by its antiparticle, it can never be modified. It can never have excited states, so that all possible states are only kinematical modifications of any one of them.

Since in the process of breaking matter we need a finite number of steps to reach this ultimate object, this implies that the states of an elementary can be described by a finite set of variables. If the state of an elementary particle changes, and we assume this fundamental principle, we can always find another inertial observer who describes the particle with the same values of all essential variables as before the change. One electron, if not annihilated, remains always as an electron under any interaction. This will imply a restriction in the kind of classical variables we shall use to describe the initial and final states in the variational dynamical description.

It is this explicit distinction between compound systems and elementary particles, considered as a basic part of the formalism, what makes sense to consider this atomic principle as a fundamental principle.

These three fundamental principles complete our classical framework. To quantize the formalism we have to replace the Variational Principle for the next Quantization Principle.

1.2.4 Quantization Principle

For the quantum description we must substitute this last variational principle by the **Quantization Principle**, in the form proposed by Feynman [3]: *All paths of the evolution of any mechanical system between some initial and final states are equally probable.* For each path a probability amplitude is defined, which is a complex number of the same magnitude but whose phase is the action of the system between the end points along the corresponding path. The probability amplitude for finding the system in any classical state, *i.e,* the quantum wave function, will be a squared integrable and normalized complex function of the variables which define the states in the variational approach. In this way, classical and quantum mechanics are described in terms of exactly the same set of classical variables.

This formalism will determine that these variables for an elementary particle, which define the initial and final states of the evolution in the variational description, are a finite set of variables which necessarily span a homogeneous space of the kinematical group. We shall call them the kinematical variables of the particle. The manifold they span is larger than the configuration space and in addition to the time and the independent degrees of freedom it also includes the derivatives of the independent degrees of freedom up to one order less the highest order they have in the Lagrangian. The Lagrangian for describing these systems will

[3] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, MacGraw Hill, N.Y., (1965).

be thus dependent on these kinematical variables and their next order time derivative. If the evolution is described in terms of some group invariant evolution parameter τ, then, when writting the Lagrangian not in terms of the independent degrees of freedom but as a function of the kinematical variables and their $\tau-$derivatives, it becomes a homogeneous function of first degree of the $\tau-$derivatives of all kinematical variables.

Feynman's path integral method seems to be inspired in a Dirac's paper [4]. In this article Dirac states, when comparing the Lagrangian approach with the canonical aproach, that: *the two formulations are, of course, closely related, but there are reasons for believing that the Lagrangian one is more fundamental.* Later, he expresses that *we ought to consider the classical Lagrangian, not as a function of the coordinates and velocities, but rather as a function of the coordinates at time t and at time $t + dt$.* Here, he is clearly suggesting the use of boundary variables, i.e, the kinematical variables for the expression of the Lagrangian.

In the Preface of Feynman and Hibbs book, it is mentioned that Feynman, in a private conversation with a European colleague, became aware of the mentioned Dirac's paper, suggesting that the wave function at time $t+\epsilon$ would be related to the wave function at time t in the form

$$\psi(t+\epsilon) \sim e^{i\epsilon L/\hbar}\psi(t).$$

What Feynman did was to postulate that the above relation is an identity. There is a quotation in the book [5] that the European colleague was Herbert Jehle, while visiting Princeton in 1941.

We shall analyze several examples of spinning particles. But we shall be surprised that, for the description of free elementary particles, in particular a Dirac particle, is not necessary to postulate any Lagrangian. The analysis of Noether's theorem and conservation laws, and the group invariants will be sufficient to describe the dynamics of a free spinning elementary particle.

1.3 Variational Principle: Lagrangian Formalism

The Lagrangian formalism postulated by Lagrange (1788) was generalized for systems depending on higher order derivatives by Ostrogradsky (1850). [6] The result is that if the Lagrangian depends on time t, the n degrees of freedom $q_i(t)$ and their first order time derivatives $L(t, q_i, \dot{q}_i)$, Euler-Lagrange equations are

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \quad i = 1, \ldots, n.$$

But if the Lagrangian depends up to the derivatives of order k-th of the degrees of freedom, the equations are

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(1)}}\right) + \cdots + (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial q_i^{(k)}}\right) = 0, \quad i = 1, \ldots, n, \tag{1.1}$$

where we use here an exponent between brackets to express the order of the time derivative of the corresponding variable. We shall arrive to these dynamical equations as a necessary condition for the action to be stationary. But it is also important to remark that the general formalism which brings us to equations (1.1), requires that the end points of the evolution, i.e., the boundary conditions for the evolution, remain fixed. In addition to obtain Euler-Lagrange

[4] P.A.M. Dirac, *The Lagrangian in quantum mechanics*, Phys. Zeitsch. der Sowjetunion, **3**, 64–72 (1933).

[5] L.M. Brown (editor), Feynman's thesis: A new approach to quantum theory, (World Scientific 2005)

[6] M. Ostrogradsky, *Mémoire sur les équations différentielles relatives au problème des isopérimètres,* Mem. Acad. St. Petersburg, **6**(4), 385-517 (1850).

equations, we are going to see what are these boundary variables which define in an essential way the initial and final states of the evolution, and which we propose to call them **kinematical variables**. In particular, it is the atomic principle which will limit what these variables are for an elementary particle.

Finally we are going to analyze the geometrical structure of the space spanned by the kinematical variables. We shall see that for any arbitrary system it is always a metric space, but not Riemannian but rather a Finsler metric space. In this way, if a mechanical system of a definite number of kinematical variables is analyzed under two different interactions, the kinematical space is the same manifold but the Finsler metrics associated to the corresponding interactions are different. In this way, gravity will be studied as a modification of the Finsler metric of the manifold which describes free elementary spinning matter, modification produced by the material content of all objects in the universe, including the analyzed elementary particles.

1.3.1 Euler-Lagrange equations

Let us consider a mechanical system of n degrees of freedom, characterized by a Lagrangian that depends on time t and on the n essential coordinates $q_i(t)$, that represent the n independent degrees of freedom, and their derivatives up to a finite order k. Because we can have time derivatives of arbitrary order we use a superindex enclosed in brackets to represent the corresponding k-th derivative, *i.e.*, $q_i^{(k)}(t) = d^k q_i(t)/dt^k$. The **action functional** is defined by:

$$\mathcal{A}[q] = \int_{t_1}^{t_2} L(t, q_i(t), q_i^{(1)}(t), \ldots, q_i^{(k)}(t))dt, \tag{1.2}$$

where $i = 1, \ldots, n$. For any trajectory $q_i(t)$ introduced into the integral (1.2), we shall obtain a real number, the action of the system along that trajectory.

> **Variational Principle:** The trajectory followed by the dynamical system is that path which passing through the fixed end points defined at times t_1 and t_2, respectively, where we fix on them the values of the variables and their time derivatives $q_i^{(s)}(t_1)$ and $q_i^{(s)}(t_2)$, $i = 1, ..., n$, $s = 0, 1, ..., k-1$, up to the maximum $(k-1)$ -th order, makes stationary the action functional (1.2), i.e., the value of the action along that path is a maximum or a minimum.

Please remark that we need to fix as boundary values of the variational principle some particular values of time t, the n degrees of freedom q_i and their time derivatives up to order $k-1$, *i.e.*, one order less than the highest derivative of each variable q_i in the Lagrangian, at both end points. Although the values we fix as boundary variables correspond to the degrees of freedom and their derivatives, their fixed values are considered as essential parameters, and therefore they are selected without constraints. They uniquely define the initial and final state.

Conversely we can say that the Lagrangian of any arbitrary generalized system is in general an explicit function of the variables we keep fixed as end points of the variational formulation and also of their next order time derivative.

Once the action functional (1.2) is defined for some particular path $q_i(t)$, to analyze its variation let us produce an infinitesimal modification of the functions $q_i(t)$, $q_i(t) \to q_i(t) + \delta q_i(t)$ while leaving fixed the end-points of the variational problem, *i.e.*, such that at t_1 and t_2 the modification of the generalized coordinates and their derivatives up to order $k-1$ vanish, and thus $\delta q_i^{(s)}(t_1) = \delta q_i^{(s)}(t_2) = 0$, for $i = 1, \ldots, n$ and $s = 0, 1, \ldots, k-1$. Then, the variation of the derivatives of the $q_i(t)$ is given by $q_i^{(s)}(t) \to q_i^{(s)}(t) + \delta q_i^{(s)}(t) = q_i^{(s)}(t) + d^s \delta q_i(t)/dt^s$, since the modification of the s-th derivative function is just the s-th derivative of the modification of the corresponding function.

This produces a variation in the action functional $\delta\mathcal{A} = \mathcal{A}[q + \delta q] - \mathcal{A}[q]$, given by:

$$\delta\mathcal{A} = \int_{t_1}^{t_2} L(t, q_i^{(s)}(t) + \delta q_i^{(s)}(t))dt - \int_{t_1}^{t_2} L(t, q_i^{(s)}(t))dt$$

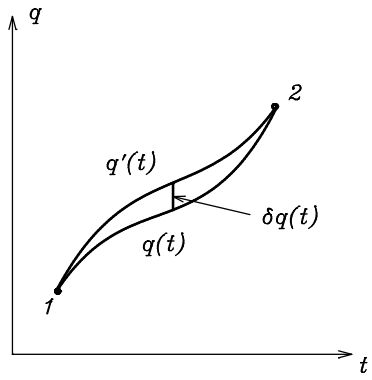


Figure 1.1: **Two close paths $q(t)$ and the modified $q'(t)$, passing through the same end points 1 and 2**

$$= \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial q_i^{(1)}} \delta q_i^{(1)} + \cdots + \frac{\partial L}{\partial q_i^{(k)}} \delta q_i^{(k)} \right], \tag{1.3}$$

after expanding to lowest order the first integral. The term

$$\frac{\partial L}{\partial q_i^{(1)}} \delta q_i^{(1)} = \frac{\partial L}{\partial q_i^{(1)}} \frac{d}{dt} \delta q_i = \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(1)}} \delta q_i \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(1)}} \right) \delta q_i,$$

and by integration of this expression between t_1 and t_2, it gives:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial q_i^{(1)}} \delta q_i^{(1)} dt = \frac{\partial L}{\partial q_i^{(1)}} \delta q_i(t_2) - \frac{\partial L}{\partial q_i^{(1)}} \delta q_i(t_1) - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(1)}} \right) \delta q_i dt$$

$$= - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(1)}} \right) \delta q_i \, dt,$$

because the variations $\delta q_i(t_1)$ and $\delta q_i(t_2)$, vanish. Similarly for the next term:

$$\frac{\partial L}{\partial q_i^{(2)}} \delta q_i^{(2)} = \frac{\partial L}{\partial q_i^{(2)}} \frac{d}{dt} \delta q_i^{(1)} = \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(2)}} \delta q_i^{(1)} \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(2)}} \right) \delta q_i^{(1)},$$

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial q_i^{(2)}} \delta q_i^{(2)} dt = - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(2)}} \right) \delta q_i^{(1)} dt = \int_{t_1}^{t_2} \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial q_i^{(2)}} \right) \delta q_i \, dt,$$

because δq_i and $\delta q_i^{(1)}$ vanish at t_1 and t_2, and finally for the last term

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial q_i^{(k)}} \delta q_i^{(k)} dt = (-1)^k \int_{t_1}^{t_2} \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q_i^{(k)}} \right) \delta q_i \, dt,$$

so that each term of (1.3) is written only in terms of the variations of the degrees of freedom δq_i and not of their higher order derivatives. Remark that to reach these final expressions, it has been necessary to assume the vanishing of all $\delta q_i^{(s)}$, for $s = 0, \ldots, k-1$, at times t_1 and t_2. By collecting all terms we get

$$\delta \mathcal{A} = \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(1)}} \right) + \cdots + (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q_i^{(k)}} \right) \right] \delta q_i.$$

If the action functional is extremal along the path $q_i(t)$, its variation must vanish, $\delta \mathcal{A} = 0$. The variations δq_i are arbitrary and therefore all terms between squared brackets cancel out. We obtain a system of n ordinary differential equations, the **Euler-Lagrange equations** (1755),

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(1)}} \right) + \cdots + (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q_i^{(k)}} \right) = 0, \quad i = 1, \ldots, n. \tag{1.4}$$

1.3.2 Kinematical space

In general, the system (1.4) is a system of n ordinary differential equations of order $2k$, and thus existence and uniqueness theorems guarantee only the existence of a solution of this system for the $2kn$ boundary conditions $q_i^{(s)}(t_1)$, $i = 1, \ldots, n$ and $s = 0, 1, \ldots, 2k-1$, at the initial instant t_1. However the variational problem has been stated by the requirement that the solution goes through the two fixed endpoints, a condition that does not guarantee neither the existence nor the uniqueness of the solution. Nevertheless, let us assume that with the fixed endpoint conditions of the variational problem, $q_i^{(s)}(t_1)$ and $q_i^{(s)}(t_2)$, $i = 1, \ldots, n$ and $s = 0, 1, \ldots, k-1$, at times t_1 and t_2, respectively, there exists a solution of (1.4) perhaps non-unique. This implies that the $2kn$ integration constants of the system (1.4), can be expressed perhaps in a non-uniform way, as functions of the kn conditions at each of the two endpoints. From now on, we shall consider systems in which this condition is satisfied. It turns out that a particular solution passing through these points will be expressed as a function of time with some explicit dependence of the end point values

$$\widetilde{q}_i(t) \equiv q_i(t;\, q_j^{(r)}(t_1), q_l^{(r)}(t_2)), \tag{1.5}$$

$i, j, l = 1, \ldots, n,\ r = 0, 1, \ldots k-1$, in terms of these boundary end point conditions.

> **Definition:** The **Action Function** [7] of the system along a classical path is the value of the action functional (1.2) when we introduce in the integrand a particular solution (1.5) of Euler-Lagrange equations (1.4) passing through those endpoints:
>
> $$\int_{t_1}^{t_2} L\left(t, \widetilde{q}_i(t)\right) dt = A\left(t_1, q_i^{(r)}(t_1); t_2, q_i^{(r)}(t_2)\right). \tag{1.6}$$

Once the time integration is performed, we see that it will be an explicit function of the $kn+1$ variables at the initial instant, $q_j^{(r)}(t_1)$, $r = 0, \ldots, k-1$ including the time t_1, and of the corresponding $kn+1$ variables at final time t_2. We write it as

$$A\left(t_1, q_i^{(r)}(t_1); t_2, q_i^{(r)}(t_2)\right) \equiv A(x_1, x_2).$$

We thus arrive at the following

> **Definition:** The **kinematical variables** of the system are the time t and the n degrees of freedom q_i and their time derivatives up to order $k-1$. The manifold X they span is the **kinematical space** or state space of the mechanical system.

The kinematical space for ordinary Lagrangians is just the configuration space spanned by variables q_i enlarged with the time variable t. It is usually called the **enlarged configuration space**. But for generalized Lagrangians it also includes higher order derivatives up to one order less than the highest derivative that appears in the Lagrangian. Thus, **the action function of a system becomes a function of the values the kinematical variables take at the end points of the trajectory**, x_1 and x_2. From now on we shall consider systems for which the action function is defined and is a continuous and differentiable function of the kinematical variables at the end points of its possible evolution. This function clearly has the property $A(x, x) = 0$.

[7] Please remark that we use the same letter $A(\ )$ for the action function, followed by normal brackets containing the variables of which it depends, and for the action functional $\mathcal{A}[\ ]$ which is followed by squared brackets to enhance that it is not a function but rather a functional over the class of all paths.

1.3.3 Replacement of time as evolution parameter

The constancy of speed of light in special relativity brings space and time variables on the same footing and time is relative to every observer. So, the next step is to remove the time observable as the evolution parameter of the variational formalism and express the evolution as a function of some arbitrary parameter, the same for all inertial observers. Then, let us assume that the trajectory of the system can be expressed in parametric form, $\{t(\tau), q_i(\tau)\}$, in terms of some arbitrary evolution parameter τ, the same for all inertial observers. The functional (1.2) can be rewritten in terms of the kinematical variables and their derivatives and becomes:

$$\mathcal{A}[t,q] = \int_{\tau_1}^{\tau_2} L\left(t(\tau), q_i(\tau), \frac{\dot{q}_i(\tau)}{\dot{t}(\tau)}, \ldots, \frac{\dot{q}_i^{(k-1)}(\tau)}{\dot{t}(\tau)}\right) \dot{t}(\tau) d\tau$$

$$= \int_{\tau_1}^{\tau_2} \widetilde{L}\left(x(\tau), \dot{x}(\tau)\right) d\tau, \tag{1.7}$$

where the dot means derivative with respect to the evolution variable τ that without loss of generality can be taken dimensionless. Therefore $\widetilde{L} \equiv L(t(\tau), \dot{q}_i^{(s)}/\dot{t}(\tau))\, \dot{t}(\tau)$ has dimensions of action.

1.3.4 Homogeneity of the Lagrangian

We can also see that the integrand $\widetilde{L}$ is a homogeneous function of first degree as a function of the $\tau-$derivatives of the kinematical variables. In fact, each time derivative function $q_i^{(s)}(t)$ has been replaced by the quotient $\dot{q}_i^{(s-1)}(\tau)/\dot{t}(\tau)$ of two derivatives with respect to τ. Even the highest order k-th derivative function $q_i^{(k)} = \dot{q}_i^{(k-1)}/\dot{t}$, is expressed in terms of the derivatives of the kinematical variables $q_i^{(k-1)}$ and t. Thus the original function L, without tilde, is a homogeneous function of zero degree of the derivatives of the kinematical variables. Finally, the last term $\dot{t}(\tau)$, gives to the new defined $\widetilde{L}$ the character of a homogeneous function of first degree.

If we replace each $\dot{x}^i$ by $y^i = \lambda \dot{x}^i$, then $\widetilde{L}(x,y) = \widetilde{L}(x(\tau), \lambda\dot{x}(\tau)) = \lambda \widetilde{L}(x(\tau), \dot{x}(\tau))$. Therefore Euler's theorem on homogeneous functions gives rise, by taking the derivative with respect to λ of both sides, and taking $\lambda = 1$, to the result

$$\widetilde{L}(x(\tau), \dot{x}(\tau)) = \sum_j \frac{\partial \widetilde{L}}{\partial y^j} \dot{x}^j \Bigg|_{\lambda=1} = \sum_j \frac{\partial \widetilde{L}}{\partial \dot{x}^j} \dot{x}^j = \sum_j F_j(x, \dot{x}) \dot{x}^j. \tag{1.8}$$

This possibility of expressing the Lagrangian as a homogeneous function of first degree of the derivatives was already considered in 1933 by Dirac [8] on aesthetical grounds. It is this homogeneity of first degree in terms of the derivatives which will allow us later to transform the variational formalism into a geodesic problem on the kinematical space X, but where the metric $g_{ij}(x, \dot{x})$ will be direction dependent, and thus the particle trajectory is a geodesic, not in a Riemannian manifold but rather in a Finsler space.[9]

The function $\widetilde{L}$ is not an explicit function of the evolution parameter τ and thus we can see that the variational problem (1.7), is invariant with respect to any arbitrary change of evolution parameter τ. [10]

[8] P.A.M. Dirac, *Proc. Cam. Phil. Soc.* **29**, 389 (1933): "a greater elegance is obtained", "a symmetrical treatment suitable for relativity."

[9] G.S. Asanov, *Finsler geometry, Relativity and Gauge theories*, Reidel Pub. Co, Dordrecht (1985).

[10] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, Interscience, N.Y. (1970); I.M. Gelfand, S.V. Fomin, *Calculus of Variations* Prentice Hall, Englewood Cliffs, N.J. (1963).

In fact, if we change the evolution parameter $\tau = \tau(\theta)$, then the derivative $\dot{t}(\tau) = (dt/d\theta)(d\theta/d\tau)$ and $\dot{q}_i^{(s)}(\tau) = (dq_i^{(s)}(\theta)/d\theta)(d\theta/d\tau)$ such that the quotients

$$\frac{\dot{q}_i^{(s)}(\tau)}{\dot{t}(\tau)} = \frac{(dq_i^{(s)}(\theta)/d\theta)\,\dot{\theta}(\tau)}{(dt(\theta)/d\theta)\,\dot{\theta}(\tau)} \equiv \frac{\dot{q}_i^{(s)}(\theta)}{\dot{t}(\theta)},$$

where once again this last dot means derivation with respect to θ. It turns out that (1.7) can be written as:

$$\begin{aligned} A[t,q] &= \int_{\tau_1}^{\tau_2} L(t(\theta), q_i(\theta), \ldots, \dot{q}_i^{(k-1)}(\theta)/\dot{t}(\theta))\frac{dt(\theta)}{d\theta}d\theta \\ &= \int_{\theta_1}^{\theta_2} \widehat{L}(x(\theta), \dot{x}(\theta))d\theta. \end{aligned} \tag{1.9}$$

1.3.5 Recovering the Lagrangian from the Action function

The formalism thus stated has the advantage that it is independent of the evolution parameter, and if we want to come back to a time evolution description, we just use the time of the corresponding inertial observer as the evolution parameter and make the replacement $\tau = t$, and therefore $\dot{t} = 1$. In this case the homogeneity of the Lagrangian disappears. From now on we shall consider those systems for which the evolution can be described in a parametric form, and we shall use the symbol $\widetilde{\ }$ over the Lagrangian, which is understood as written in terms of the kinematical variables and their first order $\tau-$derivative. In this way we shall distinguish between the Lagrangians $\widetilde{L}$, from the Lagrangians L, without the symbol $\widetilde{\ }$, when we make the analysis in a time evolution description. To pass from $\widetilde{L}$ to L is just to make $t = \tau$, and thus $\dot{t} = 1$.

If what we know is the action function of any system $A(x_1, x_2)$, as a function of the kinematical variables at the end points we can proceed conversely and recover the Lagrangian $\widetilde{L}(x, \dot{x})$ by the limiting process:

$$\widetilde{L}(x, \dot{x}) = \lim_{y \to x} \frac{\partial A(x, y)}{\partial y^j}\dot{x}^j, \tag{1.10}$$

where the usual addition convention on repeated or dummy index j, extended to the whole set of kinematical variables, has been assumed.

If in (1.7) we consider two very close points $x_1 \equiv x$ and $x_2 \equiv x + dx$, we have that the action function $A(x, x + dx) = A(x, x + \dot{x}d\tau) = \widetilde{L}(x, \dot{x})d\tau$ and making a Taylor expansion of the function A with the condition $A(x, x) = 0$ we get (1.10).

In a certain sense the knowledge of the action function $A(x_1, x_2)$ characterizes the dynamics in a global way because by means of (1.10) $\widetilde{L}$ is determined and therefore, Euler-Lagrange equations.

1.3.6 Symmetry of a dynamical system

A symmetry of a dynamical system is defined as that **mathematical transformation of the variables of the dynamical system which leaves invariant the dynamical equations**. Since the composition of symmetries produces new symmetries, and this composition is associative and there exists the trivial or identity transformation, the set of symmetries of any dynamical system forms a group. It is the **symmetry group** of the system. If we admit as a fundamental principle the Restricted Relativity Principle, then the kinematical group of space-time transformations, which define the relationship between equivalent observers, is a subgroup of the general symmetry group.

If a transformation leaves invariant the Lagrangian of a dynamical system, then that transformation represents a symmetry for this mechanical system. The opposite is not true, i.e., there can be transformations which are symmetries but they do not leave the Lagrangian invariant. If

the Lagrangian, under a transformation, changes into another Lagrangian which differs from the previous one in a function which is a total derivative, with respect to the evolution parameter τ of some arbitrary function $\lambda(x)$ of the kinematical variables, then that transformation is a symmetry.

The symmetry transformations can be continuous or discrete. A transformation is discrete if it is an element of a discrete or finite subgroup, like the transformation $t' = -t$, which represents a time reversal. This is a discrete transformation and if it is a symmetry we shall say that the mechanical system is time reversal invariant. Continuous transformations are those related to continuous or Lie groups, for instance translations and rotations. In the case of continuous groups, it is sufficient to make the analysis of the symmetries by considering only the infinitesimal transformations, i.e., what is called the Lie algebra of the group. In the appendix **1.10**, we make a short introduction to continuous groups to fix the notation and the representation of the infinitesimal transformations and the generators of the group and its Lie algebra.

1.3.7 Lagrangian gauge functions

In the variational formulation of classical mechanics

$$\mathcal{A}[\,q\,] = \int_{t_1}^{t_2} L(t, q_i^{(s)}(t))dt \equiv \int_{\tau_1}^{\tau_2} \widetilde{L}(x, \dot{x})d\tau, \tag{1.11}$$

$\mathcal{A}[\,q\,]$ is a path functional, *i.e.*, it takes in general different values for the different paths joining the fixed end points x_1 and x_2. Then it is necessary that $\widetilde{L}d\tau$ be a non-exact differential. Otherwise, if $Ldt = d\lambda$, then $\mathcal{A}[\,q\,] = \lambda_2 - \lambda_1$ and the functional does not distinguish between the different paths and the action function of the system from x_1 to x_2, $A(x_1, x_2) = \lambda(x_2) - \lambda(x_1)$, is expressed in terms of the function $\lambda(x)$, and is thus, path independent.

If $\lambda(x)$ is a real function defined on the kinematical space X of a Lagrangian system with action function $A(x_1, x_2)$, then the function $A'(x_1, x_2) = A(x_1, x_2) + \lambda(x_2) - \lambda(x_1)$ is another action function equivalent to $A(x_1, x_2)$. In fact it gives rise by (1.10) to the Lagrangian $\widetilde{L}'$ that differs from $\widetilde{L}$ in a total τ-derivative. [11]

Using (1.10), we have

$$\widetilde{L}'(x, \dot{x}) = \widetilde{L}(x, \dot{x}) + \frac{d\lambda}{d\tau}, \tag{1.12}$$

and therefore $\widetilde{L}$ and $\widetilde{L}'$ produce the same dynamical equations and $A(x_1, x_2)$ and $A'(x_1, x_2)$ are termed as equivalent action functions.

Let us assume a Lagrangian system of one degree of freedom described by the Lagrangian $L(t, q, \dot{q})$ and we modify this Lagrangian in the form $L' = L + d\lambda(t, q)/dt$. The dynamical equations derived from L' are:

$$L' = L + \frac{\partial \lambda}{\partial t} + \frac{\partial \lambda}{\partial q}\dot{q}, \quad \frac{\partial L'}{\partial q} = \frac{\partial L}{\partial q} + \frac{\partial^2 \lambda}{\partial q \partial t} + \frac{\partial^2 \lambda}{\partial q^2}\dot{q},$$

$$\frac{\partial L'}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} + \frac{\partial \lambda}{\partial q}, \quad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left(\frac{\partial \lambda}{\partial q}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial^2 \lambda}{\partial t \partial q} + \frac{\partial^2 \lambda}{\partial q^2}\dot{q},$$

and thus

$$\frac{\partial L'}{\partial q} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

and therefore L' and L produce the same dynamical equations. This result is completely general if L depends on more than one degree of freedom or even if the Lagrangian depends on higher order derivatives. The only condition is that the function λ must be a function of the kinematical variables.

[11] J.M. Levy-Leblond, *Comm. Math. Phys.* **12**, 64 (1969).

Let G be a transformation group of the enlarged configuration space (t, q_i), that can be extended to a transformation group of the kinematical space X. Let $g \in G$ be an arbitrary element of G and $x' = gx$, the transformation of x. Consider a mechanical system characterized by the action function $A(x_1, x_2)$ that under the transformation g is changed into $A(x'_1, x'_2)$. If G is a symmetry group of the system, *i.e.*, the dynamical equations in terms of the variables x' are the same as those in terms of the variables x, this implies that $A(x'_1, x'_2)$ and $A(x_1, x_2)$ are necessarily equivalent action functions, and thus they will be related by:

$$A(gx_1, gx_2) = A(x_1, x_2) + \alpha(g; x_2) - \alpha(g; x_1). \tag{1.13}$$

The function α will be in general a continuous function of g and x. This real function $\alpha(g; x)$ defined on $G \times X$ is called a **gauge function** of the group G for the kinematical space X. Because of the continuity of the group it satisfies $\alpha(e; x) = 0$, e being the neutral element of G. If the transformation g is infinitesimal, let us represent it by the coordinates δg^σ, then $\alpha(\delta g; x) = \delta g^\sigma \lambda_\sigma(x)$ to first order in the group parameters. The transformation of the action function takes the form

$$A(\delta g x_1, \delta g x_2) = A(x_1, x_2) + \delta g^\sigma \lambda_\sigma(x_2) - \delta g^\sigma \lambda_\sigma(x_1),$$

i.e., in the form required by Noether's theorem to obtain the corresponding conserved quantities, as we shall show in the next section. In general, λ_σ functions for gauge-variant Lagrangians are obtained by

$$\lambda_\sigma(x) = \left. \frac{\partial \alpha(g; x)}{\partial g^\sigma} \right|_{g=0}. \tag{1.14}$$

Because of the associative property of the group law, any gauge function satisfies the identity

$$\alpha(g'; gx) + \alpha(g; x) - \alpha(g'g; x) = \xi(g', g), \tag{1.15}$$

where the function ξ, defined on $G \times G$, is independent of x and is an exponent of the group G.

This can be seen by the mentioned associative property of the group law. From (1.13) we get:

$$A(g'gx_1, g'gx_2) = A(x_1, x_2) + \alpha(g'g; x_2) - \alpha(g'g; x_1), \tag{1.16}$$

and also

$$A(g'gx_1, g'gx_2) = A(gx_1, gx_2) + \alpha(g'; gx_2) - \alpha(g'; gx_1)$$
$$= A(x_1, x_2) + \alpha(g; x_2) - \alpha(g; x_1) + \alpha(g'; gx_2) - \alpha(g'; gx_1),$$

and therefore by identification of this with the above (1.16), when collecting terms with the same x argument we get

$$\alpha(g'; gx_2) + \alpha(g; x_2) - \alpha(g'g; x_2) = \alpha(g'; gx_1) + \alpha(g; x_1) - \alpha(g'g; x_1),$$

and since x_1 and x_2 are two arbitrary points of X, this expression is (1.15) and defines a function $\xi(g', g)$, independent of x.

It is shown by Levy-Leblond in the previous reference that if X is a homogeneous space of G, *i.e.*, if there exists a subgroup H of G such that $X = G/H$, then, the exponent ξ is equivalent to zero on the subgroup H, and the gauge functions for homogeneous spaces become:

$$\alpha(g; x) = \xi(g, h_x), \tag{1.17}$$

where h_x is any group element of the coset space represented by $x \in G/H$.

For the Poincaré group $\mathcal{P}$ all its exponents are equivalent to zero and thus the gauge functions when X is a homogeneous space of $\mathcal{P}$ are identically zero. Lagrangians of relativistic systems whose kinematical spaces are homogeneous spaces of $\mathcal{P}$ can be taken strictly invariant.

However, the Galilei group $\mathcal{G}$ has nontrivial exponents, that are characterized by a parameter m that is interpreted as the total mass of the system, and thus Galilei Lagrangians for massive systems are not in general invariant under $\mathcal{G}$. In the quantum formalism, the Hilbert space of states of a massive nonrelativistic system carries a projective unitary representation of the Galilei group instead of a true unitary representation. [12]

1.4 Generalized Noether's theorem

Noether's analysis for generalized Lagrangian systems also states the following

Theorem: To every one-parameter group of continuous transformations that leaving the dynamical equations invariant, transform the action function of the system in the form

$$A(\delta g x_1, \delta g x_2) = A(x_1, x_2) + \lambda(x_2)\delta g - \lambda(x_1)\delta g,$$

and where $\lambda(x)$ is a function defined on the kinematical space, there is associated a classical observable N, which is a constant of the motion.

This observable N is written, as we will see in (1.37), in terms of the function $\lambda(x)$, is linear in the Hamiltonian H and in the canonical momenta $p_{i(s)}$, and depends on the infinitesimal action of the group on the kinematical variables.
In addition to the function $\lambda(x)$ contains as many terms as kinematical variables.

The requirement of Noether's theorem for the transformation of the action function is equivalent to the requirement for the Lagrangian to transform under the corresponding infinitesimal transformation in the way:

$$\widetilde{L}(\delta g x, \delta g \dot{x}) = \widetilde{L}(x, \dot{x}) + \frac{d\lambda(x)}{d\tau}\delta g,$$

i.e., invariant up to a total $\tau-$derivative of a function of the kinematical variables.

Proof:
Let us assume the existence of a one-parameter continuous group of transformations G, of the enlarged configuration space (t, q_i), that can be extended as a transformation group of the whole kinematical space X. Let δg be an infinitesimal element of G and its action on these variables is given by:

$$t \to t' = t + \delta t = t + M_0(t,q)\delta g, \tag{1.18}$$

$$q_i(t) \to q_i'(t') = q_i(t) + \delta q_i(t) = q_i(t) + M_i^{(0)}(t,q)\delta g, \tag{1.19}$$

and its extension on the remaining kinematical variables by

$$q'^{(1)}_i(t') = q_i^{(1)}(t) + \delta q_i^{(1)}(t) = q_i^{(1)}(t) + M_i^{(1)}(t, q, q^{(1)})\delta g, \tag{1.20}$$

and in general

$$q'^{(s)}_i(t') = q_i^{(s)}(t) + \delta q_i^{(s)}(t) = q_i^{(s)}(t) + M_i^{(s)}(t, q, \ldots, q^{(s)})\delta g, \qquad s = 0, 1, \ldots, k-1, \tag{1.21}$$

where M_0 and $M_i^{(0)}$ are functions only of q_i and t while the functions $M_i^{(s)}$ with $s \geq 1$, obtained in terms of the derivatives of the previous ones, will be functions of the time t and of the variables q_i and their time derivatives up to order s.

[12] see ref.7 and also J.M. Levy-Leblond, *Galilei Group and Galilean Invariance*, in E.M. Loebl, *Group Theory and its applications*, Acad. Press, NY (1971), vol. 2, p. 221.

For instance,

$$q'^{(1)}_i(t') \equiv \frac{dq'_i(t')}{dt'} = \frac{d(q_i(t) + M_i^{(0)}\delta g)}{dt}\,\frac{dt}{dt'},$$

but up to first order in δg

$$\frac{dt'}{dt} = 1 + \frac{dM_0(t,q)}{dt}\delta g, \quad \frac{dt}{dt'} \approx 1 - \frac{dM_0(t,q)}{dt}\delta g,$$

and thus

$$q'^{(1)}_i(t') = q_i^{(1)}(t) + \left(\frac{dM_i^{(0)}(t,q)}{dt} - q_i^{(1)}\frac{dM_0(t,q)}{dt}\right)\delta g,$$

and comparing with (1.20) we get

$$M_i^{(1)}(t,q,q^{(1)}) = \frac{dM_i^{(0)}(t,q)}{dt} - q_i^{(1)}\frac{dM_0(t,q)}{dt},$$

where the total time derivatives

$$\frac{dM_0(t,q)}{dt} = \frac{\partial M_0}{\partial t} + \sum_j \frac{\partial M_0}{\partial q_j}\,q_j^{(1)}, \quad \frac{dM_i^{(0)}(t,q)}{dt} = \frac{\partial M_i^{(0)}}{\partial t} + \sum_j \frac{\partial M_i^{(0)}}{\partial q_j}\,q_j^{(1)}.$$

The remaining $M_i^{(s)}$ for $s > 1$, are obtained in the same way from the previous $M_i^{(s-1)}$.

Under δg the change of the action functional of the system is:

$$\begin{aligned}\delta\mathcal{A}[\,q\,] &= \int_{t'_1}^{t'_2} L(t', q'^{(s)}_i(t'))dt' - \int_{t_1}^{t_2} L(t, q_i^{(s)}(t))dt \\ &= \int_{t'_1}^{t'_2} L(t+\delta t, q_i^{(s)}(t) + \delta q_i^{(s)}(t))dt' - \int_{t_1}^{t_2} L(t, q_i^{(s)}(t))dt.\end{aligned}$$

By replacing in the first integral the integration range (t'_1, t'_2) by (t_1, t_2) having in mind the Jacobian of t' in terms of t, this implies that the differential $dt' = (1 + d(\delta t)/dt)dt$, and thus:

$$\begin{aligned}\delta\mathcal{A}[\,q\,] &= \int_{t_1}^{t_2} L(t+\delta t, q_i^{(s)} + \delta q_i^{(s)})\left(1 + \frac{d(\delta t)}{dt}\right)dt - \int_{t_1}^{t_2} L(t, q_i^{(s)})dt \\ &= \int_{t_1}^{t_2}\left(L\frac{d(\delta t)}{dt} + \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q_i^{(s)}}\delta q_i^{(s)}(t)\right)dt,\end{aligned}$$

keeping only for the Lagrangian $L(t+\delta t, q^{(s)} + \delta q^{(s)})$, first order terms in its Taylor expansion.

Now, in the total variation of $\delta q_i^{(s)}(t) = q'^{(s)}_i(t') - q_i^{(s)}(t)$ is contained a variation in the form of the function $q_i^{(s)}(t)$ and a variation in its argument t, that is also affected by the transformation of the group, *i.e.*,

$$\begin{aligned}\delta q_i^{(s)} = q'^{(s)}_i(t+\delta t) - q_i^{(s)}(t) &= q'^{(s)}_i(t) - q_i^{(s)}(t) + (dq_i^{(s)}(t)/dt)\delta t \\ &= \bar{\delta} q_i^{(s)}(t) + q_i^{(s+1)}(t)\delta t,\end{aligned}$$

where $\bar{\delta} q_i^{(s)}(t)$ is the variation in form of the function $q_i^{(s)}(t)$ at the instant of time t. Taking into account that for the variation in form

$$\bar{\delta} q_i^{(s)}(t) = d^s(\bar{\delta} q_i(t))/dt^s = d(\bar{\delta} q_i^{(s-1)}(t))/dt,$$

it follows that

$$\delta\mathcal{A}[\,q\,] = \int_{t_1}^{t_2}\left(L\frac{d(\delta t)}{dt} + \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q_i^{(s)}}\bar{\delta} q_i^{(s)}(t) + \frac{\partial L}{\partial q_i^{(s)}}\frac{dq_i^{(s)}}{dt}\delta t\right)dt$$

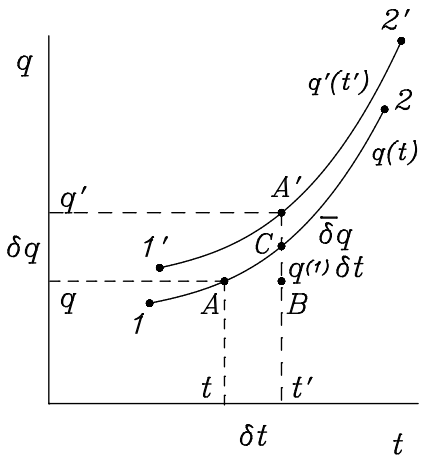


Figure 1.2: **Transformation of point A into A', and the curve $q(t)$ into $q'(t')$ under an infinitesimal transformation. the variation $\delta q = BA'$ is the sum of the part $BC = q^{(1)}\delta t$ and the part $CA' = \bar{\delta}q$, which is the variation of the function q at constant t, which we call here the "form variation" of the function.**

$$= \int_{t_1}^{t_2} \left(\frac{d(L\delta t)}{dt} + \frac{\partial L}{\partial q_i^{(s)}} \bar{\delta} q_i^{(s)}(t) \right) dt. \tag{1.22}$$

Making the replacements

$$\begin{aligned}
\frac{\partial L}{\partial q_i} \bar{\delta} q_i &= \frac{\partial L}{\partial q_i} \bar{\delta} q_i, \\
\frac{\partial L}{\partial q_i^{(1)}} \bar{\delta} q_i^{(1)} &= \frac{\partial L}{\partial q_i^{(1)}} \frac{d(\bar{\delta} q_i)}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(1)}} \bar{\delta} q_i \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(1)}} \right) \bar{\delta} q_i, \\
\frac{\partial L}{\partial q_i^{(2)}} \bar{\delta} q_i^{(2)} &= \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}} \bar{\delta} q_i^{(1)} \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}} \right) \bar{\delta} q_i^{(1)} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}} \bar{\delta} q_i^{(1)} \right) - \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}} \right) \bar{\delta} q_i \right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial q_i^{(2)}} \right) \bar{\delta} q_i, \\
\frac{\partial L}{\partial q_i^{(k)}} \bar{\delta} q_i^{(k)} &= \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(k)}} \bar{\delta} q_i^{(k-1)} \right) - \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(k)}} \right) \bar{\delta} q_i^{(k-2)} \right) + \cdots,
\end{aligned}$$

and collecting terms we get

$$\begin{aligned}
\delta \mathcal{A}[\, q \,] = \int_{t_1}^{t_2} dt \Bigg\{ & \frac{d(L\delta t)}{dt} \\
& + \bar{\delta} q_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(1)}} \right) + \cdots + (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial q_i^{(k)}} \right) \right] && (1.23) \\
& + \frac{d}{dt}\left(\bar{\delta} q_i \left[\frac{\partial L}{\partial q_i^{(1)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}} \right) + \cdots + (-1)^{k-1} \frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial q_i^{(k)}} \right) \right] \right) && (1.24) \\
& + \frac{d}{dt}\left(\bar{\delta} q_i^{(1)} \left[\frac{\partial L}{\partial q_i^{(2)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(3)}} \right) + \cdots + (-1)^{k-2} \frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial q_i^{(k)}} \right) \right] \right) && (1.25)
\end{aligned}$$

$$+\cdots+\frac{d}{dt}\left(\bar{\delta}q_i^{(k-1)}\left[\frac{\partial L}{\partial q_i^{(k)}}\right]\right)\Bigg\}. \tag{1.26}$$

The terms between squared brackets [...], (1.24-1.26) are the conjugate momenta $p_{i(s)}$ of the generalized coordinates, except the first one (1.23), which is the left-hand side of (1.4) and vanishes identically if the functions q_i satisfy the dynamical equations. Thus the integrand of the variation of the action functional, except the first term (1.23), is the time derivative of a sum of terms.

Generalized coordinates and generalized canonical-conjugate momenta:
In ordinary Lagrangian systems that depend only on first order derivatives of the independent degrees of freedom, the canonical approach associates to every generalized coordinate q_i a dynamical variable p_i, called its canonical conjugate momentum and defined by

$$p_i=\frac{\partial L}{\partial \dot{q}_i}.$$

As a generalization of this, for Lagrangian systems depending on higher order derivatives, the generalized canonical formalism defines as **generalized coordinates** the degrees of freedom q_i and their time derivatives $q_i^{(s)}$ up to order $k-1$, i.e., the generalized coordinates are the kinematical variables with the time excluded. Then each generalized coordinate has a **canonical conjugate momentum** defined according to the mentioned squared brackets terms: [13]

$$p_{i(1)}=\frac{\partial L}{\partial q_i^{(1)}}-\frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}}\right)+\cdots+(-1)^{k-1}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial q_i^{(k)}}\right) \tag{1.27}$$

$$p_{i(2)}=\frac{\partial L}{\partial q_i^{(2)}}-\frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(3)}}\right)+\cdots+(-1)^{k-2}\frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial q_i^{(k)}}\right) \tag{1.28}$$

$$\cdots$$

$$p_{i(k)}=\frac{\partial L}{\partial q_i^{(k)}} \tag{1.29}$$

We say that $p_{i(s)}$ is the canonical conjugate momentum of the coordinate $q_i^{(s-1)}$ and, as a general rule we see that the first term contains the partial derivative of L with respect to $q_i^{(s)}$, i.e., with respect to the first time derivative of the corresponding canonical conjugate generalized coordinate $q_i^{(s-1)}$. From its definition these canonical momenta satisfy

$$p_{i(s)}-\frac{\partial L}{\partial q_i^{(s)}}=-\frac{dp_{i(s+1)}}{dt},\quad i=1,\ldots,n,\quad s=1,\ldots,k-1. \tag{1.30}$$

With this definition of the canonical momenta Euler-Lagrange equations (1.4) are written as:

$$\frac{dp_{i(1)}}{dt}=\frac{\partial L}{\partial q_i},\quad i=1,\ldots,n. \tag{1.31}$$

In this way if a Lagrangian is not an explicit function of some degree of freedom q_i, the corresponding conjugate canonical momentum $p_{i(1)}$, is a constant of the motion.

Now if we introduce in the integrand the variables q_i that satisfy Euler-Lagrange equations, the variation of the action functional (1.22) is transformed into the variation of the action function along the classical trajectory, and therefore, the variation of the action function can be written as,

$$\delta A(x_1,x_2)=\int_{t_1}^{t_2}\frac{d}{dt}\left\{L\delta t+\left(\bar{\delta}q_i p_{i(1)}+\bar{\delta}q_i^{(1)}p_{i(2)}+\cdots+\bar{\delta}q_i^{(k-1)}p_{i(k)}\right)\right\}dt, \tag{1.32}$$

with $p_{i(s)}$ given in (1.27)-(1.29). If we replace in (1.32) the form variation $\bar{\delta}q_i^{(s)}=\delta q_i^{(s)}-q_i^{(s+1)}\delta t$, then

$$\delta A(x_1,x_2)=\int_{t_1}^{t_2}\frac{d}{dt}\left\{L\delta t+\delta q_i^{(s)}p_{i(s+1)}-q_i^{(s)}p_{i(s)}\delta t\right\}dt \tag{1.33}$$

[13] E.T.Whittaker, *Analytical Dynamics*, Cambridge University Press, Cambridge (1927), p. 265.

with the usual addition convention. By substitution of the variations δt and $\delta q_i^{(s)}$ in terms of the infinitesimal element of the group δg, (1.19-1.21), we get:

$$\delta A(x_1, x_2) = \int_{t_1}^{t_2} \frac{d}{dt} \left\{ \left(L - p_{i(s)} q_i^{(s)} \right) M_0 + p_{i(s)} M_i^{(s-1)} \right\} \delta g dt, \tag{1.34}$$

with the following range for repeated indexes for the addition convention, $i = 1, \ldots, n$, $s = 1, \ldots, k$, $u = 0, 1, \ldots, k-1$,

In the above integral we are using the solution of the dynamical equations, and therefore the variation of the action function is

$$\delta A(x_1, x_2) = A(\delta g x_1, \delta g x_2) - A(x_1, x_2).$$

If it happens to be of first order in the group parameters in the form

$$\delta A(x_1, x_2) = \lambda(x_2)\delta g - \lambda(x_1)\delta g, \tag{1.35}$$

which is equivalent to the Lagrangian transforming in the way

$$\widetilde{L}(\delta g x, \delta g \dot{x}) = \widetilde{L}(x, \dot{x}) + \delta g \frac{d\lambda(x)}{d\tau},$$

and therefore the dynamical equations are invariant, then equating (1.35) to (1.34) we can perform the trivial time integral on the right hand side. The group parameter δg cancel out on both sides, and rearranging terms depending on t_1 and t_2 on the left- and right-hand side, respectively, we get several observables that take the same values at the two arbitrary times t_1 and t_2. They are thus constants of the motion and represent the time conserved physical quantities,

$$N = \lambda(x) - \left(L - p_{i(s)} q_i^{(s)} \right) M_0 - p_{i(s+1)} M_i^{(s)}, \tag{1.36}$$

where the term within brackets $H = p_{i(s)} q_i^{(s)} - L$ is the generalized Hamiltonian. It is written as the product of each generalized momentum times the time derivative of the corresponding conjugate generalized variable minus the Lagrangian, and finally

$$\boxed{N = \lambda(x) + H M_0(t, q) - p_{i(s)} M_i^{(s-1)}(t, q, \ldots, q^{(s)}).} \tag{1.37}$$

If the symmetry group has r parameters, there exist r constants of the motion related to the corresponding infinitesimal transformations (1.35) of the action function under the corresponding r-parameter Lie group.

Expression (1.37) is a **linear function** of the Hamiltonian and of the canonical momenta, where the coefficients, in addition to the function $\lambda(x)$, are functions of the kinematical variables. If we consider that the Hamiltonian can be interpreted as the conjugate momentum of the time variable t, then each term contains the product of each momentum times the infinitesimal transformation of the corresponding conjugate generalized variable, $\delta t = \delta g M_0$, $\delta q_i^{(s)} = \delta g M_i^{(s)}$, $s = 0, 1, \ldots, k-1$, from the time t till the kinematical variables $q_i^{(k-1)}$. The only difference is that the Hamiltonian is preceeded by a $+$ sign while the remaining momenta are affected by the $-$ sign.

Since $\lambda(x)\delta g$ has dimensions of action, all terms in (1.37) have the same physical dimensions than $\lambda(x)$, and therefore the Noether constants of the motion **have the complementary dimension with respect to the action of the dimension of the corresponding group parameter** δg of the symmetry group.

The Hamiltonian and the momenta are written in terms of the functions $F_i(x, \dot{x})$ of the development (1.8) of the Lagrangian, as we can see in the next example.

For example, if we have a Lagrangian which depends up to the second derivative of a degree of freedom r, $L(t,r,dr/dt,d^2r/dt^2) \equiv L(t,r,u,a)$, and $\widetilde{L}(t,r,u,\dot{t},\dot{r},\dot{u})$. The Lagrangian $\widetilde{L}$ can be written as

$$\widetilde{L} = \frac{\partial \widetilde{L}}{\partial \dot{x}_i}\dot{x}_i = F_i(x,\dot{x})\dot{x}_i = T\dot{t} + R\dot{r} + U\dot{u},$$

where the functions T, R and U are those partial derivatives $F_i(x,\dot{x})$ of $\widetilde{L}$, which are homogeneous functions of zero-th degree of the derivatives $\dot{x}_i$, and therefore they are functions of (t,r,u,a). The kinematical variables are, $x \equiv \{t,r,u\}$ and the generalized variables are $q \equiv \{r,u\}$ so that we have a momentum conjugate of r, p_r and another p_u, the canonical conjugate of u and thus we have:

$$\frac{\partial L}{\partial u} = \frac{\partial(\widetilde{L}/\dot{t})}{\partial \dot{r}}\frac{\partial \dot{r}}{\partial u} = \frac{1}{\dot{t}}\frac{\partial \widetilde{L}}{\partial \dot{r}}\dot{t} = \frac{\partial \widetilde{L}}{\partial \dot{r}} = R$$

since $\dot{r} = u\dot{t}$. Similarly

$$\frac{\partial L}{\partial a} = \frac{\partial(\widetilde{L}/\dot{t})}{\partial \dot{u}}\frac{\partial \dot{u}}{\partial a} = \frac{1}{\dot{t}}\frac{\partial \widetilde{L}}{\partial \dot{u}}\dot{t} = \frac{\partial \widetilde{L}}{\partial \dot{u}} = U$$

since $\dot{u} = a\dot{t}$.

The momentum p_r is defined according to (1.27-1.29)

$$p_r = \frac{\partial L}{\partial u} - \frac{d}{dt}\left(\frac{\partial L}{\partial a}\right) = \frac{\partial \widetilde{L}}{\partial \dot{r}} - \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{u}}\right) = R - \frac{dU}{dt},$$

and the momentum p_u

$$p_u = \frac{\partial L}{\partial a} = U,$$

which are finally expressed in terms of the functions $F_i(x,\dot{x})$ and their time derivatives. The Lagrangian

$$L = \widetilde{L}/\dot{t} = T + Ru + Ua,$$

and the generalized Hamiltonian

$$H = p_r u + p_u a - L = Ru - \frac{dU}{dt}u + Ua - T - Ru - Ua = -T - \frac{dU}{dt}u.$$

The functions $F_i(x,\dot{x})$ and their time derivatives are homogeneous functions of zero degree in terms of the derivatives of the kinematical variables $\dot{x}^i$. Functions $\lambda(x)$ and $M_i^{(s)}(x)$ depend only on the kinematical variables. Consequently, Noether constants of the motion are also homogeneous functions of zero degree in terms of the derivatives of kinematical variables and thus invariant under arbitrary changes of evolution parameter. They are only functions of the time derivatives of the degrees of freedom.

1.5 Atomic Principle: Elementary particles

In Newtonian mechanics the simplest geometrical object is a point of mass m. Starting from massive points we can construct arbitrary systems of any mass and shape, and thus any distribution of matter. The massive point can be considered as the elementary particle of Newtonian mechanics. In the modern view of particle physics it corresponds to a spinless particle. We know that there exist spinning objects like electrons, muons, photons, neutrinos, quarks and perhaps many others, that can be considered as elementary particles in the sense that they cannot be considered as compound systems of other objects. Even more, we do not find in Nature any spinless elementary particle. It is clear that the Newtonian point does not give account of the spin structure of elementary particles and the existence of spin is a fundamental intrinsic attribute, which is lacking in Newtonian mechanics, but it has to be accounted for.

In quantum mechanics, Wigner's work [14] on the representations of the inhomogeneous Lorentz group provides a very precise mathematical definition of the concept of elementary

[14] E.P. Wigner, *Ann. Math.* **40** 149 (1939).

particle. An **elementary particle** is a quantum mechanical system whose Hilbert space of pure states is the representation space of a projective unitary irreducible representation of the Poincaré group. Irreducible representations of the Poincaré group are characterized by two invariant parameters m and S, the mass and the spin of the system, respectively. By finding the different irreducible representations, we can obtain the quantum description of massless and massive elementary particles of any spin.

The very important expression of the above mathematical definition, with physical consequences, lies in the term **irreducible**. Mathematically it means that the Hilbert space is an invariant vector space under the group action and that it has no other invariant subspaces. But it also means that there are no other states for a single elementary particle than those that can be obtained by just taking any arbitrary vector state, form all its possible images in the different inertial frames and finally produce the closure of all finite linear combinations of these vectors.

We see that starting from a single state and by a simple change of inertial observer, we obtain the state of the particle described in this new frame. Take the orthogonal part of this vector to the previous one and normalize it. Repeat this operation with another kinematical transformation acting on the same first state, followed by the corresponding orthonormalization procedure, as many times as necessary to finally obtain a complete orthonormal basis of the whole Hilbert space of states. We see here the idea of the atomic principle. There are no more states than the possible kinematical modifications of any one of them. If the elementary particle changes its state, it is possible to find another inertial observer who describes the particle in the same state as before the modification.

In the Lagrangian formulation if we prepare the particle in the initial state x_1 to evolve to the final state x_2, this final state and any intermediate state can always be obtained by means of a change of inertial observer, i.e., $x_2 = gx_1$, for some element g of the kinematical group G. This is not possible for any arbitrary mechanical system. This is what distinguishes an elementary system from another one which is not elementary. The manifold X, the kinematical space must fulfill this restriction, that given any two points on it it is always possible to find a kinematical transformation that links them. We thus arrive at the

> **Atomic Principle:** A **classical elementary particle** is a Lagrangian system whose kinematical space X is a **homogeneous space** of the kinematical group G.

The Galilei and Poincaré groups are ten-parameter Lie groups and therefore the largest homogeneous space we can find for these groups is a ten-dimensional manifold. The variables that define the different homogeneous spaces will share the same domains and dimensions as the corresponding variables we use to parameterize the group. Both groups, as we shall see later, are parameterized in terms of the following variables $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ with domains and dimensions respectively like $b \in \mathbb{R}$ that represents the time parameter of the time translation and $\boldsymbol{a} \in \mathbb{R}^3$, the three spatial coordinates for the space translation. Parameter $\boldsymbol{v} \in \mathbb{R}^3$ are the three components of the relative velocity between the inertial observers, restricted to $v < c$ in the Poincaré case. Finally $\boldsymbol{\alpha} \in SO(3)$ are three dimensionless variables which characterize the relative orientation of the corresponding Cartesian frames and whose compact domain is expressed in terms of a suitable parameterization of the rotation group.

In this way the maximum number of kinematical variables, for a classical elementary particle, is also ten. We represent them by $x \equiv (t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha})$ with the same domains and dimensions as above and interpret them respectively as the **time** t, **position** $\boldsymbol{r}$, **velocity** $\boldsymbol{u}$ and **orientation** $\boldsymbol{\alpha}$ of the particle.

Because the Lagrangian must also depend on the next order time derivative of the kinematical variables, we arrive at the conclusion that L must also depend on the acceleration and

angular velocity of the particle. The particle is a system of six degrees of freedom, three $\boldsymbol{r}$, represent the position of a point and other three $\boldsymbol{\alpha}$, its orientation in space. We can visualize this by assuming a system of three orthogonal unit vectors linked to point $\boldsymbol{r}$ as a body frame. But the Lagrangian will depend up to the second time derivative of $\boldsymbol{r}$, or acceleration of that point, and on the first derivative of $\boldsymbol{\alpha}$, *i.e.*, on the angular velocity. The Galilei and Poincaré groups lead to generalized Lagrangians depending up to second order derivatives of the position.

By this definition it is the kinematical group G that implements the Restricted Relativity Principle which completely determines the structure of the kinematical space where the Lagrangians that represent classical elementary particles have to be defined. Point particles are particular cases of the above definition and their kinematical space is described by the variables $(t, \boldsymbol{r})$, time and position. Given any two points $(t_1, \boldsymbol{r}_1)$ and $(t_2, \boldsymbol{r}_2)$, con $t_2 > t_1$, a spacetime translation transform one into the other, so that this kinematical space is a homogeneous space of both Galilei and Poincaré group. In this way, the proposed formalism can be accommodated to any symmetry group. It is the proper definition of this group which contains the physical information of the elementary particles, but this group is still unveiled.

Example: Galilei point particle

It is a mechanical system of three degrees of freedom $\boldsymbol{r}$, the position of the point. It has four kinematical variables, $x \equiv \{t, \boldsymbol{r}\}$. If we define the initial state by $x_1 \equiv \{t_1, \boldsymbol{r}_1\}$ and the final state of the evolution $x_2 \equiv \{t_2, \boldsymbol{r}_2\}$, we see that a spacetime translation transform one into the other, and therefore the kinematical space is a homogeneous space of the Galilei group. It is an elementary particle according to the above definition. Of course, the spacetime translation subgroup is also a subgroup of the Poincaré group, and thus this point particle is also an elementary particle from the relativistic point of view. We shall obtain in the next chapter that, if the evolution is free, the Lagrangian is

$$L_0 = \frac{1}{2} m \left(\frac{d\boldsymbol{r}}{dt} \right)^2, \quad \widetilde{L}_0 = \frac{1}{2} m \frac{\dot{\boldsymbol{r}}^2}{\dot{t}}$$

in terms of the independent degrees of freedom and also a homogeneous function of first degree in terms of the $\tau-$derivatives of the kinematical variables. We see that $\widetilde{L}_0$ depends on the $\tau-$derivatives of all kinematical variables. Euler-Lagrange dynamical equations obtained from L_0 are $d^2\boldsymbol{r}/dt^2 = 0$, and we have to use as boundary conditions that the solution goes through the initial and final states x_1 y x_2, respectively,

$$\boldsymbol{r}(t) = \boldsymbol{r}_1 + \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{t_2 - t_1}(t - t_1), \quad t \in [t_1, t_2].$$

In terms of some arbitrary evolution parameter τ, the solution is:

$$t(\tau) = t_1 + (t_2 - t_1)(\tau - \tau_1), \quad \boldsymbol{r}(\tau) = \boldsymbol{r}_1 + (\boldsymbol{r}_2 - \boldsymbol{r}_1)(\tau - \tau_1), \quad \tau \in [\tau_1, \tau_2].$$

If we redefine the evolution parameter as $\theta = (\tau - \tau_1)/(\tau_2 - \tau_1)$, we can have a dimensionless evolution parameter such that the initial and final instants correspond to $\theta_1 = 0$ and $\theta_2 = 1$, and therefore

$$t(\theta) = t_1 + (t_2 - t_1)\theta, \quad \boldsymbol{r}(\theta) = \boldsymbol{r}_1 + (\boldsymbol{r}_2 - \boldsymbol{r}_1)\theta, \quad \theta \in [0, 1].$$

The action function, i.e., the integral of the Lagrangian along the classical path is

$$A(x_1, x_2) = \frac{m}{2} \int_{t_1}^{t_2} \left(\frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{t_2 - t_1} \right)^2 dt = \frac{m}{2} \frac{(\boldsymbol{r}_2 - \boldsymbol{r}_1)^2}{t_2 - t_1},$$

which is finally expressed in terms of the end points variables and of the intrinsic characteristic parameter of this spinless object, the mass m.

The Lagrangian $\widetilde{L}_0$ can be obtained from the action function through the limiting process of (1.10) by taking the derivatives with respect to the variables t_2 and $\boldsymbol{r}_2$ and making the limit $2 \to 1$,

$$\lim_{2\to 1} \left[\frac{\partial A}{\partial t_2} \dot{t} + \frac{\partial A}{\partial r_{2i}} \dot{r}_{2i} \right] = \frac{m}{2} \lim_{2\to 1} \left[-\frac{(\boldsymbol{r}_2 - \boldsymbol{r})^2}{(t_2 - t)^2} \dot{t} + 2 \frac{(\boldsymbol{r}_2 - \boldsymbol{r}) \cdot \dot{\boldsymbol{r}}}{t_2 - t} \right] = \frac{m}{2} \frac{\dot{\boldsymbol{r}}^2}{\dot{t}} = \widetilde{L}_0.$$

Noether's theorem leads us to find that the energy and linear momentum are expressed in terms of the partial derivatives of $\widetilde{L}_0$, in the form:

$$H = -\frac{\partial \widetilde{L}_0}{\partial \dot{t}} = \frac{1}{2} m \frac{\dot{\boldsymbol{r}}^2}{\dot{t}^2} = \frac{m}{2} \left(\frac{d\boldsymbol{r}}{dt} \right)^2, \quad \boldsymbol{p} = \frac{\partial \widetilde{L}_0}{\partial \dot{\boldsymbol{r}}} = m \frac{\dot{\boldsymbol{r}}}{\dot{t}} = m \frac{d\boldsymbol{r}}{dt}.$$

They are homogeneous functions of zero degree in terms of the derivatives of the kinematical variables, and therefore functions of the time derivatives of the degrees of freedom. These conserved magnitudes are independent of the evolution parameter τ.
This Lagrangian is a homogeneous function of first degree in the derivatives $\dot{t}$ and $\dot{\boldsymbol{r}}$, so that we can write it as a sum of as many terms as kinematical variables:

$$\widetilde{L}_0 = \frac{\partial \widetilde{L}_0}{\partial \dot{t}}\dot{t} + \frac{\partial \widetilde{L}_0}{\partial \dot{r}_i}\dot{r}_i = -H\dot{t} + \boldsymbol{p}\cdot\dot{\boldsymbol{r}}.$$

Exemple: Poincaré point particle
It is a mechanical system of 3 degrees of freedom, $\boldsymbol{r}$, the position of a point. Like in the previous exemple the space generated by $t, \boldsymbol{r}$ is also a homogeneous space of the Poincaré group. It thus represents an elementary particle, the relativistic point particle. Let us consider the following free Lagrangian, which will be obtained in the next chapter,

$$L_0 = -mc^2\sqrt{1 - v^2/c^2}, \quad \widetilde{L}_0 = -mc\sqrt{c^2\dot{t}^2 - \dot{\boldsymbol{r}}^2}, \quad \boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$$

in terms of the degrees of freedom and also as a homogeneous function of degree 1 in terms of the derivatives of the kinematical variables. We can also see that $\widetilde{L}_0$ depends on the derivatives of all kinematical variables in this case. The dynamical equations reduce to $d^2\boldsymbol{r}/dt^2 = 0$, and the solution passing through the end points $x_1 \equiv (t_1, \boldsymbol{r}_1)$ and $x_2 \equiv (t_2, \boldsymbol{r}_2)$, is, as in the previous case,

$$\boldsymbol{r}(t) = \boldsymbol{r}_1 + \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{t_2 - t_1}(t - t_1), \quad t \in [t_1, t_2].$$

The action function is

$$A(x_1, x_2) = \int_{t_1}^{t_2} L_0 dt = -mc\sqrt{c^2(t_2 - t_1)^2 - (\boldsymbol{r}_2 - \boldsymbol{r}_1)^2},$$

which is finally expressed in terms of the kinematical variables of the initial and final state, of the only mechanical parameter of the spinless particle the mass m, and of the universal constant c. Since the action is a real observable, the above expression is not valid for those end points which satisfy $c^2(t_2 - t_1)^2 - (\boldsymbol{r}_2 - \boldsymbol{r}_1)^2 < 0$, because in this case the squared root will be pure imaginary. The Causality principle requires that the points causally conneted satisfy the condition

$$c^2(t_2 - t_1)^2 - (\boldsymbol{r}_2 - \boldsymbol{r}_1)^2 > 0, \quad \text{o or} \quad v < c.$$

The Lagrangian $\widetilde{L}_0$ can be obtained from the action function through the limit process (1.10) by taking the derivative with respect to the variables t_2 and $\boldsymbol{r}_2$ and taking the limit $2 \to 1$,

$$\lim_{2\to 1}\left[\frac{\partial A}{\partial t_2}\dot{t} + \frac{\partial A}{\partial r_{2i}}\dot{r}_i\right] = -mc\lim_{2\to 1}\left(\frac{c^2(t_2 - t_1)\,\dot{t}}{\sqrt{c^2(t_2 - t_1)^2 - (\boldsymbol{r}_2 - \boldsymbol{r}_1)^2}} - \frac{(\boldsymbol{r}_2 - \boldsymbol{r}_1)\cdot\dot{\boldsymbol{r}}}{\sqrt{c^2(t_2 - t_1)^2 - (\boldsymbol{r}_2 - \boldsymbol{r}_1)^2}}\right) = \widetilde{L}_0.$$

Noether's theorem gives rise to the energy and linear momentum which are expressed in terms of the derivatives in the form:

$$H = -\frac{\partial \widetilde{L}_0}{\partial \dot{t}} = mc\frac{c^2\dot{t}}{\sqrt{c^2\dot{t}^2 - \dot{\boldsymbol{r}}^2}} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma(v)mc^2,$$

$$\boldsymbol{p} = \frac{\partial \widetilde{L}_0}{\partial \dot{\boldsymbol{r}}} = -mc\frac{-\dot{\boldsymbol{r}}}{\sqrt{c^2\dot{t}^2 - \dot{\boldsymbol{r}}^2}} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}} = \gamma(v)m\boldsymbol{v}.$$

They are homogeneous functions of zero degree in terms of the derivatives of the kinematical variables, and thus they are functions of the time derivatives of the degrees of freedom. The conserved magnitudes are independent of the evolution parameter τ, like in the Galilei case.
The Lagrangian is also a homogenenous function of degree 1 in the derivatives $\dot{t}$ and $\dot{\boldsymbol{r}}$, and therefore we can express as a sum of as many terms as kinematical variables:

$$\widetilde{L}_0 = \frac{\partial \widetilde{L}_0}{\partial \dot{t}}\dot{t} + \frac{\partial \widetilde{L}_0}{\partial \dot{r}_i}\dot{r}_i = -H\dot{t} + \boldsymbol{p}\cdot\dot{\boldsymbol{r}}.$$

Example: Non-elementary System
Let us assume a material system characterized by the kinematical variables: $(t, \boldsymbol{r}_1, \boldsymbol{r}_2)$, i.e., a time variable and the position of two points. If we give two sets of values for these variables at the end points of the evolution $x_1 \equiv (t(\tau_1), \boldsymbol{r}_1(\tau_1), \boldsymbol{r}_2(\tau_1))$ and $x_2 \equiv (t(\tau_2), \boldsymbol{r}_1(\tau_2), \boldsymbol{r}_2(\tau_2))$, it is not possible to find a transformation among inertial observers which transform one into the other because if the time translation brings $t(\tau_1)$ to $t(\tau_2)$ and the space translation takes $\boldsymbol{r}_1(\tau_1)$ into $\boldsymbol{r}_1(\tau_2)$, it is impossible, in general, that this translation also transforms $\boldsymbol{r}_2(\tau_1)$ into $\boldsymbol{r}_2(\tau_2)$, except if the two points are rigidly bounded. The kinematical space is not a homogeneous space of the subgroup of spatial translations. This system of 6 degrees of freedom cannot be considered as representing an elementary particle.

Exemple: The quark.
Let us consider that a quark can be described as a classical system. In this case we have to locate its interacting properties. In the case of the electron which only interacts electromagnetically we have to locate a single point, the center of the electric charge. In the case of the quark we have to locate the center of the electric charge and the center of the color charge, and therefore to obtain the evolution of two points. By the previous example, if we have a system with two characteristic points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ among the kinematical variables, this manifold is no longer a homogeneous space of the Poincaré group. Then necessarilly if we were able to describe a quark we will only need a single point, the center of charge, the center of all charges, to describe all the interacting properties. This argument also holds for the electron if we consider that the weak interaction is different than the electromagnetic interaction, so that the center of the weak charge must be the same as the center of the electric charge.

1.5.1 Aplication to some simpler kinematical groups

Let us consider that physical laws are invariant only under spacetime translations. It is equivalent to assume that the kinematical group of spacetime transformations associated to the Restricted Relativity Principle is just the group $G \equiv \{\mathbb{R}^4, +\}$ the four-parameter group of spacetime translations:

$$t' = t + b, \quad \boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{a}.$$

In this case the largest homogenous space of this group is the group itself, and therefore the kinematical variables are $(t, \boldsymbol{r})$. We are describing the point particle localized at point $\boldsymbol{r}$. Because the only symmetries are translations, Noether's theorem only produces four conserved quantities, the observables H and $\boldsymbol{P}$, energy and linear momentum, respectively, and therefore angular momentum conservation is not described in this restricted symmetry group. The Lagrangian for this system will be a function of $(t, \boldsymbol{r}, \boldsymbol{u})$, being $\boldsymbol{u}$ the velocity of point $\boldsymbol{r}$.

Let us go further and assume that physical laws are also invariant under spatial rotations. Then the action of group G on space-time is given by

$$t' = t + b, \quad \boldsymbol{r}' = R(\boldsymbol{\alpha})\boldsymbol{r} + \boldsymbol{a},$$

which depends on seven parameters. The largest homogeneous space is the whole group and we have as kinematical variables $(t, \boldsymbol{r}, \boldsymbol{\alpha})$ and we say that the elementary particle is localized at point $\boldsymbol{r}$, and has an orientation described by the variables $\boldsymbol{\alpha}$. The Lagrangian for this particle will be a function of $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha}, \boldsymbol{\omega})$, and will depend, in addition to the velocity of point $\boldsymbol{r}$, $\boldsymbol{u} = d\boldsymbol{r}/dt$, of the velocity of the change of orientation or angular velocity $\boldsymbol{\omega}$. For this particle Noether's theorem gives us an angular momentum observable. This particle has spin. We are describing something formally equivalent to a rotating rigid body.

The next step is to consider that the kinematical group also contains pure inertial transformations of constant velocity (boosts). We have three new parameters which can enlarge our kinematical space with three new kinematical variables with physical dimensions of velocity. The Lagrangian will also depend on the acceleration. We shall analyze in the next chapters this possibility by assuming that the kinematical group is either the Galilei group $\mathcal{G}$ or the Poincaré group $\mathcal{P}$.

1.6 Metric structure of the kinematical space

The manifold X, the kinematical space of any Lagrangian mechanical system, has always a metric structure. It is a Finsler space in which the metric is a function not only of the point x, but also of the derivatives $\dot{x}$. In fact, since $\widetilde{L}(x,\dot{x})$ is a homogeneous function of first degree in terms of the variables $\dot{x}^i$, it implies that $\widetilde{L}^2$ is a homogeneous function of degree 2 of the variables $\dot{x}^i$. Then if we replace in $\widetilde{L}^2(x,\dot{x})$ each $\dot{x}^i$ by $\lambda\dot{x}^i \equiv y^i$, $\widetilde{L}^2(x,\lambda\dot{x}) = \widetilde{L}^2(x,y) = \lambda^2\widetilde{L}^2(x,\dot{x})$. If we derivate twice with respect to λ and we make afterwards $\lambda = 1$,

$$2\lambda\widetilde{L}^2(x,\dot{x}) = \frac{\partial\widetilde{L}^2(x,y)}{\partial y^i}\dot{x}^i, \quad 2\widetilde{L}^2(x,\dot{x}) = \left.\frac{\partial^2\widetilde{L}^2(x,y)}{\partial y^i\partial y^j}\dot{x}^i\dot{x}^j\right|_{\lambda=1}$$

we get

$$\widetilde{L}^2(x,\dot{x}) = g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j, \quad g_{ij}(x,\dot{x}) = \frac{1}{2}\frac{\partial^2\widetilde{L}^2}{\partial\dot{x}^i\partial\dot{x}^j} = g_{ji}$$

where the functions $g_{ij}(x,\dot{x})$ are homogeneous functions of zero-th degree of the $\dot{x}^i$ and therefore they only involve time derivatives. But in addition of being functions of the point x, they are, in general, functions of the $\dot{x}$. A metric space whose metric is also a function of the derivatives of the variables of the manifold is called a **Finsler space**[15] [16].

Since $\pm\widetilde{L} = \pm\sqrt{\widetilde{L}^2}$, the variational problem in the kinematical space X can be rewritten as

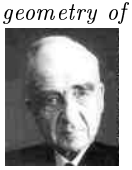
$$\int_{\tau_1}^{\tau_2}\widetilde{L}(x,\dot{x})d\tau = \int_{\tau_1}^{\tau_2}\sqrt{\widetilde{L}^2(x,\dot{x})}d\tau = \int_{\tau_1}^{\tau_2}\sqrt{g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j}d\tau =$$

$$= \int_{x_1}^{x_2}\sqrt{g_{ij}(x,\dot{x})dx^idx^j} = \int_{x_1}^{x_2}ds,$$

where we can interpret ds as the arc length of the curve joining two close points in the kinematical space, and the above integral as the length between the end points of the path followed by the system in the kinematical space X.

The variational problem of making extremal the action of the mechanical system is equivalent to consider that the distance, in the kinematical space X between x_1 and x_2, has to be a minimum, and our variational formalism is equivalent to a geodesic problem in a metric space. The evolution of any dynamical system between the initial state x_1 yo the final state x_2, follows a geodesic in the state space X. This is independent of whether the system is a free particle or any interacting arbitrary system. What happens is that the difference between a free particle and an interacting particle, is that the corresponding Lagrangians, and thus the metrics, are

[15] G.S. Asanov, *Finsler geometry, Relativity and Gauge theories*, (Reidel Pub. Co, Dordrecht 1985); H. Rund, *The Hamilton-Jacobi theory in the calculus of variations*, (Krieger Pub. Co., N.Y 1973). H. Rund, *The differential geometry of Finsler spaces*, (Springer, Berlin, 1959).



[16] **Paul Finsler** Born in Heilbronn, Neckar, Germany, the 11th of April of 1894 and died in Zurich, Switzerland, the 29th April of 1970. He devoted mainly to differential geometry and set theory. It was Elie Cartan in 1934 who published a book entitled *Les espaces de Finsler*, where he named Finsler spaces to the metric spaces related to the variational formalism we are going to consider.

different. Any **interaction modifies the metric of the kinematical space of any free particle**.

Under transformations of the kinematical space which leave the Lagrangian invariant, the magnitudes g_{ij} transform like the covariant components of a second rank symmetric tensor.

Given the Euler-Lagrange dynamical equations of a mechanical system, the variational formalism implies that we have to search for solutions of these equations passing through the extremal points x_1 and x_2. Given two arbitrary points it may happen that no solution exists joining them. If we prepare the system at the initial state x_1, we shall say that the state x_2 is **caussally connected** with x_1, if the Euler-Lagrange dynamical equations have a solution between them. Otherwise we shall say that they are caussally disconnected and therefore it is impossible to bring, by dynamical evolution, the system from state x_1 to the state x_2. Since $\widetilde{L}^2 > 0$, the metric of the space is definite positive between the states caussally connected, and if it happens that this quadratic form between two close points does not satisfy $\widetilde{L}^2 > 0$, they will be caussally disconnected and the evolution between them is physically impossible.

For the free relativistic point particle of mass m, the Lagrangian is written as

$$\widetilde{L}_0 = \pm mc\sqrt{\dot{x}_0^2 - \dot{\boldsymbol{r}}^2}, \quad x_0 = ct.$$

If we divide $\widetilde{L}_0$ by the constant mc, the Lagrangian has now dimensions of length and the metric is clearly $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$, with $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Since $\widetilde{L}^2 > 0$ this implies that at any τ it must hold that any point joining with x_1 must satisfy $\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu > 0$. Then the points caussally connected with it are those of the interior of the forward light cone. The remaining points of the kinematical space (which for the point particle is the spacetime) are caussally disconnected. These are the points of the past and those on the light cone and outside it. In these cases $\widetilde{L}^2 \leq 0$, and Euler-Lagrange equations do not fulfill physical solutions.

Given the general structure of any Lagrangian $\widetilde{L} = F_i(x,\dot{x})\dot{x}^i$, with $F_i = \partial\widetilde{L}/\partial\dot{x}^i$, is easy to see that the metric coefficients are written as

$$g_{ij} = \frac{1}{2}\frac{\partial^2\widetilde{L}^2}{\partial\dot{x}^i\partial\dot{x}^j} = \frac{\partial}{\partial\dot{x}^i}\left(\widetilde{L}\frac{\partial\widetilde{L}}{\partial\dot{x}^j}\right) = F_iF_j + \widetilde{L}\frac{\partial^2\widetilde{L}}{\partial\dot{x}^i\partial\dot{x}^j} = F_iF_j + \widetilde{L}\frac{\partial F_i}{\partial\dot{x}^j}, \tag{1.38}$$

coefficients which are symmetric in their indices because $\partial F_i/\partial\dot{x}^j = \partial F_j\partial\dot{x}^i$.

For the free point particle, the rescaled Lagrangian

$$F_\mu = \frac{\partial\widetilde{L}_0}{\partial\dot{x}^\mu} = \frac{\dot{x}_\mu}{\widetilde{L}_0}, \quad \frac{\partial F_\mu}{\partial\dot{x}^\nu} = \frac{\eta_{\mu\nu}}{\widetilde{L}_0} - \frac{\dot{x}_\mu\dot{x}_\nu}{\widetilde{L}_0^3}$$

$$g_{\mu\nu} = F_\mu F_\nu + \widetilde{L}_0\frac{\partial F_\mu}{\partial\dot{x}^\nu} = \frac{\dot{x}_\mu\dot{x}_\nu}{\widetilde{L}_0^2} + \widetilde{L}_0\left(\frac{\eta_{\mu\nu}}{\widetilde{L}_0} - \frac{\dot{x}_\mu\dot{x}_\nu}{\widetilde{L}_0^3}\right) = \eta_{\mu\nu}$$

whether we consider $\widetilde{L}_0$ as well as $-\widetilde{L}_0$.

In general, the kinematical space will have a metric which depends on x if the Lagrangian is a function of x, but in any case it will always be a function of $\dot{x}^i$. In the case of the free relativistic particle, the metric does not depend on x nor $\dot{x}$ as it corresponds to a free system on spacetime where all points and all velocities are equivalent. But if we introduce an interaction and the intensity of this interaction depends on the velocity, as is the case when we have a magnetic field, the homogeneity of spacetime is destroyed, the metric is no longer uniform, and it will be, in general, a function of the velocity of the point.

1.6.1 Geodesics on a Finsler space

The magnitudes $\dot{x}^i$ transform as the contravariant componentses of a vector on X. They are the components of the tangent vector to the trajectory followed by the characteristic point of the dynamical system on the kinematical space X. If the Lagrangian $\widetilde{L}$ is invariant under the transformation $x \to x'$, $\widetilde{L}(x', \dot{x}') = \widetilde{L}(x, \dot{x})$, then the magnitudes $F_i = \partial\widetilde{L}/\partial\dot{x}^i$, transform as the covariant components of a vector on X.

$$F_i(x, \dot{x}) = \frac{\partial\widetilde{L}(x, \dot{x})}{\partial\dot{x}^i} = \frac{\partial\widetilde{L}(x', \dot{x}')}{\partial\dot{x}'^j}\frac{\partial\dot{x}'^j}{\partial\dot{x}^i} = \frac{\partial\dot{x}'^j}{\partial\dot{x}^i}F_j(x', \dot{x}').$$

The covariant components of the tangent vector can be defined by means of the metric tensor as usual

$$\dot{x}_i = g_{ij}\dot{x}^j.$$

Since

$$\widetilde{L} = \frac{\partial\widetilde{L}(x, \dot{x})}{\partial\dot{x}^i}\dot{x}^i, \quad \widetilde{L}^2 = \widetilde{L}\frac{\partial\widetilde{L}(x, \dot{x})}{\partial\dot{x}^i}\dot{x}^i = g_{ij}\dot{x}^i\dot{x}^j = \dot{x}_i\dot{x}^i, \quad \Rightarrow \dot{x}_i = g_{ij}\dot{x}^i = \widetilde{L}\frac{\partial\widetilde{L}(x, \dot{x})}{\partial\dot{x}^i} = \widetilde{L}F_i$$

and therefore $\widetilde{L}^2$ represents the squared absolute value of the tangent vector, and the vector

$$F_i = \frac{\partial\widetilde{L}(x, \dot{x})}{\partial\dot{x}^i} = \frac{\dot{x}_i}{\widetilde{L}}, \quad F_iF^i = 1,$$

represents the tangent unit vector.

If we take the derivative of $\dot{x}_i$ with respecto to $\dot{x}^j$, we obtain

$$\frac{\partial\dot{x}_i}{\partial\dot{x}^j} = \frac{\partial}{\partial\dot{x}^j}\left(\widetilde{L}\frac{\partial\widetilde{L}}{\partial\dot{x}^i}\right) = \frac{\partial}{\partial\dot{x}^j}\left(\frac{1}{2}\frac{\partial\widetilde{L}^2}{\partial\dot{x}^i}\right) = \frac{1}{2}\frac{\partial^2\widetilde{L}^2}{\partial\dot{x}^i\partial\dot{x}^j} = g_{ij}.$$

Geodesic equations are Euler-Lagrange's equations, i.e.,

$$\frac{\partial\widetilde{L}}{\partial x^i} - \frac{d}{d\tau}\left(\frac{\partial\widetilde{L}}{\partial\dot{x}^i}\right) = 0.$$

$$\frac{\partial\widetilde{L}}{\partial x^i} = \frac{1}{2\widetilde{L}}\frac{\partial\widetilde{L}^2}{\partial x^i} = \frac{1}{2\widetilde{L}}\frac{\partial(g_{jk}\dot{x}^j\dot{x}^k)}{\partial x^i} = \frac{\dot{x}^j\dot{x}^k}{2\widetilde{L}}\frac{\partial g_{jk}}{\partial x^i},$$

and

$$\frac{\partial\widetilde{L}}{\partial\dot{x}^i} = F_i = \frac{\dot{x}_i}{\widetilde{L}} = \frac{1}{\widetilde{L}}g_{ij}\dot{x}^j.$$

$$\frac{d}{d\tau}\left(\frac{\partial\widetilde{L}}{\partial\dot{x}^i}\right) = -\frac{1}{\widetilde{L}^2}\frac{d\widetilde{L}}{d\tau}g_{ij}\dot{x}^j + \frac{1}{\widetilde{L}}\frac{dg_{ij}}{d\tau}\dot{x}^j + \frac{1}{\widetilde{L}}g_{ij}\ddot{x}^j.$$

Also

$$\frac{dg_{ij}}{d\tau} = \frac{\partial g_{ij}}{\partial x^k}\dot{x}^k + \frac{\partial g_{ij}}{\partial\dot{x}^k}\ddot{x}^k, \quad \frac{1}{\widetilde{L}^2}\frac{d\widetilde{L}}{d\tau} = \frac{1}{\widetilde{L}}\frac{d(\log\widetilde{L})}{d\tau},$$

so that we can eliminate the $\widetilde{L}$ of all denominators. If we take the evolution parameter as the arc length, $\widetilde{L} = 1$ and $\log\widetilde{L} = 0$, we arrive to the geodesic equations

$$g_{ij}\ddot{x}^j = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k - \frac{\partial g_{ij}}{\partial x^k}\dot{x}^j\dot{x}^k - \frac{\partial g_{ij}}{\partial\dot{x}^k}\dot{x}^j\ddot{x}^k. \tag{1.39}$$

The last term vanishes because of the symmetry of the metric tensor,

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{1}{2}\left(\frac{\partial^3 \widetilde{L}^2}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k}\right) = \frac{\partial g_{ik}}{\partial \dot{x}^j},$$

and taking the contraction with $\dot{x}^j$, this gives $\partial g_{ik}/\partial \dot{x}^j\, \dot{x}^j = 0$, because g_{ij} tensor is a homogeneous function of zeroeth degree in terms of the $\dot{x}$. In the second term the indices j, k are dummy indices and we can write that term as

$$\frac{\partial g_{ij}}{\partial x^k}\dot{x}^j\dot{x}^k = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j}\right)\dot{x}^j\dot{x}^k.$$

The contravariant components of the metric tensor are defined as usual as $g^{li}g_{ij} = \delta^l_i$. By making use of this tensor, contraction of (1.39) with g^{li}, we arrive to

$$\ddot{x}^l + \Gamma^l_{jk}\dot{x}^j\dot{x}^k = 0, \tag{1.40}$$

where

$$\Gamma^l_{jk} = \frac{1}{2}g^{li}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) = \Gamma^l_{kj},$$

the Finslerian Christoffel symbols are defined in the same way as in a Riemannian space in terms of the derivatives of the metric tensor with respecto to the varibles x. The only difference with the Riemannian case is that they are also functions of the $\dot{x}$.

An alternative expression of the geodesic equation is

$$g_{ij}\ddot{x}^j = \Psi_{ikl}\dot{x}^k\dot{x}^l, \quad \Psi_{ikl} = \frac{1}{2}\left(\frac{\partial g_{kl}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^l} - \frac{\partial g_{il}}{\partial x^k}\right) = \Psi_{ilk}. \tag{1.41}$$

The kinematical space X is, in general, a Finsler space with torsion. Cartan torsion tensor is the symmetric tensor

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{1}{4}\left(\frac{\partial^3 \widetilde{L}^2}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k}\right).$$

Riemannian spaces are Cartan torsion free spaces, because the metric is independent of the derivatives $\dot{x}$.

General Relativity postulates that gravity modifies the metric of space-time, i.e., the metric of the kinematical space of the test point particle, and this modification produces a new Riemannian metric, where the coefficients $g_{ij}(x)$ are only functions of the point x. This is a very strong mathematical restriction because, in general, as far as the metric structure of the kinematical space of the point particle is concerned, the new modified metric can also be a function of the derivatives $\dot{x}$.

We consider that this restriction of General Relativity is a kind of low velocity limit of a more general theory of gravitation. Another restriction is that in Nature it seems that spinless elementary particles do not exist, and therefore, gravity considered as another interaction, should modify the metric of the whole kinematical space of any elementary particle, which is a larger manifold than space-time, as we shall see along this lecture course, and not only the space-time submanifold.

As a final conclusion we could say that General Relativity seems to be a theory of gravitation of spinless matter moving at low velocity. [17]

[17] M. Rivas, *Is General Relativity a simplified theory?* J. Phys:Conference Series **437** (2013) 012008. (ArXiv:1203.4076); *Is General Relativity a v/c -> 0 limit of a Finsler geometry?* (Contribution to the Spanish Relativity Meeting 2012), Progress in Mathematical Relativity, Gravitation and Cosmology Guimaraes, Portugal Sept 3-7, 2012 Springer ISBN 978-3-642-40156-5.

1.6.2 Example: Point particle under an electromagnetic field

For example, the point particle of mass m and electric charge e in an external electromagnetic field, is described by the Lagrangian $\widetilde{L} = \widetilde{L}_0 + \widetilde{L}_I$, where the free Lagrangian $\widetilde{L}_0 = -p_\mu(x)\dot{x}^\mu = -H\dot{t} + \boldsymbol{p}\cdot\dot{\boldsymbol{r}}$, and the interaction Lagrangian $\widetilde{L}_I = -eA_\mu(x)\dot{x}^\mu$, such that the variational problem, according to (1.38), is equivalent to a geodesic problem on spacetime with a metric,

$$g_{\mu\nu}(x,\dot{x}) = m^2c^2\eta_{\mu\nu} + e^2A_\mu A_\nu + e(p_\mu A_\nu + p_\nu A_\mu) + eA_\sigma\dot{x}^\sigma\frac{\partial p_\mu}{\partial\dot{x}^\nu}. \tag{1.42}$$

The modification of the metric vanishes when $e \to 0$. Since p_μ does not depend explicitely on the variables x, the dependence of the metric on the point x, is through the dependence of the external potentials $A_\mu(x)$. But the metric depends on the variables $\dot{x}$ through the dependence on p_μ and its derivatives. In the low velocity limit, when $u/c \to 0$, $p_0 = mc$ and $p_i = 0$, we get a Riemannian metric, such that if we divide $\widetilde{L}$ by a global factor mc and calling $k = e/mc$, $\widetilde{L}_I = -kA_\mu(x)\dot{x}^\mu$, and thus

$$g_{00}(x) = 1 + k^2A_0^2 + 2kA_0 = (1 + kA_0(x))^2\,, \quad g_{ii}(x) = -1 - kA_0(x) + k^2A_i^2(x), \quad i = 1,2,3,$$

$$g_{0i}(x) = kA_i(x) + k^2A_0(x)A_i(x), \quad g_{ij}(x) = k^2A_i(x)A_j(x), \quad i \neq j = 1,2,3.$$

In a uniform electric field, $A_0 = \boldsymbol{E}\cdot\boldsymbol{r}/c$, $\boldsymbol{A} = 0$, and the nonvanishing coefficients of the Riemannian approach are $g_{00} = (1 + e\boldsymbol{E}\cdot\boldsymbol{r}/mc^2)^2$, $g_{ii} = -(1 + e\boldsymbol{E}\cdot\boldsymbol{r}/mc^2)$. If what we have is a uniform magnetic fielf, $A_0 = 0$, $\boldsymbol{A} = (\boldsymbol{r}\times\boldsymbol{B})/2$, $g_{00} = 1$, $g_{ii} = -1 + (e(\boldsymbol{r}\times\boldsymbol{B})/2mc)_i^2$, $g_{0i} = e(\boldsymbol{r}\times\boldsymbol{B})_i/2mc$ and finally $g_{ij} = (e(\boldsymbol{r}\times\boldsymbol{B})/2mc)_i(e(\boldsymbol{r}\times\boldsymbol{B})/2mc)_j$, with $i \neq j$. In some interaction with only scalar potential, like in the usual gravitational field, $mA_0 = mV(x)/c$, and $g_{00} = (1 + V(x)/c^2)^2$, $g_{ii} = -(1 + V(x)/c^2)$, as we shall see in the examples we are going to analyze in the coming section.

We have two ways of determining the dynamical equations of any mechanical system. One is by the usual Euler-Lagrange equations obtained from the Lagrangian $\widetilde{L}$. For the charged point particle of this example, we have

$$\widetilde{L} = -p_\mu\dot{x}^\mu - kA_\mu\dot{x}^\mu, \quad p_\mu = \frac{\dot{x}_\mu}{(\dot{x}\cdot\dot{x})^{1/2}}, \quad \frac{\partial\widetilde{L}}{\partial x^\sigma} = -k\frac{\partial A_\mu}{\partial x^\sigma}\dot{x}^\mu, \quad \frac{\partial\widetilde{L}}{\partial\dot{x}^\sigma} = -p_\sigma - kA_\sigma,$$

$$\frac{d}{d\tau}\left(\frac{\partial\widetilde{L}}{\partial\dot{x}^\sigma}\right) = -\dot{p}_\sigma - k\dot{A}_\sigma = -\dot{p}_\sigma - k\frac{\partial A_\sigma}{\partial x^\mu}\dot{x}^\mu, \quad F_{\sigma\mu}(x) = \partial_\sigma A_\mu - \partial_\mu A_\sigma = -F_{\mu\sigma}(x),$$

$$\dot{p}_\sigma = kF_{\sigma\mu}(x)\dot{x}^\mu, \quad \text{or} \quad \frac{\ddot{x}_\sigma}{(\dot{x}\cdot\dot{x})^{1/2}} - \frac{(\dot{x}\cdot\ddot{x})\dot{x}_\sigma}{(\dot{x}\cdot\dot{x})^{3/2}} = kF_{\sigma\mu}(x)\dot{x}^\mu$$

If in some inertial frame the observer takes the time as evolution parameter $\tau = t$, the dynamical equations become:

$$\frac{dH}{dt} = e\boldsymbol{E}\cdot\boldsymbol{u}, \quad \frac{d\boldsymbol{p}}{dt} = e\left(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}\right).$$

The first equation is a consequence of the second and that for an elementary particle $H^2 - \boldsymbol{p}^2c^2 = m^2c^4$, is invariant. The second equation for the position of the point is transformed into

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{e}{m\gamma(u)}\left[\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} - \frac{\boldsymbol{u}}{c^2}(\boldsymbol{u}\cdot\boldsymbol{E})\right]$$

Another alternative are the geodesic equations constructed from the metric g_{ij}, given in (1.42), which is obtained from $\widetilde{L}^2$ by taking the second order derivatives with respect to $\dot{x}^\mu$. The metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + k^2A_\mu A_\nu + \frac{k}{(\dot{x}\cdot\dot{x})^{1/2}}(\dot{x}_\mu A_\nu + \dot{x}_\nu A_\mu) + \frac{k(A\cdot\dot{x})}{(\dot{x}\cdot\dot{x})^{1/2}}\left(\eta_{\mu\nu} - \frac{\dot{x}_\mu\dot{x}_\nu}{(\dot{x}\cdot\dot{x})}\right)$$

$$g_{00} = 1 + k^2 A_0^2 + 2k\gamma(u)A_0 + k\gamma(1-\gamma^2)(A_0 - \boldsymbol{u}\cdot\boldsymbol{A}/c),$$

$$g_{ii} = -1 + k^2 A_i^2 + 2k\gamma(u)u_i A_i/c - k\gamma(u)(A_0 - \boldsymbol{u}\cdot\boldsymbol{A}/c)(1+\gamma^2 u_i^2/c^2)$$

$$g_{0i} = k^2 A_0 A_i - k\gamma(u)(A_i + A_0 u_i/c) + k\gamma^3(A_0 - \boldsymbol{u}\cdot\boldsymbol{A}/c)u_i/c,$$

$$g_{ij} = k^2 A_i A_j + k\gamma(u)(u_i A_j + u_j A_i)/c - k\gamma(u)^3(A_0 - \boldsymbol{u}\cdot\boldsymbol{A})u_i u_j/c^2, \quad i \neq j.$$

The geodesic equations are (1.41), and for space-time become

$$g_{\mu\nu}\ddot{x}^\nu = \Psi_{\mu\lambda\sigma}\dot{x}^\lambda \dot{x}^\sigma.$$

The difference with the Riemannian case, these coefficients $\Psi_{\mu\lambda\sigma}$ are functions of x and $\dot{x}$, and are homogeneous functions of zero degree of the $\dot{x}$.

If, as before, we take the evolution parameter as $\tau = t$, since $\dot{t} = 1$, the first equation is $\ddot{t} = 0$, and we have to solve the equations:

$$\frac{d^2 r_i}{dt^2} + \Gamma^i_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0.$$

.

Cartan torsion tensor of space-time in the presence of an electromagnetic field is,

$$2C_{\mu\nu\lambda} = \frac{\partial g_{\mu\nu}}{\partial \dot{x}^\lambda} = eA_\mu \frac{\partial p_\nu}{\partial \dot{x}^\lambda} + eA_\nu \frac{\partial p_\mu}{\partial \dot{x}^\lambda} + eA_\lambda \frac{\partial p_\mu}{\partial \dot{x}^\nu} + eA_\sigma \dot{x}^\sigma \frac{\partial^2 p_\mu}{\partial \dot{x}^\nu \partial \dot{x}^\lambda}.$$

For the point particle Lagrangian $\widetilde{L} = -(\dot{x}\cdot\dot{x})^{1/2} - k(A\cdot\dot{x})$, Cartan torsion tensor is linear in the potentials A_μ, and take the form:

$$2C_{\mu\nu\lambda} = \frac{k}{(\dot{x}\cdot\dot{x})^{1/2}}(\eta_{\mu\lambda}A_\nu + \eta_{\mu\nu}A_\lambda + \eta_{\nu\lambda}A_\mu) - \frac{k(A\cdot\dot{x})}{(\dot{x}\cdot\dot{x})^{3/2}}(\eta_{\mu\lambda}\dot{x}_\nu + \eta_{\mu\nu}\dot{x}_\lambda + \eta_{\nu\lambda}\dot{x}_\mu)$$

$$-\frac{k}{(\dot{x}\cdot\dot{x})^{3/2}}(\dot{x}_\mu\dot{x}_\lambda A_\nu + \dot{x}_\mu\dot{x}_\nu A_\lambda + \dot{x}_\nu\dot{x}_\lambda A_\mu) - \frac{3k(A\cdot\dot{x})}{(\dot{x}\cdot\dot{x})^{5/2}}\dot{x}_\mu\dot{x}_\nu\dot{x}_\lambda.$$

1.6.3 Another examples of Finsler spaces

In the figure 1.3 we show possible motions of a charged point particle in its kinematical space, which in this case is spacetime, under four different dynamical situations. [18]

The four trajectories are geodesics of spacetime but with respect to four different Finslerian metrics. In (a) the motion is free, the trajectory is a straight line; in (b) the particle is under a uniform magnetic field, and the trajectory has curvature and torsion. In this case the Finsler metric of spacetime is different than Minkowski metric. The presence of a magnetic field has modified the metric. In (c) it is the same free trajectory but as seen by an accelerated observer. According to the equivalence principle, it is equivalent to the description in the presence of a uniform gravitational field. Also in this case the metric has been modified. Finally, in (d) we analyze the motion a point particle under the Newtonian potential produced by a mass M located at the origin of the inertial reference frame in which the analysis is done.

In these examples, relative to the motion of a point particle of mass m, we are going to change the scale of the Lagrangian by dividing by the factor mc, and thus $\widetilde{L}$ will have now dimensions of length. We like to mention that if the evolution is expressed in terms of some dimensionless

[18] This subject corresponds to a talk lectured by the author at IAC in November 2014, in Spanish. (http://iactalks.iac.es/talks/view/703) and a videoconference, in English, at VIA, (http://viavca.in2p3.fr/site.html) in January 2015.

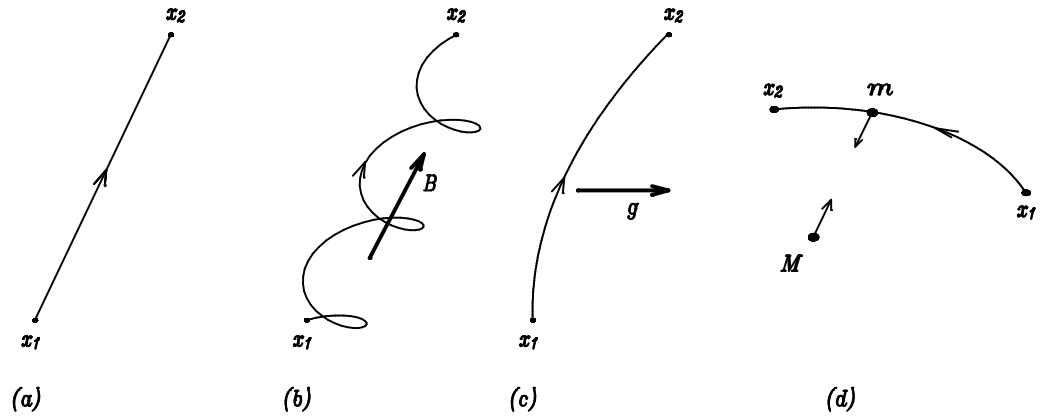


Figure 1.3: **Four possible motions of the point particle in its kinematical space between the points x_1 and x_2, (a) free case, (b) under a uniform magnetic field B, (c) free motion as seen by an accelerated observer or motion under a uniform gravitational field g. The example (d) is the particle under the Newtonian gravitational field of a point mass M located at the origin of a reference frame. In the four cases the kinematical space is the same, spacetime, but with four different Finslerian metrics, which produce different geodesics and which in three-dimensional space are, respectively, (a) a straight line with no curvature and torsion, (b) a line with curvature and torsion, and in (c) and (d) a flat trajectory with curvarture.**

parameter τ, the metric coefficients $g_{\mu\nu}$ are dimensionless, since spacetime coordinates have dimension of length.

In the case (a) the Lagrangian of the free particle is:

$$\widetilde{L}_0 = \pm\sqrt{\dot{x}_0^2 - \dot{\boldsymbol{r}}^2} = F_\mu \dot{x}^\mu, \quad \widetilde{L}_0^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = c^2\dot{t}^2 - \dot{\boldsymbol{r}}^2 > 0,$$

the metric is $g_{\mu\nu} = \eta_{\mu\nu}$ with $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. It is constant and corresponds to the Minkowski metric.

In the case (b), let us assume a uniform magnetic field of intensity B along the direction of OZ axis. We can take as the vector potential $\boldsymbol{A} = (0, Bx, 0)$ and scalar potencial $A_0 = 0$. The Lagrangian for the point particle under this field is

$$\widetilde{L}_B = -\sqrt{\dot{x}_0^2 - \dot{\boldsymbol{r}}^2} + \frac{eB}{mc}x\dot{y} = F_\mu\dot{x}^\mu, \quad F_0 = -p_0, \quad F_1 = -p_1, \quad F_2 = -p_2 + (eB/mc)x, \quad F_3 = -p_3.$$

which leads to the dynamical equation under the external Lorentz force in a magnetic field:

$$\frac{d\boldsymbol{p}}{dt} = e\boldsymbol{u}\times\boldsymbol{B}.$$

$$p_0 = \frac{\dot{x}_0}{\sqrt{\dot{x}_0^2 - \dot{\boldsymbol{r}}^2}} = \frac{c}{\sqrt{c^2-u^2}}, \quad p_i = \frac{-\dot{x}_i}{\sqrt{\dot{x}_0^2 - \dot{\boldsymbol{r}}^2}} = \frac{-u_i}{\sqrt{c^2-u^2}}$$

According to (1.42) with $A_0 = A_1 = A_3 = 0$, $A_2 = Bx$, if we call $k = eB/mc$, the variational formulation implies that spacetime has a Finsler metric:

$$g_{00} = 1 + \frac{kxu^2u_y}{(c^2-u^2)^{3/2}}, \quad g_{11} = -1 + \frac{kxu_y}{(c^2-u^2)^{3/2}}\left(c^2 - u_y^2 - u_z^2\right),$$

$$g_{22} = -1 + k^2x^2 + \frac{kxu_y}{(c^2-u^2)^{3/2}}\left(3c^2 - 3u_x^2 - 2u_y^2 - 3u_z^2\right),$$

$$g_{33} = -1 + \frac{kxu_y}{(c^2-u^2)^{3/2}}\left(c^2 - u_x^2 - u_y^2\right),$$

$$g_{01} = -\frac{kxcu_xu_y}{(c^2-u^2)^{3/2}}, \quad g_{02} = -\frac{kxc}{(c^2-u^2)^{3/2}}(c^2 - u_x^2 - u_z^2), \quad g_{03} = -\frac{kxcu_yu_z}{(c^2-u^2)^{3/2}}$$

$$g_{12} = \frac{kxu_x}{(c^2-u^2)^{3/2}}\left(c^2 - u_x^2 - u_z^2\right), \quad g_{13} = \frac{kx}{(c^2-u^2)^{3/2}}u_xu_yu_z, \quad g_{23} = \frac{kxu_z}{(c^2-u^2)^{3/2}}\left(c^2 - u_x^2 - u_z^2\right),$$

We see that the metric coefficients are functions of the point, i.e, of the variable x, but they are also functions of the velocity of the particle u_x, u_y, u_z, i.e., $g_{\mu\nu}(x, \dot{x})$. If the velocity of the point is negligible with respect to the speed of light c, the coefficientes of the metric become:

$$g_{00} = 1, \quad g_{02} = -kx, \quad g_{11} = -1, \quad g_{22} = -1 + k^2x^2, \quad g_{33} = -1,$$

vanishing the remaining ones. The dependence on the velocity of the metric coefficients has dissapeared and the metric is now a Riemannian metric. With this restricted metric the variational problem is related to the restricted Lagrangian $\widetilde{L}_R$

$$\widetilde{L}_R^2 = c^2\dot{t}^2 - \dot{\boldsymbol{r}}^2 + k^2x^2\dot{y}^2 - 2kxc\dot{t}\dot{y},$$

which, when compared with the original, it lacks an extra term:

$$\widetilde{L}_B^2 = \widetilde{L}_R^2 - 2kx\dot{y}\left(\sqrt{c^2\dot{t}^2 - \dot{\boldsymbol{r}}^2} - c\dot{t}\right),$$

and therefore the force acting on the particle is no longer the Lorentz force. This metric is not a vacuum solution of Einstein's equations in General relativity, but it leads to a curvature scalar and Einstein's tensor

$$R = \frac{-k^2}{2}, \quad G_{tt} = \frac{3k^2}{4}, \quad G_{ty} = -\frac{3k^3x}{4}, \quad G_{xx} = \frac{k^2}{4}, \quad G_{yy} = \frac{1}{4}(k^2 + 3k^4x^2), \quad G_{zz} = \frac{-k^2}{4},$$

and the nonvanishing Christoffel symbols are

$$\Gamma^t_{tx} = k^2x/2, \quad \Gamma^t_{xy} = -\frac{1}{2}k(1+k^2x^2), \quad \Gamma^x_{ty} = -k/2, \quad \Gamma^x_{yy} = k^2x, \quad \Gamma^y_{tx} = k/2, \quad \Gamma^y_{xy} = -k^2x/2.$$

With the Lorentz force, dynamical equations are

$$\frac{du_x}{dt} = \frac{eB}{m\gamma(u)}u_y = \frac{1}{\gamma(u)}kcu_y, \quad \frac{du_y}{dt} = -\frac{eB}{m\gamma(u)}u_x = -\frac{1}{\gamma(u)}kcu_x, \quad \frac{du_z}{dt} = 0,$$

which lead to $u_xdu_x/dt + u_ydu_y/dt + u_zdu_z/dt = \boldsymbol{u}\cdot d\boldsymbol{u}/dt = 0$, and thus the motion is at a velocity of constant modulus, the factor $\gamma(u)$ is constant and the particle goes along OZ axis with a constant velocity and also rotates on the plane XOY, with constant angular velocity $\omega = eB/\gamma(u)m$. However the geodesic equations obtained from the restricted metric associated to $\widetilde{L}_R$ are

$$\frac{du_x}{dt} = kcu_y(1 - kxu_y/c), \quad \frac{du_y}{dt} = -kcu_x(1 - kxu_y/c), \quad \frac{du_z}{dt} = 0,$$

which also lead to a motion of velocity of constant modulus u. Because we are taking the low velocity limit we have to replace in these equations $u/c \to 0$, and $\gamma(u) \to 1$, and in this case they approximate to the previous ones . For the restricted Lagrangian $\widetilde{L}_R$, the force acting on the particle becomes the Lorentz force in the low velocity limit.

In the example (c) in a uniform gravitational field, the dynamics is described by the Lagrangian

$$\widetilde{L}_g = \widetilde{L}_0 + \frac{\boldsymbol{g}\cdot\boldsymbol{r}}{c^2}\,c\dot{t},$$

which leads to the dynamical equations $d\boldsymbol{p}/dt = \boldsymbol{g}$, with $\boldsymbol{p} = \gamma(u)\boldsymbol{u}$, independent of the mass of the particle. This Lagrangian, from the geodesic point of view corresponds to an evolution on spacetime with a Finsler metric given by:

$$g_{00} = 1 + \left(\frac{\boldsymbol{g}\cdot\boldsymbol{r}}{c^2}\right)^2 - (3\gamma - \gamma^3)\frac{(\boldsymbol{g}\cdot\boldsymbol{r})}{c^2},$$

$$g_{ii} = -1 + \gamma(1 + \gamma^2\frac{u_i^2}{c^2})\frac{(\boldsymbol{g}\cdot\boldsymbol{r})}{c^2}, \quad i = 1,2,3$$

$$g_{0i} = -(\gamma^3 - \gamma)\frac{u_i}{c}\frac{(\boldsymbol{g}\cdot\boldsymbol{r})}{c^2}, \quad i = 1,2,3$$

$$g_{ij} = \gamma^3\frac{u_i u_j}{c^2}\frac{(\boldsymbol{g}\cdot\boldsymbol{r})}{c^2}, \quad i \neq j = 1,2,3$$

The term $\boldsymbol{g}\cdot\boldsymbol{r}$ has dimensions of velocity squared. If the velocity of the point is negligible when compared with c, the nonvanishing coefficients are

$$g_{00} = 1 + \left(\frac{\boldsymbol{g}\cdot\boldsymbol{r}}{c^2}\right)^2 - 2(\boldsymbol{g}\cdot\boldsymbol{r})/c^2, \quad g_{ii} = -1 + (\boldsymbol{g}\cdot\boldsymbol{r})/c^2, \quad i = 1,2,3$$

i.e.,

$$g_{00} = \left(1 - \frac{\boldsymbol{g}\cdot\boldsymbol{r}}{c^2}\right)^2, \quad g_{ii} = -\left(1 - \frac{\boldsymbol{g}\cdot\boldsymbol{r}}{c^2}\right), \quad i = 1,2,3,$$

where the component g_{00} is the same as that of the Rindler metric, corresponding to an uniformly accelerated observer, or to the presence of a uniform gravitational field.

The last example (d) represents the point particle under the gravitational Newtonian potential of a point mass M located at the origin of the reference frame. The Lagrangian is

$$\widetilde{L}_N = \widetilde{L}_0 + \frac{GM}{c^2 r}\,c\dot{t}.$$

As usual, taking into account (1.38) we get the metric of a point particle under a central potential. This metric is

$$g_{00} = 1 + \left(\frac{GM}{c^2 r}\right)^2 - (3\gamma - \gamma^3)\frac{GM}{c^2 r},$$

$$g_{ii} = -1 + \gamma(1 + \gamma^2\frac{u_i^2}{c^2})\frac{GM}{c^2 r}, \quad i = 1,2,3$$

$$g_{0i} = -(\gamma^3 - \gamma)\frac{u_i}{c}\frac{GM}{c^2 r}, \quad i = 1,2,3,$$

$$g_{ij} = \gamma^3\frac{u_i u_j}{c^2}\frac{GM}{c^2 r}, \quad i \neq j = 1,2,3.$$

It is a Finsler metric, which in the case of a low velocity with respect to c, the only coefficients which survive are the diagonal components.

$$g_{00} = \left(1 - \frac{2GM}{c^2 r} + \frac{G^2M^2}{c^4 r^2}\right) = \left(1 - \frac{GM}{c^2 r}\right)^2,$$

the last term goes as G^2/c^4 and if it is considered negligible, this metric coefficient is that of the Schwarzschild's metric. The remaining terms are

$$g_{ii} = -\left(1 - \frac{GM}{c^2 r}\right),$$

while in the metric of Schwarzschild they will be $(1-2GM/c^2r)^{-1}$. We see that the modification of the metric coefficients, in the low velocity limit, differ from the Minkowski metric in a term which is the gravitational potential of the central mass M, divided by c^2.

This low velocity limit of the Finsler metric in a Newtonian potential looks

$$ds^2 = \left(1 - \frac{GM}{c^2 r}\right)^2 c^2 dt^2 - \left(1 - \frac{GM}{c^2 r}\right)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2))$$

which is a rotation invariant, static Riemannian metric. If we call $R_s = 2GM/c^2$ to the Schwarzschild radius, the curvature scalar and Einstein tensor become:

$$R = \frac{R_s^2}{r(2r - R_s)^3},$$

$$G_{tt} = \frac{3R_s^2}{8r^3(2r - R_s)}, \quad G_{rr} = \frac{(24r - 7R_s)R_s}{4r^2(2r - R_s)^2}, \quad G_{\theta\theta} = \frac{(R_s - 3r)R_s}{(2r - R_s)^2}, \quad G_{\phi\phi} = \frac{(R_s - 3r)R_s \sin^2\theta}{(2r - R_s)^2},$$

and therefore it is not a vacuum solution of Einstein's equations of General Relativity.

In the two gravitational examples, the Riemanian approach of the metric has produced that the Minkowski coefficient g_{00} of the free particle has been transformed into $g'_{00} = g_{00}(1 + V(r)/c^2)^2$ and the g_{ii} in the form $g'_{ii} = g_{ii}(1 + V(r)/c^2)$, where in both cases $V(r)$ is the gravitational potential.

1.7 Causality Principle

Among the fundamental principles analyzed, the **Causality Principle** has not been included. Basically, the contents of this principle is the idea that things do not happen by themselves, but rather that any physical effect is the result of a previous cause which determines it. We shall see that, in a certain sense, this principle is already contained in the Variational Principle.

We can consider that the Causality Principle is the restriction on the kinematical space X that the Finsler metric should be definite positive. This condition defines in the kinematical space X, once a point is fixed, two submanifolds, one causally connected with that point and another disconnected. If we select an initial point for the variational description, one cannot arbitrarily select another point as the final state. Only those points belonging to the submanifold causally connected. First of all we have the arrow of time, so that $\dot{t}(\tau) > 0$, or that $t_2 > t_1$, and another that $g_{ij}dx^i dx^j = \widetilde{L}^2 d\tau^2 > 0$. If the Atomic Principle determines that the kinematical space X, for an elementary particle is necessarily a homogeneous space of the kinematical group, the Causality Principle restricts this space, once the initial state is fixed, to a submanifold. For instance, for the point particle, once the state x_1 is fixed, the evolution takes place inside the future light cone of point x_1. Given two points x_1 and x_2 of the kinematical space X, there exists a group element $g \in G$, such that $x_2 = gx_1$, but this does not imply that they are causally connected. The two points $x_1 \equiv (t_1, \boldsymbol{r}_1)$ and $x_2 \equiv (t_1, \boldsymbol{r}_2)$ with the same time, are linked by a space translation, but we cannot arrive dynamically to x_2 coming from x_1, because the velocity should be infinite. Between these two points the Minkowski distance $\int \eta_{\mu\nu}dx^\mu dx^\nu < 0$. Their separation is space-like.

This confirms that only between those points where $g_{ij}dx^i dx^j > 0$ is definite positive, the evolution is possible, while the remaining points will be causally disconected. For a massless point particle, the connected manifold is the future light cone, where the metric vanishes.

The homogenenity of X means that all points represent physically equivalent states. When one is fixed, the remaining states represent the description of the particle for all other inertial observers. Is the definite positive character of the action squared between two points what justifies that the evolution between those points is allowed.

1.8 Generalized canonical formalism

Euler-Lagrange's equations for the n degrees of freedom q_i, (1.4) are a system of n ordinary differential equations of order $2k$, so that to single out a unique solution, according to the theorems of existance and uniqueness of solutions, we have to give as boundary conditions the $2kn$ conditions at the initial time t_1, $q_i^{(s)}(t_1)$, $i = 1, \ldots, n$, $s = 0, 1, 2, \ldots 2k-1$.

Nevertheless, by defining some intermediate variables, every differential equation of order m can be reduced to a system of m ordinary differential equations of first order. In our case to a system of $2kn$ equations of first order. Defining as the new intermediate variables $q_i^{(s)}$, $s = 1, \ldots, 2k-1$:

$$\frac{dq_i}{dt} = q_i^{(1)}, \quad \frac{dq_i^{(1)}}{dt} = q_i^{(2)}, \quad \frac{dq_i^{(2)}}{dt} = q_i^{(3)}, \quad \ldots, \quad \frac{dq_i^{(2k-2)}}{dt} = q_i^{(2k-1)}$$

and the last n final equations (1.4), which have the form as in (1.31)

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} p_{i(1)}(t, q_i, q_i^{(1)}, \ldots, q_i^{(2k-1)}) = 0.$$

To this system of equations we shall suply as boundary conditions the above mentioned $2kn$ conditions at time t_1.

The canonical formalism takes as intermediate variables the kn generalized coordinates $q_i^{(s)}$, $i = 1, \ldots, n$, $s = 0, 1, 2, \ldots, k-1$, i.e., the kinematical variables with the time excluded, and the kn canonical conjugate momenta $p_{i(s)}$. The corresponding set of ordinary differential equations of first order we are going to obtain are known as Hamilton's equations.

The generalized Hamiltonian is defined as

$$H = \sum_{s=1}^{k} p_{i(s)} q_i^{(s)} - L(t, q_i, \ldots, q_i^{(k)}). \tag{1.43}$$

If from the definition of the momentum of order k, $p_{i(k)} = \partial L / \partial q_i^{(k)}$ we can eliminate the n derivatives of higher order $q_i^{(k)}$ in terms of the n $p_{i(k)}$, the the Lagrangian will be written in terms of the time t, of the kn canonical coordinates $q_i^{(s)}$, $s = 0, \ldots, k-1$ and of the n momenta $p_{i(k)}$. Therefore the Hamiltonian will be a function $H(t, q, p)$, of the time, of the canonical coordinates and their canonical conjugate momenta. Therefore

$$dH = \sum_{s=0}^{k-1} \frac{\partial H}{\partial q_i^{(s)}} dq_i^{(s)} + \sum_{r=1}^{k} \frac{\partial H}{\partial p_{i(r)}} dp_{i(r)} + \frac{\partial H}{\partial t} dt, \tag{1.44}$$

But from its definition (1.43)

$$dH = \sum_{s=1}^{k} q_i^{(s)}\, dp_{i(s)} + \sum_{s=1}^{k} p_{i(s)}\, dq_i^{(s)} - \frac{\partial L}{\partial t} dt - \sum_{s=0}^{k} \frac{\partial L}{\partial q_i^{(s)}} dq_i^{(s)},$$

The first term can be written as:

$$\sum_{s=1}^{k} q_i^{(s)}\, dp_{i(s)} = \sum_{r=1}^{k} \frac{dq_i^{(r-1)}}{dt}\, dp_{i(r)},$$

where the coefficients of the $dp_{i(r)}$ are the time derivatives of their corresponding conjugate coordinates $q_i^{(r-1)}$, $r = 1, \ldots, k$. The second term can be rewritten as:

$$\sum_{s=1}^{k} p_{i(s)}\, dq_i^{(s)} = \sum_{s=1}^{k-1} p_{i(s)}\, dq_i^{(s)} + p_{i(k)}\, dq_i^{(k)}.$$

and in the first sums there are only the differentials $dq_i^{(s)}$ of the generalized coordinates, with $s \leq k-1$. If we make the same with the last term

$$\sum_{s=0}^{k} \frac{\partial L}{\partial q_i^{(s)}} dq_i^{(s)} = \frac{\partial L}{\partial q_i} dq_i + \sum_{s=1}^{k-1} \frac{\partial L}{\partial q_i^{(s)}} dq_i^{(s)} + \frac{\partial L}{\partial q_i^{(k)}} dq_i^{(k)},$$

where we have separated the last term which contains the differential $dq_i^{(k)}$. This last term cancels out with the last term of the previous expression, and thus these two expressions together become

$$\sum_{s=1}^{k} p_{i(s)}\, dq_i^{(s)} - \sum_{s=0}^{k} \frac{\partial L}{\partial q_i^{(s)}} dq_i^{(s)} = -\frac{\partial L}{\partial q_i} dq_i + \sum_{s=1}^{k-1} \left(p_{i(s)} - \frac{\partial L}{\partial q_i^{(s)}} \right) dq_i^{(s)},$$

From the definition of the canonical momenta we have the expressions (1.30) and (1.31) and this last expression takes the form:

$$\sum_{s=1}^{k} p_{i(s)}\, dq_i^{(s)} - \sum_{s=0}^{k} \frac{\partial L}{\partial q_i^{(s)}} dq_i^{(s)} = -\frac{dp_{i(1)}}{dt} dq_i - \sum_{s=1}^{k-1} \frac{dp_{i(s+1)}}{dt} dq_i^{(s)} = -\sum_{s=0}^{k-1} \frac{dp_{i(s+1)}}{dt} dq_i^{(s)},$$

where the differentials $dq_i^{(s)}$ are extended to the differentials of all generalized variables. In this way, by collecting all terms, we have:

$$dH = -\sum_{s=0}^{k-1} \frac{dp_{i(s+1)}}{dt} dq_i^{(s)} + \sum_{r=1}^{k} \frac{dq_i^{(r-1)}}{dt}\, dp_{i(r)} - \frac{\partial L}{\partial t} dt, \quad s = 0, 1, \ldots, k-1$$

and by identifying with (1.44) it implies $\partial H/\partial t = -\partial L/\partial t$, and

$$\frac{\partial H}{\partial q_i^{(s)}} = -\frac{dp_{i(s+1)}}{dt}, \quad s = 0, 1, \ldots, k-1, \quad i = 1, \ldots, n,$$

$$\frac{\partial H}{\partial p_{i(r)}} = \frac{dq_i^{(r-1)}}{dt}, \quad r = 1, \ldots, k, \quad i = 1, \ldots, n,$$

so that we arrive to the $2kn$ equations of first order for the canonical variables $q's$ and $p's$, where $q_i^{(s)}$ is the canonical conjugate of the $p_{i(s+1)}$, $s = 0, 1, \ldots, k-1$,

$$\frac{dq_i^{(s)}}{dt} = \frac{\partial H}{\partial p_{i(s+1)}}, \quad \frac{dp_{i(s+1)}}{dt} = -\frac{\partial H}{\partial q_i^{(s)}}, \quad i = 1, \ldots, n, \quad s = 0, 1, \ldots, k-1. \tag{1.45}$$

which are known as **Hamilton's canonical equations** (1833).

The time derivative of each canonical variable is the partial derivative of the Hamiltonian with respect to the corresponding canonical conjugate variable. As far as the variables $q's$ are concerned the partial derivative of the Hamiltonian is preceded with a + sign, while the time derivatives of the $p's$ the partial derivative of the Hamiltonian is affected with a − sign. The knowledge of the Hamiltonian instead of the knowledge of the Lagrangian, will produce the dynamical equations of the material system as equations of first order.

The **Poisson bracket** of two functions $A(q,p)$ and $B(q,p)$ of the conjugate variables, is defined as

$$\{A(q,p),B(q,p)\}=\sum_{i=1}^{n}\sum_{s=0}^{k-1}\left(\frac{\partial A}{\partial q_i^{(s)}}\frac{\partial B}{\partial p_{i(s+1)}}-\frac{\partial A}{\partial p_{i(s+1)}}\frac{\partial B}{\partial q_i^{(s)}}\right)$$

This bracket operation is antisymmetric $\{A,B\}=-\{B,A\}$ and satisfy the distributive properties

$$\{A,B+C\}=\{A,B\}+\{A,C\},\quad \{A,BC\}=\{A,B\}C+B\{A,C\}.$$

For any tree functions A,B,C, Jacobi's identities are fulfilled:

$$\{A,\{B,C\}\}+\{B,\{C,A\}\}+\{C,\{A,B\}\}=0.$$

With this notation, Hamilton's canonical equations are written as:

$$\{p_{i(s)},H\}=\frac{dp_{i(s)}}{dt},\quad \{q_i^{(s)},H\}=\frac{dq_i^{(s)}}{dt}.$$

The dynamical equation of each canonical variable is the Poisson bracket of that variable with the Hamiltonian. There is no distinction between generalized coordinates and conjugate momenta.

If we have an arbitrary observable which is also an explicit function of time $A(t,q,p)$, its time derivative is obtained as:

$$\frac{dA}{dt}=\frac{\partial A}{\partial t}+\frac{\partial A}{\partial q_i^{(s)}}\frac{dq_i^{(s)}}{dt}+\frac{\partial A}{\partial p_{i(s+1)}}\frac{dp_{i(s+1)}}{dt},$$

$$=\frac{\partial A}{\partial t}+\frac{\partial A}{\partial q_i^{(s)}}\frac{\partial H}{\partial p_{i(s+1)}}-\frac{\partial A}{\partial p_{i(s+1)}}\frac{\partial H}{\partial q_i^{(s)}}=\frac{\partial A}{\partial t}+\{A,H\}.$$

The manifold of dimension $2kn$ generated by the $q's$ and $p's$ is known as the **phase space** of the dynamical system, such that when fixing a point in this space, dynamical equations supply a unique solution passing through it. In this sense Euler-Lagrange's equations with boundary conditions at the initial time t_1 are equivalent to Hamilton's canonical equations.

Although we have included in these notes the generalized canonical formulation is simply by consistency. To show that even when the Lagrangians depend on higher order derivatives we also have the associated Hamiltonian formalism. The system of Euler-Lagrange's equations can be reduced to a first order system in the generalized variables $q's$ and $p's$. However, in what follows in our formalism, it is not necessary the use of the canonical formalism, not even for the quantization. Generalized momenta are used for the construction of the Noether constants of the motion, but what we want to stress is that we are interested in finding solutions of the dynamical equations not by giving boundary conditions at time t_1, but rather to find solutions passing through the initial and final points of the variational formalism. What we want is to enhance the role of the kinematical variables as the variables which define at any instant τ, the state of any dynamical system.

1.8.1 Hamilton-Jacobi's equations

Hamilton's principal function is a function defined on the kinematical space of the dynamical system. It is defined as the action function of the final point of the evolution. Usually is represented as $S(x)$, and although we are going to use the character S to represent the observable spin, we are going to mantain this notation in this complementary section. According to this definition:

$$S(x) \equiv A(x_1, x),$$

where the initial point x_1 is fixed, but arbitrary. Hamilton's principal function is a function of time t and of the generalized variables q's. We are going to obtain Hamilton-Jacobi's equations in the case of an ordinary Lagrangian which depends on the first order derivatives of the degrees of freedom q_i, $L(t, q, \dot{q})$, but the proof can be extended to generalized Lagrangians.

If at the final instant of the evolution t the boundary conditions $q(t)$ are modified in an infinitesimal way $q + \delta q$, how this change modifies the function S?

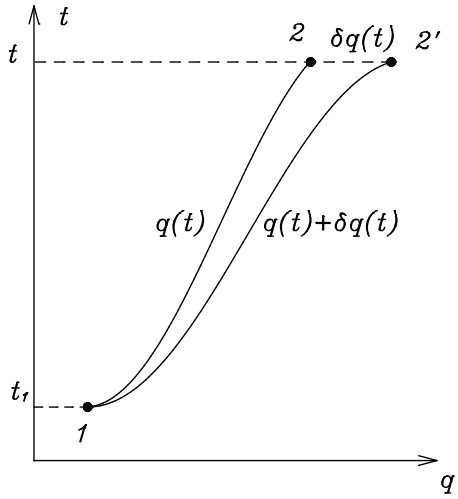


Figure 1.4: **Modification of the final point of the evolution at the instant t, and also of the path $q(t)$ while remaining fixed the initial point 1. The variation of the δq is its form variation $\bar{\delta} q$, at constant t.**

The variation of the action functional between the paths $q(t)$ and $q(t) + \delta q(t)$ is

$$\mathcal{A}(q + \delta q) - \mathcal{A}(q) = \delta \mathcal{A} = \int_{t_1}^{t} dt \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]$$

but the variation of δq is its variation in form at constant t

$$\frac{\partial L}{\partial \dot{q}} \delta \dot{q} = \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q,$$

and thus

$$\delta \mathcal{A} = \int_{t_1}^{t} dt \left\{ \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \right\}$$

If the trajectory $q(t)$ is the one which makes extremal to the action functional the term between squared brackets vanishes and the variation of the action functional is reduced to the variation of the action function

$$\delta A = \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t} - \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t_1} = \frac{\partial L}{\partial \dot{q}}(t)\, \delta q(t),$$

since $\delta q(t_1) = 0$. But this is the variation of Hamilton's principal function δS at fixed time t, when we modify the remaining kinematical variables q,

$$S(t, q + \delta q) - S(t, q) = \delta S(t, q) = \frac{\partial L}{\partial \dot{q}_i} \delta q_i = p_i \delta q_i, \quad \Rightarrow \quad \frac{\partial S}{\partial q_i} = p_i. \tag{1.46}$$

where p_i is the conjugate momentum of the variable q_i at the instant t.

What is left is how Hamilton's principal function is modified when we change the time of the final point while keeping fixed the remaining variables q at the final instant and, of course, the initial point 1. The variation of the action functional is

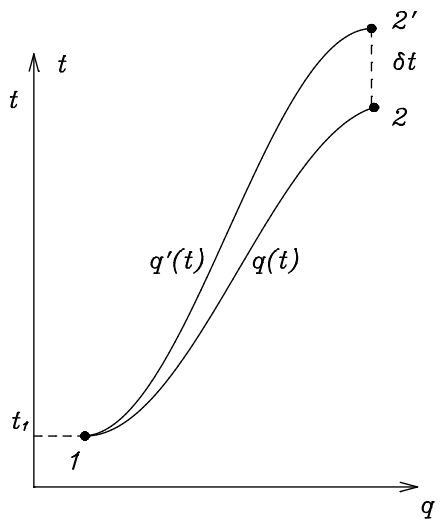


Figure 1.5: **Modification of the final point of the evolution at instant t to the instant $t + \delta t$, keeping constants at this time the values of the variables q.**

$$\mathcal{A}(q') - \mathcal{A}(q) = \delta \mathcal{A} = \int_{t_1}^{t+\delta t} L(t, q', \dot{q}')dt - \int_{t_1}^{t} L(t, q, \dot{q})dt.$$

In some intermediate instant of the evolution $\delta q = \bar{\delta} q + \dot{q}\delta t$ and $\delta \dot{q} = \bar{\delta}\dot{q} + \ddot{q}\delta t$, where $\bar{\delta}$ represents the variation in form, at constant t, of the corresponding function. But

$$\bar{\delta}\dot{q} = \frac{d}{dt}\bar{\delta} q.$$

The proof follows the same method as the one we used when analyzing Noether's theorem, including the change of integration interval from $(t_1, t + \delta t)$ to (t_1, t) and we finally arrive to the expression (1.22) which for the indices $s = 0, 1$, becomes:

$$\delta \mathcal{A} = \int_{t_1}^{t} \left(\frac{d(L\delta t)}{dt} + \frac{\partial L}{\partial q_i} \bar{\delta} q_i(t) + \frac{\partial L}{\partial \dot{q}_i} \bar{\delta}\dot{q}_i(t) \right) dt.$$

If the last term is written as

$$\frac{\partial L}{\partial \dot{q}_i} \bar{\delta}\dot{q}_i(t) = \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \bar{\delta} q_i = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \bar{\delta} q_i \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \bar{\delta} q_i,$$

we get

$$\delta \mathcal{A} = \int_{t_1}^{t} dt \left\{ \frac{d(L\delta t)}{dt} + \bar{\delta} q_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \bar{\delta} q_i \right) \right\}.$$

By recovering the total variations $\bar{\delta}q_i = \delta q_i - \dot{q}_i \delta t$, and if $q(t)$ is the path $1 \to 2$ which makes extremal the action, the term between squared brackets vanishes, and what remains is the variation of the action function with the initial point fixed, i.e., the variation of Hamilton's principal function:

$$\delta S = \delta A(x_1, x) = \int_{t_1}^{t} dt \left\{ \frac{d(L\delta t)}{dt} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\, \delta q_i - \frac{\partial L}{\partial \dot{q}_i}\, \dot{q}_i \delta t \right) \right\}.$$

It is the integral of a total derivative, and the result is the value of the integrand evaluated at time t minus the same at initial time t_1; but $\delta t(t_1) = 0$, $\delta t(t) = \delta t$, $\delta q_i(t_1) = \delta q_i(t) = 0$, and thus

$$S(t + \delta t, q) - S(t, q) = \delta S(t, q) = (L - p_i \dot{q}_i)\, \delta t = -H \delta t, \quad \Rightarrow \quad \frac{\partial S}{\partial t} = -H.$$

Since the Hamiltonian is a function of the momenta p_i which are expresed by (1.46) in terms of the partial derivatives $p_i = \partial S / \partial q_i$, Hamilton's principal function satisfies the partial differential equation with respect to the kinematical variables:

$$\frac{\partial S(t, q)}{\partial t} = -H\left(t, q, \frac{\partial S}{\partial q_i} \right), \tag{1.47}$$

which is known as **Hamilton-Jacobi equation** (1834).

Hamilton's principal function satisfies a wave equation, in general non linear, in the kinematic space, X, because (1.47) represents how this function S changes with time at the point $x \equiv \{t, q\}$, as a function of its changes in the directions of the generalized variables q_i.

1.8.2 About the equivalence between the canonical and the variational formulation

Many texts books state that both formulations are equivalent[19]. However this is not correct. Given Euler-Lagrange's equations the variational formulation tries to find a solution with the boundary values $q_i(t_1)$ and $q_i(t_2)$, at the times t_1 and t_2, respectively, while the canonical formulation the boundary conditions are $q_i(t_1)$ and $p_i(t_1)$, at the same initial time t_1. For regular systems of differential equations, to fix at initial time t_1 a point of the phase space $(q_i, p_j)(t_1)$ singles out a unique solution passing through it, while fixing two points of the kinematical space $x_1 \equiv (t_1, q_i(t_1))$ and $x_2 \equiv (t_2, q_i(t_2))$ does not guarantee the existence of solution, compatible with the mentioned causality principle in the section **1.7**, and in the case that it exists, the uniqueness is not guaranteed.

Let us see with some example: the motion of a rigid body. From the variational point of view we have to fix at time t_1 the center of mass position $\boldsymbol{r}_1$ and the orientation $\boldsymbol{\alpha}_1$ of the inertia principal frame and the same magnitudes for the final time t_2. From the point of view of the canonical formalism we have to fix at time t_1 the center of mass position $\boldsymbol{r}_1$ the center of mass velocity $\boldsymbol{v}_1 = \boldsymbol{p}_1/m$, the orientation $\boldsymbol{\alpha}_1$ and the angular velocity $\boldsymbol{\omega}_1$. If the motion is free $\boldsymbol{v}_1 = (\boldsymbol{r}_2 - \boldsymbol{r}_1)/(t_2 - t_1)$, but the angular velocity cannot be expressed in terms of the two orientations, because we do not know how many turns and about what axis, has rotated the rigid body. In the canonical formulation the solution is unique, but in the variational formulation we have infinite solutions, because the angular velocity remains undetermined. It is the presence

[19] D.A. Wells, Theory and Problems of Lagrangian dynamics, Schaum McGraw Hill, NY (1967) p.1:...*the basic laws of dynamics can be formulated in several ways other than that given by Newton. The most important of these are referred to as: a)D'Alembert's principle, b) Lagrange's equations, c) Hamilton's equations, d) Hamilton's principle. All are basically equivalent.*

E.C.G. Sudarshan and N. Mukunda, Classical Dynamics: A modern perspective, John Willey NY (1974), p.24 ... *this completes the demonstration, in the standard case, of the complete equivalence of the Lagrangian and Hamiltonian forms of dynamics.*

of the compact variables, which determine the orientation in space, which makes that both boundary conditions and therefore both formalisms are not equivalent.

1.9 Appendix: Summary of the formalism

1. For a system of n degrees of freedom q_i whose Lagrangian depends up to the time derivatives of order k, $q_i^{(k)} = d^k q_i/dt^k$, $L(t, q_i, q_i^{(1)}, \ldots, q_i^{(k)})$, the kinematical variables are $x_j \equiv \{t, q_i, q_i^{(1)}, \ldots, q_i^{(k-1)}\}$, i.e., the time, the degrees of freedom and their time derivatives up to order $k-1$. The generalized variables are $\{q_i, q_i^{(1)}, \ldots, q_i^{(k-1)}\}$, i.e., the kinematical variables with the time excluded.

2. Each generalized variable has associated a canonical conjugate momentum, defined by

$$p_{i(s)} = \frac{\partial L}{\partial q_i^{(s)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(s+1)}}\right) + \cdots + (-1)^{k-s}\frac{d^{k-s}}{dt^{k-s}}\left(\frac{\partial L}{\partial q_i^{(k)}}\right), \quad s = 1, \ldots, k$$

   $p_{i(1)}$ is the conjugate momentum of q_i, $p_{i(2)}$ is the conjugate momentum of $q_i^{(1)}$ and finally $p_{i(k)}$ is the conjugate momentum of the $q_i^{(k-1)}$.

3. In a parametric description of the evolution, $t(\tau)$, $q_i(\tau)$, the Lagrangian $\widetilde{L} = L\dot{t}$, where $\dot{}$ represents the derivative with respect to the parameter τ, is a function of the kinematical variables x and their first order τ-derivative, $\dot{x}$, $\widetilde{L}(x, \dot{x})$.

4. The action function is the value of the action functional along the path that satisfies Euler-Lagrange equations.

5. The action function is an explicit function of **all** kinematical variables x_1 and x_2 at the boundary points of the trajectory on the kinematical space X, $A(x_1, x_2)$.

6. The evolution parameter τ can be taken dimensionless, and therefore $\widetilde{L}$ has dimensions of action.

7. The Lagrangian $\widetilde{L}$ can be obtained from the action function through the limit

$$\widetilde{L}(x, \dot{x}) = \lim_{y \to x} \frac{\partial A(x, y)}{\partial y_i}\dot{x}_i.$$

8. The Lagrangian $\widetilde{L}$ is not an explicit function of τ, but it is a homogeneous function of degree 1 of the derivatives $\dot{x}_i$ of **all** kinematical variables. This allows us to write the Lagrangian as a sum of as many terms as kinematical variables

$$\widetilde{L}(x, \dot{x}) = \frac{\partial \widetilde{L}(x, \dot{x})}{\partial \dot{x}_i}\dot{x}_i = F_i(x, \dot{x})\dot{x}_i.$$

9. The functions $F_i(x, \dot{x})$ are homogeneous functions of zero degree of the $\dot{x}_i$, and thus they are functions of the time derivatives of the generalized variables. Since each term $F_i\dot{x}_i$ has dimensions of action, each F_i has the complementary dimension of the corresponding variable x_i.

10. The definite positive function $\widetilde{L}^2$, can always be written as

$$\widetilde{L}^2 = g_{ij}(x, \dot{x})\dot{x}_i\dot{x}_j, \quad g_{ij}(x, \dot{x}) = \frac{1}{2}\frac{\partial^2 \widetilde{L}^2}{\partial \dot{x}_i \partial \dot{x}_j} = g_{ji},$$

    where the coefficients $g_{ij} = g_{ji}$, are homogeneous functions of degree 0 of the derivatives $\dot{x}_i$.

11. The kinematical space is always a Finsler metric space. Since

$$\int_{\tau_1}^{\tau_2} \widetilde{L} d\tau = \pm \int_{\tau_1}^{\tau_2} \sqrt{\widetilde{L}^2}\, d\tau = \pm \int_{\tau_1}^{\tau_2} \sqrt{g_{ij}(x,\dot{x})\dot{x}_i\dot{x}_j}\, d\tau = \pm \int_{\tau_1}^{\tau_2} \sqrt{g_{ij}dx_i dx_j} = \pm \int ds$$

the variational problem is equivalent to a geodesic problem on the kinematical space X, with a metric $g_{ij}(x,\dot{x})$ which is a function of the point x and of the derivatives $\dot{x}_i$.

12. If the mechanical system is an elementary particle, then it is necessary that the kinematical space X be a homogeneous space of the kinematical group G associated to the Restricted Relativity Principle.

13. The kinematical space of the point particle is spacetime. This manifold is always a metric space with a metric more general than a Riemannian metric. To admit, as is done in General Relativity, that the spacetime manifold of the test particle is a Riemannian manifold, is a restriction about a more general situation. The kinematical space of the free point particle is Minkowski spacetime.

14. The invariance of dynamical equations under a symmetry group of transformations does not imply that the Lagrangian and the action function are invariant. Noether's theorem gives the relationship between the transformation of the action function $A(x_1, x_2)$, under a group which leaves invariant the dynamical equations, and the explicit construction of the constants of the motion. These constants of the motion are written in terms of the Lagrangian, its partial derivatives $F_i(x,\dot{x})$, and of the functions $M(x)$ of how the kinematical variables transform, $\delta t = M_0(x)\delta g$, $\delta q_i^{(s)} = M_i^{(s)}(x)\delta g$, under some infinitesimal transformation of the group of parameter δg.

$$N = \lambda(x) - (L - p_{i(s)}q_i^{(s)})M_0 - p_{i(s)}M_i^{(s-1)} = \lambda(x) + HM_0 - p_{i(s)}M_i^{(s-1)},$$

where $p_{i(s)}$ is the canonical conjugate momentum of the generalized variable $q_i^{(s-1)}$ and $\lambda(x)$ the function associated to the non-invariance of the Lagrangian under the group.

1.10 Appendix: Lie groups of transformations

Let us introduce the notation and general features of the action of Lie groups on continuous manifolds to analyze the transformation properties of the different magnitudes we can work with in either classical or quantum mechanics. We shall use these features all throughout this book.

Let us consider the transformation of an n-dimensional manifold X, $x' = gx$ given by n continuous and differentiable functions depending on a set $g \in G$ of r continuous parameters of the form

$$x'^i = f^i(x^j; g^\sigma), \quad \forall x \in X, \quad \forall g \in G, \quad i,j = 1,\ldots,n, \quad \sigma = 1,\ldots,r.$$

This transformation is said to be the action of a Lie group of transformations if it fulfils the two conditions:
(i) G is a Lie group, *i.e.*, there exists a group composition law $c = \phi(a,b) \in G$, $\forall a,b \in G$, in terms of r continuous and differentiable functions ϕ^σ.
(ii) The transformation equations satisfy

$$x'' = f(x';b) = f(f(x;a);b) = f(x;c) = f(x;\phi(a,b)).$$

The group parametrization can be chosen such that the coordinates that characterize the neutral element e of the group are $e \equiv (0,\ldots,0)$, so that an infinitesimal element of the group is the one with infinitesimal coordinates $\delta g^\sigma, \sigma = 1,\ldots,r$.

Under the action of an infinitesimal element δg of the group G, the change in the coordinates x^i of a point $x \in X$ is given by

$$x^i + dx^i = f^i(x;\delta g) = x^i + \left.\frac{\partial f^i(x;g)}{\partial g^\sigma}\right|_{g=e} \delta g^\sigma,$$

after a Taylor expansion up to first order in the group parameters and with $x^i = f^i(x;0)$. There are nr auxiliary functions of the group that are defined as

$$u^i_\sigma(x) = \left.\frac{\partial f^i(x;g)}{\partial g^\sigma}\right|_{g=e}, \tag{1.48}$$

and therefore to first order in the group parameters, $dx^i = u^i_\sigma(x)\delta g^\sigma$.

The group action on the manifold X can be extended to the action on the set $\mathcal{F}(X)$ of continuous and differentiable functions defined on X by means of:

$$g : h(x) \to h'(x) \equiv h(gx). \tag{1.49}$$

If the group element is infinitesimal, then

$$h'(x) = h(x^i + dx^i) = h(x^i + u^i_\sigma(x)\delta g^\sigma) = h(x) + \frac{\partial h(x)}{\partial x^i}\, u^i_\sigma(x)\delta g^\sigma,$$

after a Taylor expansion to first order in the infinitesimal group parameters. The infinitesimal transformation on $\mathcal{F}(X)$ can be represented by the action of a differential operator in the form

$$h'(x) = \left(\mathbb{I} + \delta g^\sigma\, u^i_\sigma(x)\frac{\partial}{\partial x^i}\right) h(x) = \left(\mathbb{I} + \delta g^\sigma X_\sigma\right) h(x) = U(\delta g)h(x),$$

where $\mathbb{I}$ is the identity operator and the linear differential operators

$$X_\sigma = u^i_\sigma(x)\frac{\partial}{\partial x^i}. \tag{1.50}$$

In particular, when acting with the operator $U(\delta g) \equiv (\mathbb{I} + \delta g^\sigma X_\sigma)$ on the coordinate x^j we get $x^j + dx^j = x^j + u^j_\sigma(x)\delta g^\sigma$.

The operators X_σ are called the **generators** of the infinitesimal transformations. They are r linearly independent operators that span an r-dimensional real vector space such that its commutator $[X_\sigma, X_\lambda]$ also belongs to the same vector space, *i.e.*,

$$[X_\sigma, X_\lambda] = c^\alpha_{\sigma\lambda}\, X_\alpha, \quad \alpha, \sigma, \lambda = 1, \ldots, r. \tag{1.51}$$

The coefficients $c^\alpha_{\sigma\lambda}$ are a set of real constant numbers, called the **structure constants** of the group, and the vector space spanned by the generators is named the **Lie algebra** $\mathcal{L}(G)$, associated to the Lie group G. The structure constants are antisymmetric in their lower indexes $c^\alpha_{\sigma\lambda} = -c^\alpha_{\lambda\sigma}$, and satisfy Jacobi's indentitites:

$$c^\alpha_{\sigma\lambda}c^\beta_{\mu\alpha} + c^\alpha_{\lambda\mu}c^\beta_{\sigma\alpha} + c^\alpha_{\mu\sigma}c^\beta_{\lambda\alpha} = 0, \qquad \forall\sigma, \lambda, \mu, \beta = 1, \ldots, r.$$

Equations (1.51) are the commutation relations that characterize the structure of the Lie algebra of the group.

If a finite group transformation of parameters g^σ can be done in n smaller steps of parameters g^σ/n, with n sufficiently large, then a finite transformation $U(g)h(x)$ can be obtained as

$$U(g)h(x) \equiv \lim_{n\to\infty}\left(\mathbb{I} + \frac{g^\sigma}{n}X_\sigma\right)^n h(x) = \exp(g^\sigma X_\sigma)\, h(x).$$

This defines the exponential mapping and in this case the group parameters g^σ are called **normal** or **canonical** parameters. In the normal parameterization the composition law of one-parameter subgroups reduces to the addition of the corresponding parameters of the involved group elements.

Let us consider that $\mathcal{F}(X)$ is a Hilbert space of states of a quantum system; (1.49) can be interpreted as the transformed wave function under the group element g. Then if the operator $U(g)$ is unitary it is usually written in the explicit form

$$U(g) = \exp\left(\frac{i}{\hbar}\, g^\sigma \widetilde{X}_\sigma\right),$$

in terms of the imaginary unit i and Planck's constant $\hbar$, such that in this case the new $\widetilde{X}_\sigma$ above are self-adjoint operators and therefore represent certain observables of the system. The physical dimensions of these observables depend on the dimensions of the group parameters g^σ, since the argument of the exponential function is dimensionless and because of the introduction of Planck's constant $\hbar$, this implies that $g^\sigma\widetilde{X}_\sigma$ has dimensions of action. These observables, taking into account (1.50), are represented in a unitary representation by the differential operators

$$\widetilde{X}_\sigma = \frac{\hbar}{i}\, u^i_\sigma(x)\frac{\partial}{\partial x^i}. \tag{1.52}$$

However, (1.49) is not the most general form of transformation of the wave function of a quantum system, as we shall see in Chapter 3, but once we know the way it transforms we shall be able to obtain the explicit expression of the group generators by a similar procedure as the one developed so far. In general the wave function transforms under continuous groups with what is called a projective unitary representation of the group, which involves in general some additional phase factors.

1.10.1 Casimir operators

When we have a representation of a Lie group either by linear operators or by matrices acting on a linear space, we can define there what are called the Casimir operators. They are operators C that can be expressed as functions of the generators X_σ of the Lie algebra with the property that they commute with all of them, *i.e.*, they satisfy $[C, X_\sigma] = 0, \quad \forall \sigma = 1, \ldots, r$. In general they are not expressed as real linear combinations of the X_σ and therefore they do not belong to the Lie algebra of the group. They belong to what is called the **group algebra**, *i.e.*, the associative, but in general non-commutative algebra, spanned by the real or complex linear combinations of products of the X_σ, in the corresponding group representation.

In those representations where the X_σ are represented by self-adjoint operators as in a quantum formalism, the Casimir operators may be also self-adjoint and will represent those observables that remain invariant under the group transformations. In particular, when we consider later the kinematical groups that relate the space-time measurements between inertial observers, the Casimir operators of these groups will represent the intrinsic properties of the system. They are those properties of the physical system whose measured values are independent of the inertial observers.

For semisimple groups, *i.e.*, for groups that do not have Abelian invariant subgroups like the rotation group $SO(3)$, the unitary groups $SU(n)$ and many others, it is shown that the Casimir operators are real homogeneous polynomials of the generators X_σ, but this is no longer the case for general Lie groups. Nevertheless, for most of the interesting Lie groups in physics, like Galilei, Poincaré, De Sitter, $SL(4,\mathbb{R})$, the inhomogeneous $ISL(4,\mathbb{R})$ and Conformal $SU(2,2)$ groups, the Casimir operators can be taken as real polynomial functions of the generators.

1.10.2 Homogeneous space of a group

A manifold X is called a homogeneous space of a group G, if $\forall\, x_1,\, x_2 \in X$ there exists at least one element $g \in G$ such that $x_2 = g x_1$. In that case it is said that G acts on X in a transitive way. The term homogeneous reminds us that the local properties of the manifold at a point x are translated to any other point of the manifold by means of the group action, and therefore all points of X share the same local properties.

The **orbit** of a point x is the set of points of the form gx, $\forall g \in G$, such that if X is a homogeneous space of G, then the whole X is the orbit of any of its points.

Given a point $x_0 \in X$, the **stabilizer group** (little group) of x_0 is the subgroup H_{x_0} of G, that leaves invariant the point x_0, *i.e.*, $\forall h \in H_{x_0}$, $h x_0 = x_0$.

If H is a subgroup of G, then every element $g \in G$ can be written as $g = g'h$, where $h \in H$, and g' is an element of G/H, the set of left cosets generated by the subgroup H. If X is a homogeneous space of G, it can be generated by the action of G on an arbitrary point $x_0 \in X$. Then $\forall x \in X$, $x = g x_0 = g' h x_0 = g' x_0$, and thus the homogeneous space X is isomorphic to the manifold G/H_{x_0}.

The homogeneous spaces of a group can be constructed as quotient manifolds of the group by all its possible continuous subgroups. Conversely, it can also be shown that if X a homogeneous space of a group G, then there exists a subgroup H of G such that X is isomorphic to G/H. Therefore, the largest homogeneous space of a group is the group itself.

1.10.3 Examples of continuous groups

1. Let us consider the group of translations of the straight line:

$$x' = x + a.$$

With $a = 0$ we have the neutral element and $-a$ represents the inverse element. The transformation is infinitesimal if a is infinitesimal and we write as $x' = x + \delta a$. If $f(x)$ is a function of x, the infinitesimal action of the group on f is defined as

$$f'(x) = f(x + \delta a) = f(x) + \delta a \frac{\partial f(x)}{\partial x} = (\mathbb{I} + \delta a P) f(x).$$

The operator $P = \partial/\partial x$ is called the generator of the infinitesimal transformation and the infinitesimal element of the group becomes the differential operator $\delta g \equiv \mathbb{I} + \delta a P$ when acting on the variables and also on functions of these variables. If $f(x)$ is an invariant function under this group, then $Pf = \partial f/\partial x = 0$, and f is independent of x. For a finite transformation of parameter a, the exponential mapping holds e^{aP}, because

$$e^{aP} \equiv \mathbb{I} + \frac{aP}{1!} + \frac{(aP)}{2!} + \ldots + \frac{(aP)^n}{n!} + \ldots = \mathbb{I} + \frac{a}{1!}\frac{\partial}{\partial x} + \frac{a^2}{2!}\frac{\partial^2}{\partial x^2} + \ldots$$

When we apply this operator on the variable x

$$x' = e^{aP} x = x + a,$$

and on any function of x

$$e^{aP} f(x) = f(x) + \frac{a}{1!}\frac{\partial f(x)}{\partial x} + \frac{a^2}{2!}\frac{\partial^2 f(x)}{\partial x^2} + \ldots = f(x + a)$$

because the left hand side is the Taylor expansion of $f(x)$ from the point x to the point $x + a$. The parameter a is the canonical parameter of the group.

2. Let us consider the rotations of the plane

$$x' = x\cos\alpha - y\sin\alpha, \quad y' = x\sin\alpha + y\cos\alpha.$$

with $\alpha = 0$ we have the neutral element and $-\alpha$ is the inverse. If α is infinitesimal, of value $\delta\alpha$, to first order in this parameter, the transformation equations are:

$$x' = x - y\delta\alpha, \quad y' = y + x\delta\alpha.$$

If $f(x, y)$ is a function of these variables, it transforms under the group

$$f'(x,y) \equiv f(x',y') = f(x - y\delta\alpha, y + x\delta\alpha) = f(x,y) + \delta\alpha \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) f(x,y) = (\mathbb{I} + \delta\alpha J) f(x,y)$$

where the differential operator

$$J = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

is the generator of the infinitesimal rotations. If $f(x, y)$ is invariant under rotations, then $Jf = 0$, and f is a solution of the differential equation

$$-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0, \quad \Rightarrow \frac{dx}{-y} = \frac{dy}{x}, \quad x dx + y dy = 0$$

since the arc element of components (dx, dy) is orthogonal to the gradient of f and therefore f must be an arbitrary function of the curves $x^2 + y^2 =$cte, i.e., $f(x^2 + y^2)$.
If what we want is to analyze a finite rotation of value $\boldsymbol{\alpha}$, the exponential mapping gives us the corresponding rotation operator

$$e^{\alpha J} \equiv \mathbb{I} + \frac{\alpha}{1!}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + \frac{\alpha^2}{2!}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)^2 + \ldots + \frac{\alpha^n}{n!}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)^n + \ldots$$

and when applied to the variable x gives rise to:

$$e^{\alpha J}x = x - \frac{\alpha}{1!}y - \frac{\alpha^2}{2!}x + \frac{\alpha^3}{3!}y + \ldots = x\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \ldots\right) + y\left(-\frac{\alpha}{1!} + \frac{\alpha^3}{3!} + \ldots\right)$$

i.e.,

$$x' = e^{\alpha J}x = x\cos\alpha - y\sin\alpha.$$

Simmilarly, if we apply to the variable y we obtain

$$y' = e^{\alpha J}y = x\sin\alpha + y\cos\alpha.$$

and when acting on any function $f(x, y)$ we get

$$f'(x, y) = e^{\alpha J}f(x, y) = f(x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha).$$

The parameter α is the canonical parameter of the group.

3. Let us consider a Galilei boost along axis OX,

$$t' = t, \quad x' = x + vt.$$

With $v = 0$ we have the neutral element and $-v$ represents the inverse element. The infinitesimal transformation is with δv infinitesimal and it looks:

$$t' = t, \quad x' = x + \delta vt.$$

The action of the infinitesimal element on the function $f(t, x)$ is given by

$$f'(t, x) = f(t', x') = f(t, x) + \delta vt\frac{\partial f(t, x)}{\partial x} = (\mathbb{I} + \delta vK)\, f(t, x),$$

where $K = t\partial/\partial x$ is the generator of the boosts along the axis OX.
To analyze a finite boost the exponential mapping gives rise to the operator

$$e^{vK} = \mathbb{I} + \frac{v}{1!}K + \frac{v^2}{2!}K^2 + \ldots = \mathbb{I} + \frac{v}{1!}\left(t\frac{\partial}{\partial x}\right) + \frac{v^2}{2!}\left(t\frac{\partial}{\partial x}\right)^2 + \ldots,$$

and when applied to the variable t

$$t' = e^{vK}t = t,$$

and applied to the variable x

$$x' = e^{vK}x = x + vt.$$

Acting on any function $f(t, x)$ produces

$$f'(t, x) = e^{vK}f(t, x) = f(t, x + vt).$$

The parameter v is the canonical parameter of the group.

Chapter 2

Examples of spinning particles

NONRELATIVISTIC PARTICLES

2.1 Nonrelativistic point particle

See the Appendix about the Galilei group $\mathcal{G}$ at the end of this chapter for the notation used through this section.

Let us consider a mechanical system whose kinematical space is the four-dimensional manifold spanned by the variables $(t, \boldsymbol{r}) \equiv x$, with domains $t \in \mathbb{R}$, $\boldsymbol{r} \in \mathbb{R}^3$, similar to the group parameters b and $\boldsymbol{a}$ respectively. We assume that they are functions of some evolution parameter τ and at any instant τ of the evolution two different inertial observers relate their measurements by:

$$t'(\tau) = t(\tau) + b, \tag{2.1}$$
$$\boldsymbol{r}'(\tau) = R(\boldsymbol{\mu})\boldsymbol{r}(\tau) + \boldsymbol{v}t(\tau) + \boldsymbol{a}. \tag{2.2}$$

Because of the way they transform, we can interpret them respectively as the time and position of the particle. If we assume that the evolution parameter τ is group invariant, by taking the $\tau-$derivative of both sides of the above expressions, it turns out that the derivatives of the kinematical variables at any instant τ transform as:

$$\dot{t}'(\tau) = \dot{t}(\tau), \tag{2.3}$$
$$\dot{\boldsymbol{r}}'(\tau) = R(\boldsymbol{\mu})\dot{\boldsymbol{r}}(\tau) + \boldsymbol{v}\dot{t}(\tau). \tag{2.4}$$

If we define the velocity of the point as $\boldsymbol{u} = d\boldsymbol{r}/dt = \dot{\boldsymbol{r}}/\dot{t}$, the velocity of the particle transforms in the way

$$\boldsymbol{u}'(\tau) = R(\boldsymbol{\mu})\dot{\boldsymbol{u}}(\tau) + \boldsymbol{v}.$$

We can obtain simmilarly the transformation equations of other derivatives. The Lagrangian for describing this particle will be a function $L(t, \boldsymbol{r}, \boldsymbol{u})$, and in the parametric τ-description $\widetilde{L}(t, \boldsymbol{r}, \dot{t}, \dot{\boldsymbol{r}}) \equiv \widetilde{L}(x, \dot{x})$, and homogeneous of degree 1 in terms of the $\dot{x}_i$. This homogeneity leads to the general form:

$$\widetilde{L} = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}}, \tag{2.5}$$

where $T = \partial\widetilde{L}/\partial\dot{t}$ and $R_i = \partial\widetilde{L}/\partial\dot{r}_i$ are still some unknown functions of the kinematical variables and their derivatives, which are homogeneous functions of zero degree in terms of the derivatives. This homogeneity is independent whether the particle is free or not. The functions T and R_i are not independent. In fact, if in the expression (2.5) we take the derivative of both members with respect to $\dot{r}_i$, we get

$$R_i = \frac{\partial\widetilde{L}}{\partial\dot{r}_i} = \dot{t}\frac{\partial T}{\partial u_j}\frac{\partial u_j}{\partial\dot{r}_i} + R_i + \dot{r}_j\frac{\partial R_j}{\partial u_k}\frac{\partial u_k}{\partial\dot{r}_i} = \frac{\partial T}{\partial u_i} + R_i + u_j\frac{\partial R_j}{\partial u_i},$$

because $\partial u_j/\partial \dot{r}_i = \delta_{ij}/\dot{t}$, and we thus arrive to

$$\frac{\partial T}{\partial u_i} + u_j \frac{\partial R_j}{\partial u_i} = 0. \tag{2.6}$$

We also know that because T and R_i are partial derivatives of $\widetilde{L}$ and this is a continuous and derivable function, the following equality of crossed derivatives holds

$$\frac{\partial^2 \widetilde{L}}{\partial \dot{t} \partial \dot{r}_i} = \frac{\partial^2 \widetilde{L}}{\partial \dot{r}_i \partial \dot{t}}, \quad \frac{\partial R_i}{\partial \dot{t}} = \frac{\partial T}{\partial \dot{r}_i},$$

but these functions depend on these derivatives through the variables variables $u_i = \dot{r}_i/\dot{t}$, and therefore

$$\frac{\partial R_i}{\partial \dot{t}} = \frac{\partial R_i}{\partial u_j}\frac{\partial u_j}{\partial \dot{t}} = \frac{\partial R_i}{\partial u_j}\left(-\frac{\dot{r}_j}{\dot{t}^2}\right) = \frac{\partial T}{\partial \dot{r}_i} = \frac{\partial T}{\partial u_j}\frac{\partial u_j}{\partial \dot{r}_i} = \frac{\partial T}{\partial u_j}\frac{\delta_{ij}}{\dot{t}} \quad \Rightarrow \quad \frac{\partial T}{\partial u_i} + u_j \frac{\partial R_i}{\partial u_j} = 0,$$

which is another relation of the type (2.6), which suggests that $\partial R_i/\partial u_j = \partial R_j/\partial u_i$.

2.1.1 Free point particle

If the particle is free, dynamical equations must be invariant for the set of equivalent inertial observers, since a change of reference frame cannot modify its dynamical laws. If it is under some interaction, the dynamical equations will not be invariant under the kinematical group because the group transformations affect the kinematical variables and their derivatives, but not to the mechanisms which produce the interaction, like fields, magnets, etc.

Associated to this manifold X, the gauge function for this system is

$$\alpha(g;x) = \xi(g,x) = m\left(v^2 t/2 + \boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{r}\right), \tag{2.7}$$

where the parameter m is interpreted as the mass of the system and $\xi(g,g')$ is the exponent of $\mathcal{G}$. If the transformation is infinitesimal δg, to first order in the group parameters gives

$$\alpha(\delta g;x) = mR(\delta\mu)\boldsymbol{r}\cdot\delta\boldsymbol{v} = \lambda_i(x)\delta v_i,$$

and $\alpha(\delta g;x)$ is different from zero if the transformation is a Galilei boost, and thus the Lagrangian is invariant under traslations and rotations. Under Galilei boosts the function $\lambda_i(x) = mr_i$.

If instead of making that infinitesimal analysis we make the analysis under finite Galilei transformations the transformation of the free Lagrangian under a general finite transformation of the Galilei group is

$$\widetilde{L}(x',\dot{x}') = \widetilde{L}(x,\dot{x}) + m\left(v^2\dot{t}/2 + \boldsymbol{v}\cdot R(\boldsymbol{\mu})\dot{\boldsymbol{r}}\right). \tag{2.8}$$

Then

$$T' = \frac{\partial \widetilde{L}'}{\partial \dot{t}'} = \left(\frac{\partial \widetilde{L}}{\partial \dot{t}} + \frac{1}{2}mv^2\right)\frac{\partial \dot{t}}{\partial \dot{t}'} + \left(\frac{\partial \widetilde{L}}{\partial \dot{r}_i} + mv_j R(\boldsymbol{\mu})_{ji}\right)\frac{\partial \dot{r}_i}{\partial \dot{t}'}, \tag{2.9}$$

but from (2.3) and (2.4) we get $\partial\dot{t}/\partial\dot{t}' = 1$ and $\partial\dot{r}_i/\partial\dot{t}' = -R^{-1}(\boldsymbol{\mu})_{ik}v_k$, respectively, and thus

$$T' = T - \frac{1}{2}mv^2 - \boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{R}. \tag{2.10}$$

Similarly

$$\boldsymbol{R}' = R(\boldsymbol{\mu})\boldsymbol{R} + m\boldsymbol{v}. \tag{2.11}$$

The conjugate momenta of the independent degrees of freedom $q_i = r_i$, are $p_i = \partial\widetilde{L}/\partial\dot{r}_i$, and consequently Noether's theorem leads to the following constants of the motion:
a) Under time translations the gauge function (2.7) vanishes, $\delta t = \delta b$, $M = 1$, while $\delta r_i = 0$ and the constant reduces to the following expression $\boldsymbol{R}\cdot d\boldsymbol{r}/dt - L/\dot{t} = -T$.
b) Under space translations also $\alpha(g;x) \equiv 0$, $\delta t = 0$, $M = 0$, while $\delta r_i = \delta a_i$, $M_{ij} = \delta_{ij}$ and the conserved observable is $\boldsymbol{R}$.
c) Under pure Galilei transformations $\delta t = \delta b$ and $M = 0$, while $\delta r_i = t\delta v_i$ and $M_{ij} = t\delta_{ij}$, but now the gauge function to first order in the velocity parameters is $\alpha(\delta\boldsymbol{v};x) = m\boldsymbol{r}\cdot\delta\boldsymbol{v}$, and we get $m\boldsymbol{r} - \boldsymbol{P}t$.
d) Under rotations $\alpha(g;x) \equiv 0$, $\delta t = 0$ and $M = 0$, while $\delta r_i = -\varepsilon_{ijk}r_j n_k\delta\alpha$ and $M_{ik} = -\varepsilon_{ijk}r_j$ the conserved quantity is $\boldsymbol{r}\times\boldsymbol{R}$.

Collecting all terms we can give them the following names:

$$\text{temporal momentum} \quad H = -T, \tag{2.12}$$
$$\text{linear momentum} \quad \boldsymbol{P} = \boldsymbol{R} = \boldsymbol{p}, \tag{2.13}$$
$$\text{kinematical momentum} \quad \boldsymbol{K} = m\boldsymbol{r} - \boldsymbol{P}t, \tag{2.14}$$
$$\text{angular momentum} \quad \boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P}. \tag{2.15}$$

We reserve for these observables the same symbols in capital letters as the corresponding group generators which produce the space-time transformations that leave dynamical equations invariant. Even their names make reference to the corresponding group transformation parameter.

> In general, what we have defined as the temporal momentum, usually takes the name of **energy** or **Hamiltonian** of the system. However, all observables associated to the uniparametric symmetry groups are never definite positive. All of them can take both signs, but by energy we understand an observable which is definite positive. Actually, the energy should be defined as $E = |H|$. This is important in order to classify the different particles we are going to find, in particular in the relativistic formulation, where the sign of H, is another intrinsic property, independent of the inertial observer. In the relativistic formulation we call particle a mechanical system for which $H > 0$ and antiparticle when $H < 0$. In both cases, if particle and antiparticle have mass m and they are at rest, $H_p = mc^2$ and $H_a = -mc^2$, but its energy is $E = mc^2 = |H|$. By abuse of language and because historically this observable has been denoted by energy, it is possible that along these notes we shall use the name of energy for this observable H.
> For the kinematical momentum we can find in the literature alternative names. Levy-Leblond calls it *Galilei momentum* and sometimes it is called *static momentum* because it has dimensions of mass×distance. Being consistent with this notation, we should call it 'Poincaré or Lorentz momentum' in a relativistic approach. Nevertheless we shall use the name of kinematical momentum for this observable $\boldsymbol{K}$ in either the relativistic or non-relativistic formalism.

If we take the τ-derivative in (2.14) of the kinematical momentum $\dot{\boldsymbol{K}} = 0$, because it is a constant of the motion, it implies that $\boldsymbol{P} = m\dot{\boldsymbol{r}}/\dot{t} = m\boldsymbol{u} = \boldsymbol{R}$, where $\boldsymbol{u}$ is the velocity of the particle. If we take into account the relation (2.6), then

$$\frac{\partial T}{\partial u_i} = -u_j\frac{\partial R_j}{\partial u_i} = -u_j m\delta_{ij} = -mu_i, \quad \Rightarrow \quad T(\boldsymbol{u}) = -\frac{1}{2}m\boldsymbol{u}^2 + T_0,$$

where T_0 is constant. Now the Lagrangian of the free Galilei point particle is

$$\widetilde{L}_0 = T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}} = \frac{1}{2}m\frac{\dot{\boldsymbol{r}}^2}{\dot{t}} + T_0\dot{t} \quad \Rightarrow \quad L_0 = \frac{1}{2}m\boldsymbol{u}^2 + T_0.$$

> From the point of view of infinitesimal trasnformations, since $\widetilde{L}_0(t,\boldsymbol{r},\dot{t},\dot{\boldsymbol{r}})$ depends on these variables, they transform according to (2.1-2.4), and the different generators when acting on these variables are
> $$H = \frac{\partial}{\partial t}, \quad \boldsymbol{P} = \nabla, \quad \boldsymbol{J} = \boldsymbol{r}\times\nabla + \dot{\boldsymbol{r}}\times\nabla_{\dot{r}}, \quad \boldsymbol{K} = t\nabla + \dot{t}\nabla_{\dot{r}}.$$

If the Lagrangian is invariant under translations, then $H\widetilde{L}_0 = 0$ and $\boldsymbol{P}\widetilde{L}_0 = 0$, which imply that $\widetilde{L}_0$ is not a function of t and $\boldsymbol{r}$, respectively. Under rotations $\boldsymbol{J}\widetilde{L}_0 = 0$, and this implies that it is a function of $\dot{\boldsymbol{r}}^2$ and of $\dot{t}$ and must be homogeneous of first degree in these derivatives. Finally, if it is invariant under Galilei boosts $\boldsymbol{K}\widetilde{L}_0 = 0$, and thus $\partial\widetilde{L}_0/\partial\dot{\boldsymbol{r}} = 0$, and will be independent of $\dot{\boldsymbol{r}}$. Since this is not possible because the Lagrangian always has to be a function of all derivatives of the kinematical variables, implies that $\boldsymbol{K}\widetilde{L}_0 = d(f(t,\boldsymbol{r}))/d\tau$, i.e., a total τ-derivative, with dimensions of mass×distance, and thus dynamical equations are invariant. According to the structure of the gauge function (2.7), we have

$$\boldsymbol{K}\widetilde{L}_0 = m\dot{\boldsymbol{r}} = \frac{d}{d\tau}(m\boldsymbol{r}). \quad \dot{t}\nabla_{\dot{\boldsymbol{r}}}\widetilde{L}_0 = m\dot{\boldsymbol{r}}, \quad \Rightarrow \quad \widetilde{L}_0 = \frac{1}{2}m\frac{\dot{\boldsymbol{r}}^2}{\dot{t}} + F(\dot{t}),$$

where $F(\dot{t})$ is an arbitrary function of $\dot{t}$ which has to be homogeneous of degree 1. It has the form $F = -T_0\dot{t}$, with T_0 a constant, which can be interpreted as the internal energy.

2.1.2 Center of mass observer

The six conditions $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$, imply $\boldsymbol{u} = 0$ and $\boldsymbol{r} = 0$, such that the particle is at rest and located at the origin of the observer's frame. To uniquely define an observer we need also to fix an arbitrary rotation and time translation. Nevertheless, we shall call to the class of observers to whom $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$, the center of mass observer. These six conditions will also be used to define the center of mass observer in the relativistic case.

From (2.10) and (2.11) we see that the energy and linear momentum transform as:

$$H' = H + \boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{P} + \frac{1}{2}mv^2, \tag{2.16}$$

$$\boldsymbol{P}' = R(\boldsymbol{\mu})\boldsymbol{P} + m\boldsymbol{v}. \tag{2.17}$$

Then, if H_0 and $\boldsymbol{P} = 0$ are the energy and linear momentum measured by the center of mass observer, for any arbitrary observer who sees the particle moving with velocity $\boldsymbol{u}$, it follows from (2.16) and (2.17) that

$$H = H_0 + \frac{1}{2}mu^2 = H_0 + \boldsymbol{P}^2/2m, \quad \boldsymbol{P} = m\boldsymbol{u}.$$

The Lagrangian for the point particle is thus

$$L = T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}} = -H\dot{t} + \boldsymbol{P}\cdot\dot{\boldsymbol{r}} = -H_0\dot{t} + \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}}, \tag{2.18}$$

with H_0 an arbitrary constant which plays no role in the dynamics and can be taken $H_0 = 0$. It will be related to the mc^2 term of the relativistic point particle.

If we define the spin of the system, as the angular momentum with respect to the point $\boldsymbol{r}$, which represents the location of the center of mass of the particle, then

$$\boldsymbol{S} \equiv \boldsymbol{J} - \frac{1}{m}\boldsymbol{K}\times\boldsymbol{P} = \boldsymbol{J} - \boldsymbol{r}\times\boldsymbol{P} = 0. \tag{2.19}$$

It vanishes, so that the point particle is a spinless system.

2.1.3 Interaction with some external source

The most general Lagrangian of the point particle is of the form $\widetilde{L} = T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}}$, where the functions T and $\boldsymbol{R}$ are functions of $t, \dot{t}, \boldsymbol{r}, \dot{\boldsymbol{r}}$ and homogeneous of zero degree of the derivatives $\dot{t}$ and $\dot{\boldsymbol{r}}$, and therefore they are functions of $\boldsymbol{u} = \dot{\boldsymbol{r}}/\dot{t}$. In the free case, the Lagrangian is invariant under translations and thus independent of t and $\boldsymbol{r}$, and take the form in the Galilei case, as

$$T_0 = -\frac{1}{2}m\boldsymbol{u}^2 = -H_m, \quad \boldsymbol{R}_0 = m\boldsymbol{u} = \boldsymbol{P}_m$$

while in the Poincaré case, as we shall see in section 2.3, they are

$$T_0 = \frac{-mc^2}{\sqrt{1-u^2/c^2}} = -H_m, \quad \boldsymbol{R}_0 = \frac{m\boldsymbol{u}}{\sqrt{1-u^2/c^2}} = \boldsymbol{P}_m.$$

The free Lagrangian either relativistic or nonrelativistic, can be written as $\widetilde{L}_0 = T_0\dot{t} + \boldsymbol{R}_0 \cdot \dot{\boldsymbol{r}} = -H_m\dot{t} + \boldsymbol{P}_m \cdot \dot{\boldsymbol{r}}$. We have denoted all these magnitudes related to the free Lagrangian, which depend on the mass of the particle, with a subindex $_m$, to indicate that they are mechanical properties.

In the general case, if the particle is interacting with some external source, the dynamical equations are not invariant under translations, because if we translate the particle but not the external source the dynamics will be different. The general Lagrangian will be a function of t and $\boldsymbol{r}$, but the homogeneity of $\widetilde{L}$ in terms of $\dot{t}$ and $\dot{\boldsymbol{r}}$ will still hold, and also its difference with $\widetilde{L}_0$. We can define this difference of these two homogeneous functions as the interacting Lagrangian $\widetilde{L}_I = \widetilde{L} - \widetilde{L}_0$. This homogeneous structure of this function implies that $\widetilde{L}_I = A_0\dot{t} + \boldsymbol{A}\cdot\dot{\boldsymbol{r}}$, where $A_0 = \partial\widetilde{L}_I/\partial\dot{t}$, and $\boldsymbol{A} = \partial\widetilde{L}_I/\partial\dot{\boldsymbol{r}}$.

The functions A_0 and $\boldsymbol{A}$, which depend on the external source, will be in general, functions of the variables of the particle $t, \boldsymbol{r}, \boldsymbol{u}$. It is clear that these terms modify the above definitions of H and $\boldsymbol{P}$ of the free particle, and now $H = -\partial\widetilde{L}/\partial\dot{t} = H_m - A_0$ and $\boldsymbol{P} = \partial\widetilde{L}/\partial\dot{\boldsymbol{r}} = \boldsymbol{P}_m + \boldsymbol{A}$. The function $-A_0$ is the modification of the mechanical temporal momentum H_m, and $\boldsymbol{A}$ is the modification of the mechanical linear momentum $\boldsymbol{P}_m$, due to the external interaction. Also the other observables $\boldsymbol{K}$ and $\boldsymbol{J}$ are modified by the external source.

We are going to see that the dependence on $\boldsymbol{u}$, of the functions A_0 and $\boldsymbol{A}$, is unnecessary. Those fields, in general, will be functions of the spacetime variables and independent of the velocity. Let us consider the Galilei case. The dynamical equations from the Lagrangian

$$L = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 + A_0(t,\boldsymbol{r}) + \boldsymbol{A}(t,\boldsymbol{r})\cdot\boldsymbol{u},$$

are

$$\frac{\partial A_0}{\partial r_i} + u_j\frac{\partial A_j}{\partial r_i} - \frac{d}{dt}\left(mu_i + A_i\right) = 0, \quad i = 1,2,3$$

i.e.,

$$m\frac{d^2r_i}{dt^2} = \frac{\partial A_0}{\partial r_i} - \frac{\partial A_i}{\partial t} + u_j\left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)$$

where the last term in brackets, is an antisymmetric function in i and j, and thus it can be written as $\epsilon_{ijk}B_ku_j$, and therefore the time variation of the mechanical linear momentum of the point particle is

$$\frac{d\boldsymbol{P}_m}{dt} = m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}, \tag{2.20}$$

with

$$\boldsymbol{E} = \nabla A_0 - \frac{\partial\boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \nabla\times\boldsymbol{A},$$

is the Lorentz force associated to the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ which are functions only of t and $\boldsymbol{r}$. In the relativistic case we shall also obtain $d\boldsymbol{P}_m/dt = \boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}$, but the expression of $\boldsymbol{P}_m = \gamma(u)m\boldsymbol{u}$, is different, as we shall see.

In the case that A_0 and $\boldsymbol{A}$ are functions of $\boldsymbol{u}$, the dynamical equations are:

$$\frac{\partial A_0}{\partial r_i} + u_j\frac{\partial A_j}{\partial r_i} - \frac{d}{dt}\left(mu_i + \frac{\partial A_0}{\partial u_i} + A_i + u_j\frac{\partial A_j}{\partial u_i}\right) = 0.$$

But because of the homogeneity of $\widetilde{L}_I = A_0\dot{t} + A_j\dot{r}_j$, if we derivate both sides with respect to $\dot{r}_j$, we get:

$$A_j = \dot{t}\frac{\partial A_0}{\partial u_j}\frac{1}{\dot{t}} + \dot{r}_i\frac{\partial A_i}{\partial u_j}\frac{1}{\dot{t}} + A_j,$$

so that the additional term of the dynamical equations

$$\frac{\partial A_0}{\partial u_i} + u_j\frac{\partial A_j}{\partial u_i} = 0,$$

vanishes and does not take part in the dynamics, simmilarly as if A_0 and $\boldsymbol{A}$, were independent of $\boldsymbol{u}$, as we assumed before. The same argument can be used in the relativistic case, and therefore the most general force, defined as the time derivative of the linear momentum, is a Lorentz force with only spacetime fields.

For the time variation of the mechanical energy, only the force related to the field $\boldsymbol{E}$ produces work. In fact, in the nonrelativistic case,

$$H_m = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2, \quad \frac{dH_m}{dt} = m\frac{d\boldsymbol{r}}{dt}\cdot\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{u}\cdot\boldsymbol{E}.$$

In the relativistic case, $H_m = \gamma(u)mc^2$, $\boldsymbol{P}_m = \gamma(u)m\boldsymbol{u}$, but because it is an elementary particle, the atomic principle requires that the invariant expression which defines the mass by $H_m^2/c^2 - P_m^2 = m^2c^2$, does not change under the interaction. If we take the time derivative of this expression, we have:

$$\frac{2}{c^2}H_m\frac{dH_m}{dt} - 2\boldsymbol{P}_m\cdot\frac{d\boldsymbol{P}_m}{dt} = 0, \quad \frac{dH_m}{dt} = \boldsymbol{u}\cdot\frac{d\boldsymbol{P}_m}{dt} = \boldsymbol{u}\cdot\boldsymbol{E}.$$

In both cases, the time variation of the mechanical energy of the particle is the work done by the force $\boldsymbol{E}$ along the trajectory of the center of mass of the particle. Because the external fields are defined at the position $\boldsymbol{r}$, this point is also the location of the center of charge of the particle.

Since $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, satisfies $\nabla\cdot\boldsymbol{B} = 0$, we have a pseudovector field with no sources and of null divergence. If we take the curl of $\boldsymbol{E}$, because the curl of $\nabla\times(\nabla A_0)$, vanishes, these fields satisfy the following equations:

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \qquad \nabla\cdot\boldsymbol{B} = 0, \tag{2.21}$$

evaluated at least in the region where the particle is located, and they are part of Maxwell's equations of the electromagnetic field. They are vector fields and therefore we need to know, to completely define them, $\nabla\cdot\boldsymbol{E}$ and $\nabla\times\boldsymbol{B}$ and the corresponding boundary conditions. These extra equations relate the fields with the external sources. In the case of Maxwell's equations they are:

$$\nabla\cdot\boldsymbol{E} = \frac{1}{\epsilon_0}\rho, \quad \nabla\times\boldsymbol{B} = \frac{1}{\epsilon_0 c^2}\boldsymbol{j} + \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} \tag{2.22}$$

and they do not appear until we establish that part of the total Lagrangian which describes the sources which generate the interaction, i.e., the free Lagrangian of the external fields and how they interact with the particle.

In the case of the electromagnetic field ρ represents the electric charge density and $\boldsymbol{j}$ the vector current density. If we take the divergence of the second equation and using the first we arrive to:

$$\nabla\cdot\boldsymbol{j} + \frac{\partial\rho}{\partial t} = 0,$$

which is the fundamental conservation law of the electric charge.

For a point particle of charge e, localized at point $\boldsymbol{r}$ at time t these densities are $\rho = e\delta^{(3)}(\boldsymbol{r}-\boldsymbol{x})\delta(t-T)$ and $\boldsymbol{j} = e\delta^{(3)}(\boldsymbol{r}-\boldsymbol{x})\delta(t-T)\boldsymbol{u}$, where $\boldsymbol{x}$ is another point of space and T any other instant of time and $\delta(x-a)$ the usual Dirac's delta-function. Maxwell's equations (2.21) do not depend on the particle, while those of (2.22) show how the presence of the particle, and therefore the charge and current associated to it modifies locally the fields in the surrounding area. We have to remark that what appear here are spacetime derivatives of the fields with respect to the kinematical variables of the particle, and thus they refer to how these fields, generated by some external sources, are changing in the neibourhood of the particle. The conservation law of the electric charge shows the existence of a scalar property linked to the particle, which is carried by the particle along its trajectory, This enhances the interpretation that the point $\boldsymbol{r}$ is the support or localization of the charge e.

This formalism does not guarantee that the fields A_0 and $\boldsymbol{A}$, or their derived vector fields $\boldsymbol{E}$ and $\boldsymbol{B}$, satisfy all Maxwell's equations, but that the interaction is invariant under the transformation (2.23), as we shall see in a minute. It seems to indicate that the possible interaction of a point particle can undergone is through a Lorentz type force, in terms of the vector fields $\boldsymbol{E}$ and $\boldsymbol{B}$ without any restriction on its scope and range.

Gravity, as a possible interaction, is left aside by the definition of the Restricted Relativity Principle. In this way, without further restrictions, it is not possible to determine classically the other short range interactions like the weak and strong interactions, which are confined to regions of order of 10^{-15} to 10^{-18} m, around the particles where the quantum phenomena are relevant. These other interactions are described usually in a quantum context, through a local gauge invariance hypothesis and they are not predicted in a classical formalism.

The fields A_0 and $\boldsymbol{A}$, are not uniquelly determined, because what appears in the dynamical equations are their spacetime derivatives. If we modify them in the form

$$A_0 \to A_0 + \frac{\partial\Lambda(t,\boldsymbol{r})}{\partial t}, \quad A_i \to A_i + \frac{\partial\Lambda(t,\boldsymbol{r})}{\partial r_i}, \tag{2.23}$$

where $\Lambda(t,\boldsymbol{r})$ is an arbitrary function of the kinematical variables, the Lagrangian $\widetilde{L}_I$ is modified in the form

$$\frac{\partial\Lambda(t,\boldsymbol{r})}{\partial t}\dot{t} + \frac{\partial\Lambda(t,\boldsymbol{r})}{\partial r_i}\dot{r}_i = \frac{d\Lambda}{d\tau},$$

which is a total derivative and can be deleted because do not modify the dynamical equations. The transformation (2.23), which leaves invariant the dynamical equations, while modifying the external fields at any point of spacetime, is called a **local gauge transformation**.

It seems that if we have a transformation that leaves invariant the dynamical equations we can obtain some conservation law by using Noether's theorem. But this transformation is not related to any one-parameter group of transformations but it is a general transformation generated by an arbitrary function Λ, which transforms the Lagrangian with the addition of a total derivative.

2.2 Galilei free spinning particle

The most general nonrelativistic particle [1] is the system whose kinematical space X is the largest homogeneous space of the Galilei group $\mathcal{G}$, i.e., the Galilei group itself. We shall describe the state of the elementary particle at any instant τ, by the knowledge of the time $t(\tau)$, the position of a point $\boldsymbol{r}(\tau)$, the velocity of this point $\boldsymbol{u}(\tau) = d\boldsymbol{r}/dt$ and the orientation of a Cartesian frame of unit vectors $\boldsymbol{e}_i(\tau)$, $i = 1,2,3$, linked to that point. These nine components

[1] M. Rivas, Classical Particle Systems: I. Galilei free particles, *J. Phys.* **A 18**, 1971 (1985).

$(\boldsymbol{e}_i)_j$ can be expressed in terms of three essential parameters $\boldsymbol{\rho}(\tau)$, as we can see in the appendix **2.8** about a parameterization of rotations, which are given by:

$$(\boldsymbol{e}_i)_j = R(\boldsymbol{\rho})_{ji} = \frac{1}{1+\rho^2}[(1-\rho^2)\,\delta_{ji} + 2\rho_j\rho_i + 2\varepsilon_{ikj}\rho_k] \tag{2.24}$$

This selection of the orientation variables in any inertial reference frame is completely arbitrary, because these unit vectors have no physical reality. This means that the Lagrangian cannot be an explicit function of them, since any other arbitary selection would produce the same value of the action. But the important feature is that in the dynamical evolution the orientation changes, the particle rotates, and therefore the Lagrangian is going to be an explicit function of the angular velocity of the particle, and this angular velocity is independent of the initial selection of the unit vectors. This means that any observer who changes at any time the orientation unit vectors, does not modify the value of the angular velocity in that frame, as we shall see below.

In addition to the kinematical group as a symmetry group we shall have another symmetry group, the group of rotations of the local frame associated to the particle. We shall call it the **local rotation group** and we shall denote by $SO(3)_L$. It commutes with the whole Galilei group and therefore the spacetime symmetry group is at least $\mathcal{G} \otimes SO(3)_L$. The result is that the Lagrangian $\widetilde{L}$ has to be a function of the orientation variables ρ and $\dot{\rho}$ through its dependence of the angular velocity $\boldsymbol{\omega}$, which is expressed in terms of ρ and $\dot{\rho}$ in the form (2.35), as we shall see.

Then the kinematical variables are the ten real variables $x(\tau) \equiv (t(\tau), \boldsymbol{r}(\tau), \boldsymbol{u}(\tau), \boldsymbol{\rho}(\tau))$ with domains $t \in \mathbb{R}$, $\boldsymbol{r} \in \mathbb{R}^3$, $\boldsymbol{u} \in \mathbb{R}^3$ and $\boldsymbol{\rho} \in \mathbb{R}^3_c$ similarly as the corresponding group parameters. The relationship between the values $x'(\tau)$ and $x(\tau)$ they take at any instant τ for two arbitrary inertial observers, and in the passive representation of rotations, is given by:

$$t'(\tau) = t(\tau) + b, \tag{2.25}$$

$$\boldsymbol{r}'(\tau) = R(\boldsymbol{\mu})\boldsymbol{r}(\tau) + \boldsymbol{v}t(\tau) + \boldsymbol{a}, \tag{2.26}$$

$$\boldsymbol{u}'(\tau) = R(\boldsymbol{\mu})\boldsymbol{u}(\tau) + \boldsymbol{v}, \tag{2.27}$$

$$\boldsymbol{\rho}'(\tau) = \frac{\boldsymbol{\mu} + \boldsymbol{\rho}(\tau) - \boldsymbol{\mu} \times \boldsymbol{\rho}(\tau)}{1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}(\tau)}. \tag{2.28}$$

Among these kinematical variables there exist the differential constraints $\boldsymbol{u}(\tau) = \dot{\boldsymbol{r}}(\tau)/\dot{t}(\tau)$, that together with the homogeneity condition of the Lagrangian $\widetilde{L}$ in terms of the derivatives of the kinematical variables:

$$\widetilde{L}(x, \dot{x}) = (\partial \widetilde{L}/\partial \dot{x}_i)\dot{x}_i, \tag{2.29}$$

reduce from ten to six the essential degrees of freedom of the system.

These degrees of freedom are the position $\boldsymbol{r}(t)$ and the orientation $\boldsymbol{\rho}(t)$. The Lagrangian depends on the second derivative of $\boldsymbol{r}(t)$ and the first derivative of $\boldsymbol{\rho}(t)$. Expression (2.29) is explicitly given by:

$$\widetilde{L} = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{U} \cdot \dot{\boldsymbol{u}} + \boldsymbol{V} \cdot \dot{\boldsymbol{\rho}}, \tag{2.30}$$

where the functions $T = \partial \widetilde{L}/\partial \dot{t}$, $R_i = \partial \widetilde{L}/\partial \dot{r}^i$, $U_i = \partial \widetilde{L}/\partial \dot{u}^i$, $V_i = \partial \widetilde{L}/\partial \dot{\rho}^i$, will be in general functions of the ten kinematical variables $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho})$ and homogeneous functions of zero degree in terms of the derivatives $(\dot{t}, \dot{\boldsymbol{r}}, \dot{\boldsymbol{u}}, \dot{\boldsymbol{\rho}})$.

The generalized coordinates are $\boldsymbol{r}$, $\boldsymbol{u}$ and $\boldsymbol{\rho}$, and their canonical cojugate momenta are:

$$\boldsymbol{p}_r = \frac{\partial L}{\partial(d\boldsymbol{r}/dt)} - \frac{d}{dt}\left(\frac{\partial L}{\partial(d^2\boldsymbol{r}/dt^2)}\right) = \frac{\partial \widetilde{L}}{\partial \dot{\boldsymbol{r}}} - \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{\boldsymbol{u}}}\right) = \boldsymbol{R} - \frac{d\boldsymbol{U}}{dt},$$

$$\boldsymbol{p}_u = \frac{\partial L}{\partial(d\boldsymbol{u}/dt)} = \frac{\partial \widetilde{L}}{\partial \dot{\boldsymbol{u}}} = \boldsymbol{U},$$

$$\boldsymbol{p}_\rho = \frac{\partial L}{\partial(d\boldsymbol{\rho}/dt)} = \frac{\partial \widetilde{L}}{\partial \dot{\boldsymbol{\rho}}} = \boldsymbol{V}.$$

As canonical conjugate variables, $\boldsymbol{p}_r$ is the conjugate momentum of $\boldsymbol{r}$, $\boldsymbol{p}_u$ is that of $\boldsymbol{u}$ anf $\boldsymbol{p}_\rho$ is the conjugate momentum of the orientation variables $\boldsymbol{\rho}$.

By assuming that the evolution parameter τ is group invariant, the derivatives of the kinematical variables transform under $\mathcal{G}$:

$$\begin{aligned}
\dot{t}'(\tau) &= \dot{t}(\tau), && (2.31)\\
\dot{\boldsymbol{r}}'(\tau) &= R(\boldsymbol{\mu})\dot{\boldsymbol{r}}(\tau) + \boldsymbol{v}\dot{t}(\tau), && (2.32)\\
\dot{\boldsymbol{u}}'(\tau) &= R(\boldsymbol{\mu})\dot{\boldsymbol{u}}(\tau), && (2.33)\\
\dot{\boldsymbol{\rho}}'(\tau) &= \frac{(\dot{\boldsymbol{\rho}}(\tau) + \boldsymbol{\mu}\times\dot{\boldsymbol{\rho}}(\tau))(1-\boldsymbol{\mu}\cdot\boldsymbol{\rho}(\tau))}{(1-\boldsymbol{\mu}\cdot\boldsymbol{\rho}(\tau))^2} + \\
&\quad \frac{\boldsymbol{\mu}\cdot\dot{\boldsymbol{\rho}}(\tau)(\boldsymbol{\mu}+\boldsymbol{\rho}(\tau)+\boldsymbol{\mu}\times\boldsymbol{\rho}(\tau))}{(1-\boldsymbol{\mu}\cdot\boldsymbol{\rho}(\tau))^2}. && (2.34)
\end{aligned}$$

Instead of the derivative $\dot{\boldsymbol{\rho}}(\tau)$, which transforms in a complicated way, we can define the angular velocity of the particle $\boldsymbol{\omega}$ as a linear function of it in the passive representation, in the form

$$\boldsymbol{\omega} = \frac{2}{1+\boldsymbol{\rho}^2}(-\dot{\boldsymbol{\rho}} + \boldsymbol{\rho}\times\dot{\boldsymbol{\rho}}). \tag{2.35}$$

It is a linear function of $\dot{\boldsymbol{\rho}}$, and transforms as:

$$\boldsymbol{\omega}'(\tau) = R(\boldsymbol{\mu})\boldsymbol{\omega}(\tau). \tag{2.36}$$

Two inertial observers measure the same absolute value of the angular velocity. The inverse transformation of (2.35) is

$$\dot{\boldsymbol{\rho}} = \frac{1}{2}\left(-\boldsymbol{\omega} - \boldsymbol{\rho}\times\boldsymbol{\omega} + \boldsymbol{\rho}(\boldsymbol{\rho}\cdot\boldsymbol{\omega})\right), \tag{2.37}$$

We interpret the rotation matrix $R(\boldsymbol{\rho})$ as the rotation that carries the initial frame linked to the body at instant $\tau = 0$ to the frame at instant τ, as in a rigid body. Then, the three columns of matrix $R(\boldsymbol{\rho})$ represent the Cartesian components of the three unit vectors linked to the body when chosen parallel to the laboratory frame at instant $\tau = 0$.

Expression of the angular velocity.
If at instant $\tau = 0$ we have the orientation axes $\boldsymbol{e}_i(0)$, which define by columns the rotation matrix $R(\boldsymbol{\rho}(0))$, at any instant τ they will be

$$((\boldsymbol{e}_1(\tau))(\boldsymbol{e}_2(\tau))(\boldsymbol{e}_3(\tau))) = R(\boldsymbol{\rho}(\tau))R(\boldsymbol{\rho}(0))$$

where $R(\boldsymbol{\rho}(\tau))$ is the global rotation experienced by the particle, and the change per unit time τ

$$((\dot{\boldsymbol{e}}_1(\tau))(\dot{\boldsymbol{e}}_2(\tau))(\dot{\boldsymbol{e}}_3(\tau))) = \dot{R}(\boldsymbol{\rho}(\tau))R(\boldsymbol{\rho}(0)) = \dot{R}(\boldsymbol{\rho}(\tau))R^{-1}(\boldsymbol{\rho}(\tau))((\boldsymbol{e}_1(\tau))(\boldsymbol{e}_2(\tau))(\boldsymbol{e}_3(\tau)))$$

and thus the velocity of any axis, considered as a vector column, is the action on the vector, at the instant τ, of the matrix

$$\frac{d\boldsymbol{e}_i}{d\tau} = \dot{R}(\boldsymbol{\rho}(\tau))R^{-1}(\boldsymbol{\rho}(\tau))\boldsymbol{e}_i(\tau) = \Omega\boldsymbol{e}_i(\tau),$$

where $\Omega = \dot{R}R^{-1} = \dot{R}R^T$ is an antisymmetric matrix. In fact, at any instant τ any rotation matrix satisfies, $R(\boldsymbol{\rho}(\tau))R^T(\boldsymbol{\rho}(\tau)) = \mathbb{I}$, where the superindex T means the transpose matrix, and $\mathbb{I}$ is the 3×3 unit matrix. If we take the τ-derivative of this expression, $\dot{R}R^T + R\dot{R}^T = \Omega + \Omega^T = 0$, and thus the three essential components of the antisymmetric matrix Ω define a three-vector $\boldsymbol{\omega}$

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix},$$

such that we can write the dynamics of any unit vector as

$$\frac{d\boldsymbol{e}_i}{d\tau} = \boldsymbol{\omega} \times \boldsymbol{e}_i. \tag{2.38}$$

and $\boldsymbol{\omega}$ is interpreted as the angular velocity of rotation of the local frame associated to the particle. The components of $\boldsymbol{\omega}$, expressed as functions of the variables $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ are given in (2.35).

If any inertial observer changes the matrix of orientation $R(\tau)$, made of the three unit vectors, at the instant τ, by any other matrix $R'(\tau) = R(\tau)M$, where M is any orthogonal matrix, then $R'^T(\tau) = M^T R^T(\tau)$, and for this observer $\Omega' = \dot{R}'R'^T = \dot{R}MM^T R^T = \dot{R}R^T = \Omega$, and thus any selection of the orientation produces the same expression of the angular velocity in the corresponding reference frame. This justifies that the Lagrangian does not depend explicitely on the variables $\boldsymbol{\rho}$, and depends only on them through its dependence of the angular velocity.

Expression (2.28) corresponds to $R(\boldsymbol{\rho}'(\tau)) = R(\boldsymbol{\mu})R(\boldsymbol{\rho}(\tau))$. Therefore

$$\begin{aligned} \Omega' &= \dot{R}(\boldsymbol{\rho}'(\tau))R^T(\boldsymbol{\rho}'(\tau)) = R(\boldsymbol{\mu})\dot{R}(\boldsymbol{\rho}(\tau))R^T(\boldsymbol{\rho}(\tau))R^T(\boldsymbol{\mu}) \\ &= R(\boldsymbol{\mu})\Omega R^{-1}(\boldsymbol{\mu}), \end{aligned}$$

and this leads to the equation (2.36) in terms of the essential components $\boldsymbol{\omega}$ of the antisymmetric matrix Ω.

In this way the last part of the Lagrangian $(\partial\widetilde{L}/\partial\dot{\rho}^i)\dot{\rho}^i$ can be writen as

$$\boldsymbol{V} \cdot \dot{\boldsymbol{\rho}} \equiv \frac{\partial\widetilde{L}}{\partial\dot{\rho}^i}\,\dot{\rho}^i = \frac{\partial\widetilde{L}}{\partial\omega^j}\,\frac{\partial\omega^j}{\partial\dot{\rho}^i}\,\dot{\rho}^i = \boldsymbol{W} \cdot \boldsymbol{\omega}, \tag{2.39}$$

due to the linearity of $\boldsymbol{\omega}$ in terms of $\dot{\boldsymbol{\rho}}$ and where $W_i = \partial\widetilde{L}/\partial\omega^i$. Thus the most general form of the Lagrangian of a nonrelativistic particle can also be written instead of (2.30) as:

$$\widetilde{L} = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{U} \cdot \dot{\boldsymbol{u}} + \boldsymbol{W} \cdot \boldsymbol{\omega} \tag{2.40}$$

and where the functions T, $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{W}$ are unknown functions of the variables $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{a}, \boldsymbol{\Omega})$, to be determined. Here $\boldsymbol{a} = \dot{\boldsymbol{u}}/\dot{t} = d\boldsymbol{u}/dt$, is the acceleration of the point (please do not confuse with the space translation parameter), and the variable $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$ is the angular velocity in the time evolution description. But these observables are not explicit functions of the orientation variables $\boldsymbol{\rho}$. All these features are independent of whether the particle is free or it is under some interaction.

Because we are ussing an arbitrary evolution parameter τ, the same for all inertial observers, and we can take it as dimensionless, the Lagrangian $\widetilde{L}$ has dimensions of action, and therefore each one of the terms of its expansion (2.40) also has dimension of action. This means that every one of the unknown functions $F_i = \partial\widetilde{L}/\partial\dot{x}_i$ has dimensions of action divided by the dimension of the corresponding kinematical variable x_i, because if τ is dimensionless x_i and $\dot{x}_i$, have the same dimension. Therefore, T will have dimension of (action/time)=energy, $\boldsymbol{R}$ dimension of (action/length)=mass×velocity, i.e., linear momentum, $\boldsymbol{U}$ that of (action/velocity)=mass×distance or static momentum, and finally $\boldsymbol{W}$ dimension of action or angular momentum, because $\boldsymbol{\omega}$ is dimensionless.

2.2.1 Free spinning particle

Since X is the whole Galilei group $\mathcal{G}$ the most general gauge function is just the group exponent:

$$\alpha(g; x) = \xi(g, h_x) = m(\boldsymbol{v}^2 t(\tau)/2 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r}(\tau)), \tag{2.41}$$

similar to (2.7), and this allows us to interpret the parameter m as the mass of the system. Under the action of an arbitrary element of the Galilei group, the Lagrangian $\widetilde{L}$ transforms according to:

$$\widetilde{L}(gx(\tau), d(gx(\tau))/d\tau) = \widetilde{L}(x(\tau), \dot{x}(\tau)) + d\alpha(g; x(\tau))/d\tau. \tag{2.42}$$

This leads through some straightforward calculations, similar to the ones performed in (2.9)-(2.11), to the following form of transformation of the functions:

$$\begin{aligned} T'(\tau) &= T(\tau) - \boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{R}(\tau) - m\boldsymbol{v}^2/2, && (2.43)\\ \boldsymbol{R}'(\tau) &= R(\boldsymbol{\mu})\boldsymbol{R}(\tau) + m\boldsymbol{v}, && (2.44)\\ \boldsymbol{U}'(\tau) &= R(\boldsymbol{\mu})\boldsymbol{U}(\tau), && (2.45)\\ \boldsymbol{W}'(\tau) &= R(\boldsymbol{\mu})\boldsymbol{W}(\tau). && (2.46) \end{aligned}$$

2.2.2 Noether constants of the motion

Using the action of the Galilei group on the kinematical space given by (2.25)-(2.28), Noether's theorem defines the following constants of the motion for the free particle:

a) Under time translation the action function is invariant, $\lambda(x) = 0$, and as usual we call the corresponding conserved quantity, the **total temporal momentum** of the particle H. Since $\delta t = \delta b$ and $\delta q_i^{(s)} = 0$, $M_0 = 1$ and $M_i^{(s)} = 0$, by applying (1.37) we have:

$$H = -(L - p^i_{(s)}q_i^{(s)})M_0 = -(\widetilde{L}/\dot{t} - p^i_{(s)}q_i^{(s)}) = -T - \boldsymbol{R}\cdot\boldsymbol{u} - \boldsymbol{U}\cdot\dot{\boldsymbol{u}}/\dot{t} - \boldsymbol{W}\cdot\boldsymbol{\omega}/\dot{t}$$

$$+(\boldsymbol{R} - d\boldsymbol{U}/dt)\cdot\boldsymbol{u} + \boldsymbol{U}\cdot\dot{\boldsymbol{u}}/\dot{t} + \boldsymbol{V}\cdot\dot{\boldsymbol{\rho}}/\dot{t},$$

and since $\boldsymbol{W}\cdot\boldsymbol{\omega} = \boldsymbol{V}\cdot\dot{\boldsymbol{\rho}}$, it turns out that

$$H = -T - \frac{d\boldsymbol{U}}{dt}\cdot\boldsymbol{u}. \qquad (2.47)$$

b) Under spatial translations, $A(x_1, x_2)$ is invariant, $\lambda_i(x) = 0$, and this defines the **total linear momentum** of the system. We have now:

$$\delta t = 0,\ M_0 = 0,\ \delta r_i = \delta a_j,\ M_{ij}^{(0)} = \delta_{ij},\ \delta u_i = 0,\ M_{ij}^{(1)} = 0,$$

$$\delta\rho_i = 0,\ M_{ij}^{(\rho)} = 0,$$

and then

$$\boldsymbol{P} = \boldsymbol{R} - \frac{d\boldsymbol{U}}{dt} = \boldsymbol{p}_r. \qquad (2.48)$$

c) Under a pure Galilei transformation of velocity $\delta\boldsymbol{v}$, $A(x_1, x_2)$ is no longer invariant but taking into account (1.13) and the gauge function (2.41), it transforms as $\delta A = m\boldsymbol{r}_2\cdot\delta\boldsymbol{v} - m\boldsymbol{r}_1\cdot\delta\boldsymbol{v}$ and thus, $\lambda_i(x) = mr_i$, and this defines the **total kinematical momentum** $\boldsymbol{K}$, in the following way:

$$\delta t = 0,\ M_0 = 0,\ \delta r_i = \delta v_i t,\ M_{ij}^{(0)} = \delta_{ij}t,\ \delta u_i = \delta v_i,\ M_{ij}^{(1)} = \delta_{ij},$$

$$\delta\rho_i = 0,\ M_{ij}^{(\rho)} = 0,$$

and thus

$$\boldsymbol{K} = m\boldsymbol{r} - \boldsymbol{P}\,t - \boldsymbol{U}. \qquad (2.49)$$

From $\dot{\boldsymbol{K}} = 0$, this leads to $\boldsymbol{P} = m\boldsymbol{u} - d\boldsymbol{U}/dt$, and thus by identification with (2.48), the function $\boldsymbol{R} = m\boldsymbol{u}$ irrespective of the particular Lagrangian. The total linear momentum does not lie along the velocity of the point $\boldsymbol{r}$.

d) Finally, under rotations $A(x_1, x_2)$ remains invariant, $B_i(x) = 0$, and the corresponding constant of the motion, the **total angular momentum** of the system, with respect to the

origin of observer's frame, comes from the infinitesimal transformation of value $\delta\mu_i = \delta\alpha_i/2$, *i.e.*, half of the rotated infinitesimal angle, and then

$$\delta t = 0,\ M_{0i} = 0, \delta r_i = \epsilon_{ikj}\delta\alpha_j r_k,\ M^{(r)}_{ij} = \epsilon_{ikj} r_k,$$

$$\delta u_i = \epsilon_{ikj}\delta\alpha_j u_k, \quad M^{(u)}_{ij} = \epsilon_{ikj} u_k,$$

$$\delta\rho_i = \delta\alpha^j(\delta_{ij} + \epsilon_{ikj}\rho^k + \rho_i\rho_j)/2, \quad M^{(\rho)}_{ij} = (\delta_{ij} + \epsilon_{ikj}\rho^k + \rho_i\rho_j)/2.$$

Because the constant of the motion is

$$J_j = -p_{ri}M^{(r)}_{ij} - p_{ui}M^{(u)}_{ij} - p_{\rho i}M^{(\rho)}_{ij}$$

which leads to

$$-P_i\epsilon_{ikj}r_k = \epsilon_{jki}r_k P_i, \qquad -U_i\epsilon_{ikj}u_k = \epsilon_{jki}u_k U_i,$$

from (2.35)

$$\frac{\partial\omega_k}{\partial\dot\rho_i} = \frac{2}{1+\rho^2}(-\delta_{ki} + \epsilon_{kli}\rho_l)$$

$$-V_i M^{(\rho)}_{ij} = -\frac{\partial\widetilde{L}_0}{\partial\omega_k}\frac{\partial\omega_k}{\partial\dot\rho_i}M^{(\rho)}_{ij} = W_j,$$

because

$$\frac{\partial\omega_k}{\partial\dot\rho_i}M^{(\rho)}_{ij} = -\delta_{kj},$$

and therefore, in vector notation

$$\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{u}\times\boldsymbol{U} + \boldsymbol{W} = \boldsymbol{L} + \boldsymbol{Z} + \boldsymbol{W} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{S}, \tag{2.50}$$

where all terms have dimension of action or angular momentum. $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{P}$ represents the orbital angular momentum, $\boldsymbol{Z} = \boldsymbol{u}\times\boldsymbol{U}$ is the angular momentum associated to the dependence of the Lagrangian on the acceleration and, as we shall see, comes from the relative orbital motion of the center of charge around the center of mass (or *Zitterbewegung*) and $\boldsymbol{W}$ comes from the dependence of the Lagrangian on the angular velocity and we interpret as the rotative part of the angular momentum.

Since $\boldsymbol{J}$ represents the angular momentum of the particle with respect to the origin of the reference frame, $\boldsymbol{S}$ represents the angular momentum of the particle with respect to the point $\boldsymbol{r}$. Because $d\boldsymbol{J}/dt = 0$, the function $\boldsymbol{S}$ satisfies $d\boldsymbol{S}/dt = \boldsymbol{P}\times\boldsymbol{u}$ and it is not a constant of the motion, even for a free particle. It is the classical angular momentum which satisfies the same dynamical equation as Dirac's spin operator in the quantum case.

e) We have mentioned at the beginning of this section, that in addition to the invariance of dynamical equations under the Galilei group, we also have the invariance of the Lagrangian under the local rotation group $SO(3)_L$. This group only transforms the kinematical orientation variables leaving the rest untouched. The kinematical variables transform under this group:

$$t' = t, \quad \boldsymbol{r}' = \boldsymbol{r}, \quad \boldsymbol{u}' = \boldsymbol{u}, \quad R(\boldsymbol{\rho}') = R(\boldsymbol{\rho})M(\boldsymbol{\alpha}), \quad \forall M(\boldsymbol{\alpha}) \in SO(3)_L,$$

The transformation of $\boldsymbol{\rho}$ variables, in the infinitesimal case is

$$\boldsymbol{\rho}' = \frac{\boldsymbol{\rho} + \delta\boldsymbol{\alpha}/2 - \boldsymbol{\rho}\times\delta\boldsymbol{\alpha}/2}{(1 - \boldsymbol{\rho}\cdot\delta\boldsymbol{\alpha}/2)}, \quad \delta\rho_i = \delta\alpha_j\frac{1}{2}\left(\delta_{ij} + \rho_i\rho_j - \epsilon_{ilj}\rho_l\right) = M^{(L)}_{ij}\delta\alpha_j.$$

The conserved magnitudes come from the momenta $\boldsymbol{p}_\rho = \boldsymbol{V}$, and they are:

$$T_j = -V_i M^{(L)}_{ij} = -\frac{\partial \widetilde{L}}{\partial \omega_k}\frac{\partial \omega_k}{\partial \dot{\rho}_i} M^{(L)}_{ij},$$

but

$$\frac{\partial \omega_k}{\partial \dot{\rho}_i} M^{(L)}_{ij} = \frac{-1}{1+\rho^2}\left((1-\rho^2)\delta_{kj} + 2\rho_k\rho_j + 2\epsilon_{kjs}\rho_s\right)$$

and this term is in fact the k-component, of opposite sign, of the unit vector $\boldsymbol{e}_j$,i.e., $(\boldsymbol{e}_j)_k$, given in (2.24), and thus these constants of the motion are

$$T_j = -W_k(-\boldsymbol{e}_j)_k = \boldsymbol{W}\cdot\boldsymbol{e}_j, \tag{2.51}$$

the **projection, on the particle unit vectors, of the angular momentum $\boldsymbol{W}$ associated to the rotation**.

From a different point of view, the conservation of the linear momentum $\boldsymbol{P}$ comes from the invariance of L under translations and thus because it is independent of the position variables $\boldsymbol{r}$. Then from the dynamical equations with respect to these degrees of freedom, we can obtain:

$$\frac{\partial L}{\partial r_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial (dr_i/dt)}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial (d^2r_i/dt^2)}\right) = 0, \quad \frac{d}{dt}\left[\frac{\partial L}{\partial (dr_i/dt)} - \frac{d}{dt}\left(\frac{\partial L}{\partial (d^2r_i/dt^2)}\right)\right] = 0,$$

since $\partial L/\partial r_i = 0$, and we get again (2.48).

The conservation of the projections T_i can be obtained from the dynamical equationes related to the orientation degrees of freedom. Since $\widetilde{L}$ depends on $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ through its dependence on the angular velocity $\boldsymbol{\omega}$, these dynamical equations can be rewritten as

$$\frac{\partial \widetilde{L}}{\partial \rho_i} - \frac{d}{d\tau}\left(\frac{\partial \widetilde{L}}{\partial \dot{\rho}_i}\right) = 0, \quad \frac{\partial \widetilde{L}}{\partial \omega_j}\frac{\partial \omega_j}{\partial \rho_i} - \frac{d}{d\tau}\left(\frac{\partial \widetilde{L}}{\partial \omega_j}\frac{\partial \omega_j}{\partial \dot{\rho}_i}\right) = 0, \quad \frac{\partial \widetilde{L}}{\partial \omega_j} = W_j,$$

and they lead to

$$\frac{d\boldsymbol{W}}{d\tau} = \boldsymbol{\omega}\times\boldsymbol{W}. \tag{2.52}$$

For the dynamics of the unit vector $\boldsymbol{e}_i$, we have seen in (2.38) that

$$\frac{d\boldsymbol{e}_i}{d\tau} = \boldsymbol{\omega}\times\boldsymbol{e}_i,$$

and therefore for $T_i = \boldsymbol{W}\cdot\boldsymbol{e}_i$, taking the derivative with respect to τ,

$$\frac{dT_i}{d\tau} = (\boldsymbol{\omega}\times\boldsymbol{W})\cdot\boldsymbol{e}_i + \boldsymbol{W}\cdot(\boldsymbol{\omega}\times\boldsymbol{e}_i) = 0.$$

We shall see the importance of these conserved components of the spin in the quantum case, to classify the states of the electron.

Exercise: Show that if a Lagrangian depends on the orientation variables $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ in terms of the angular velocity $\boldsymbol{\omega}(\boldsymbol{\rho},\dot{\boldsymbol{\rho}})$, through (2.35), then the dynamical equations related to the orientation degrees of freedom,

$$\frac{\partial \widetilde{L}}{\partial \rho_i} - \frac{d}{d\tau}\left(\frac{\partial \widetilde{L}}{\partial \dot{\rho}_i}\right) = 0,$$

can be transformed into

$$d\boldsymbol{W}/d\tau = \boldsymbol{\omega}\times\boldsymbol{W}, \qquad \text{where} \quad W_i = \frac{\partial \widetilde{L}}{\partial \omega_i}.$$

2.2.3 Spin with respect to the center of mass

We can also consider the spin for a free particle with respect to its center of mass, once we accurately identify the center of mass of the particle.

The center of mass observer is defined as that inertial observer for whom $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$. These six conditions do not define uniquely an inertial observer but rather a class of them up to a rotation and an arbitrary time translation. In fact, the condition $\boldsymbol{P} = 0$ establishes the class of observers for which the center of mass is at rest, and $\boldsymbol{K} = 0$ is the additional condition to locate it at the origin of coordinates, at least for the point particle. We are going to see that the same happens for the general spinning particle.

This comes from the analysis of (2.49), where $\boldsymbol{k} = \boldsymbol{U}/m$ is an observable with dimensions of length, and taking the derivative with respect to τ of both sides, taking into account that $\dot{\boldsymbol{P}} = 0$, we have:

$$\dot{\boldsymbol{K}} = 0 = m\dot{\boldsymbol{r}} - \boldsymbol{P}\,\dot{t} - m\dot{\boldsymbol{k}}, \qquad i.e., \qquad \boldsymbol{P} = m\frac{d(\boldsymbol{r}-\boldsymbol{k})}{dt}. \tag{2.53}$$

Then the point $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k}$ is moving at constant speed and we say that it represents the position of the center of mass of the system. Thus, the observable $\boldsymbol{k} = \boldsymbol{r} - \boldsymbol{q}$ is just the relative position of point $\boldsymbol{r}$ with respect to the center of mass, which is defined as

$$\boldsymbol{q} = \boldsymbol{r} - \frac{1}{m}\boldsymbol{U}. \tag{2.54}$$

Therefore $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$ give rise to $d\boldsymbol{q}/dt = 0$, and $\boldsymbol{r} = \boldsymbol{k}$, *i.e.*, $\boldsymbol{q} = 0$, as we pointed out. With this definition, the kinematical momentum can be written as $\boldsymbol{K} = m\boldsymbol{q} - \boldsymbol{P}t$, in terms of the center of mass position $\boldsymbol{q}$ and the total linear momentum $\boldsymbol{P}$.

The spin of the system, with respect to the center of mass, is defined as the difference between the total angular momentum $\boldsymbol{J}$ and the orbital angular momentum of the center of mass motion $\boldsymbol{q} \times \boldsymbol{P}$, and thus

$$\boldsymbol{S}_{CM} = \boldsymbol{J} - \boldsymbol{q} \times \boldsymbol{P} = \boldsymbol{J} - \frac{1}{m}\boldsymbol{K} \times \boldsymbol{P} = \boldsymbol{S} + \boldsymbol{k} \times \boldsymbol{P} = -m\boldsymbol{k} \times \frac{d\boldsymbol{k}}{dt} + \boldsymbol{W}. \tag{2.55}$$

We see that can also be written as the angular momentum $\boldsymbol{S}$ with respecto to the point $\boldsymbol{r}$, plus the orbital angular momentum of this point $\boldsymbol{k} \times \boldsymbol{P}$ with respecto to the center of mass. The spin $\boldsymbol{S}_{CM}$, is expressed in terms of the constants of the motion $\boldsymbol{J}$, $\boldsymbol{K}$ and $\boldsymbol{P}$, and is also a constant of the motion. Alternatively we can describe the spin with respect to the center of mass $\boldsymbol{S}_{CM}$, according to the last expression in terms of the rotational part $\boldsymbol{W}$ and the term $-\boldsymbol{k} \times m d\boldsymbol{k}/dt$ which suggests a contribution of (anti)orbital type coming from the motion of point $\boldsymbol{r}$ around the center of mass. It is related to the zitterbewegung or more precisely to the function $\boldsymbol{U} = m\boldsymbol{k}$ which reflects the dependence of the Lagrangian on the acceleration. The other term $\boldsymbol{W}$ comes from the dependence on the other three degrees of freedom ρ_i, and thus on the angular velocity. This zitterbewegung is the motion of the center of charge around the center of mass. Point $\boldsymbol{r}$, as representing the position of the center of charge, has been also suggested in previous works for the relativistic electron. [2]

Because $\dot{\boldsymbol{J}} = 0$, and that $d\boldsymbol{W}/d\tau = \boldsymbol{\omega} \times \boldsymbol{W}$ and the expression of $\boldsymbol{P}$, (2.48), this implies the general relation for a free particle

$$\dot{\boldsymbol{r}} \times \boldsymbol{R} + \dot{\boldsymbol{u}} \times \boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{W} = 0, \tag{2.56}$$

which is also valid in the relativistic case and which reflects the fact that velocity, acceleration and angular velocity are not independent magnitudes. In a certain sense, we can take as the local

[2] A.O. Barut and A.J. Bracken, *Phys. Rev. D* **23**, 2454 (1981).

frame linked to point $\boldsymbol{r}$, the Frenet-Serret triad. From the derivatives $\dot{\boldsymbol{r}}$ y $\ddot{\boldsymbol{r}}$, we can determine the tangent and normal vector, and their cross product defines the binormal, and therefore the derivative of this triad will also produce the angular velocity which will be a function of the other derivatives.

In the nonrelativistic case $\boldsymbol{R}$ and $\dot{\boldsymbol{r}}$ have the same direction, the above relation reduces to

$$\dot{\boldsymbol{u}} \times \boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{W} = 0. \tag{2.57}$$

2.2.4 Spin dynamics

Since the angular momentum is an observable defined with respect to a definite point, and the elementary particle has two characteristic points $\boldsymbol{r}$ and the center of mass $\boldsymbol{q}$, we can analyze the dynamics of the angular momenta with respect to these points, $\boldsymbol{S}$ and $\boldsymbol{S}_{CM}$, respectively. In any case, if we know the angular momentum with respect to a point, we can compute the angular momentum with respect to another point. For the free particle, the angular momentum with respect to the origin of the inertial reference frame, is written alternatively as:

$$\boldsymbol{J} = \boldsymbol{q} \times \boldsymbol{P} + \boldsymbol{S}_{CM} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{S}$$

By taking the time derivative we get,

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{P} \times \boldsymbol{u}, \qquad \frac{d\boldsymbol{S}_{CM}}{dt} = 0.$$

However, as we mentioned in the Preamble , if an external force $\boldsymbol{F}$ applied at point $\boldsymbol{r}$ is acting on the particle, the torque of this force with respect to the origin will produce the variation of the total angular momentum $\boldsymbol{J}$,

$$\frac{d\boldsymbol{J}}{dt} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{u} \times \boldsymbol{P} + \boldsymbol{r} \times \frac{d\boldsymbol{P}}{dt} + \frac{d\boldsymbol{S}}{dt}$$

but $d\boldsymbol{P}/dt = \boldsymbol{F}$, and therefore the spin $\boldsymbol{S}$ satisfies exactly the same dynamical equation than in the free case,

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{P} \times \boldsymbol{u},$$

but now $\boldsymbol{P}$ is not a constant of the motion. For the other

$$\frac{d\boldsymbol{J}}{dt} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{v} \times \boldsymbol{P} + \boldsymbol{q} \times \frac{d\boldsymbol{P}}{dt} + \frac{d\boldsymbol{S}_{CM}}{dt}$$

and thus

$$\frac{d\boldsymbol{S}_{CM}}{dt} = (\boldsymbol{r} - \boldsymbol{q}) \times \boldsymbol{F},$$

If the spin with respect to the center of mass is not conserved, this means that for an elementary particle $\boldsymbol{q} \neq \boldsymbol{r}$, and thus the center of mass and center of charge will be two different points.

2.2.5 Transformation of several observables

The different functions of the expansion of the Lagrangian $\widetilde{L}$, transform under the Galilei group according to (2.43)-(2.46). If we derivate the third equation with respect to τ and divide by $\dot{t}' = \dot{t}$, it gives

$$\frac{d\boldsymbol{U}'}{dt'} = R(\boldsymbol{\mu})\frac{d\boldsymbol{U}}{dt}, \quad \boldsymbol{u}' \cdot \frac{d\boldsymbol{U}'}{dt'} = \boldsymbol{u} \cdot \frac{d\boldsymbol{U}}{dt} + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\frac{d\boldsymbol{U}}{dt}$$

This implies that the linear momentum $\boldsymbol{P}$ and temporal momentum H, transform between Galilei observers in the same form (2.16-2.17) as in the case of the free point particle.

$$H' = H + \boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{P} + \frac{1}{2}mv^2, \tag{2.58}$$

$$\boldsymbol{P}' = R(\boldsymbol{\mu})\boldsymbol{P} + m\boldsymbol{v}. \tag{2.59}$$

In this way, if H_0 and $\boldsymbol{P}_0 = 0$, are the values they take for the center of mass observer, then for any other observer who sees the center of mass moving at the speed $\boldsymbol{v}$

$$H = H_0 + \frac{1}{2}mv^2 = H_0 + \frac{\boldsymbol{P}^2}{2m}, \quad \boldsymbol{P} = m\boldsymbol{v}.$$

Therefore, the magnitude $H - \boldsymbol{P}^2/2m = H_0$ is a constant and invariant property, independent of the inertial observer. It defines an intrinsic property of the particle. The spacetime part of $\widetilde{L}$, which is related to the gauge variant part which defines the mass, takes the general form

$$T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}} = -H\dot{t} + \boldsymbol{P}\cdot\dot{\boldsymbol{r}}.$$

In fact

$$-H'\dot{t}' + \boldsymbol{P}'\cdot\dot{\boldsymbol{r}}' = -H\dot{t} + \boldsymbol{P}\cdot\dot{\boldsymbol{r}} + \frac{1}{2}m\boldsymbol{v}^2\dot{t} + m\boldsymbol{v}\cdot R(\boldsymbol{\mu})\dot{\boldsymbol{r}}.$$

In this way, the second part of the expansion of the Lagrangian $\boldsymbol{U}\cdot\dot{\boldsymbol{u}} + \boldsymbol{W}\cdot\boldsymbol{\omega}$, is necessarily invariant under the Galilei group. The other intrinsic parameter of the elementary particle, the spin or internal rotation, will be related to that part. If we express the Hamiltonian in terms of the invariants H_0 and m, the first part remain

$$-H\dot{t} + \boldsymbol{P}\cdot\dot{\boldsymbol{r}} = -H_0\dot{t} + \frac{m\dot{\boldsymbol{r}}^2}{2\dot{t}} - \frac{1}{2m}\left(\frac{d\boldsymbol{U}}{dt}\right)^2\dot{t}.$$

The first term is a total derivative and can be deleted, the second term gives the gauge variation of the Lagrangian, and the third is necessarily Galilei invariant.

The transformation os the spin with respect to the center of mass $\boldsymbol{S}_{CM}$ defined in (2.55), comes from the transformation of $\boldsymbol{k} = \boldsymbol{U}/m$ and $\boldsymbol{W}$,

$$\boldsymbol{k}' = R(\boldsymbol{\mu})\boldsymbol{k}, \quad \frac{d\boldsymbol{k}'}{dt'} = R(\boldsymbol{\mu})\frac{d\boldsymbol{k}}{dt}, \quad \boldsymbol{W}' = R(\boldsymbol{\mu})\boldsymbol{W}$$

and this leads to

$$\boldsymbol{S}'_{CM} = R(\boldsymbol{\mu})\boldsymbol{S}_{CM}.$$

Therefore $S'^2_{CM} = S^2_{CM}$, is a constant and invariant property between inertial observers. It is another intrinsic property of the elementary particle. The Lagrangian of an spinning elementary particle will depend explicitely of these two invariants mass m and center of mass spin S_{CM}.

We cannot say the same about the spin with respect to the point $\boldsymbol{r}$, $\boldsymbol{S}$. $\boldsymbol{S} = \boldsymbol{u}\times\boldsymbol{U} + \boldsymbol{W}$ transforms in the way:

$$\boldsymbol{S}' = \boldsymbol{u}'\times\boldsymbol{U}' + \boldsymbol{W}' = (R(\boldsymbol{\mu})\boldsymbol{u} + \boldsymbol{v})\times R(\boldsymbol{\mu})\boldsymbol{U} + R(\boldsymbol{\mu})\boldsymbol{W} = R(\boldsymbol{\mu})\boldsymbol{S} + \boldsymbol{v}\times R(\boldsymbol{\mu})\boldsymbol{U},$$

and its absolute value depends on the relative velocity v among observers and, therefore, it is not an intrinsic property.

The center of mass $\boldsymbol{q}$ transforms like the point $\boldsymbol{r}$:

$$\boldsymbol{q}'(\tau) = R(\boldsymbol{\mu})\boldsymbol{q}(\tau) + \boldsymbol{v}t(\tau) + \boldsymbol{a}.$$

This feature does not hold in the relativistic case and the center of mass does not transform like the position of the point $\boldsymbol{r}$. This is because $\boldsymbol{q}$ and $\boldsymbol{r}$ are considered simultaneously in a reference frame and therefore their transformed points $\boldsymbol{q}'$ and $\boldsymbol{r}'$ are not considered simultaneous in the other relativistic reference frame. In the relativistic case the definiton of the center of mass $\boldsymbol{q}$ depends also on the acceleration of the point $\boldsymbol{r}$.

2.2.6 Galilei spinning particle of (anti)orbital spin

To analyze the spin structure of the particle, and therefore the different contributions to the spin coming from these functions $\boldsymbol{U}$ and $\boldsymbol{W}$, let us consider the following simpler example.

Consider a Galilei particle whose kinematical space is $X = \mathcal{G}/SO(3)$, so that any point $x \in X$ can be characterized by the seven variables $x \equiv (t, \boldsymbol{r}, \boldsymbol{u})$, $\boldsymbol{u} = d\boldsymbol{r}/dt$, which are interpreted as time, position and velocity of the particle respectively. In this example we have no orientation variables. The Lagrangian will also depend on the next order derivatives, *i.e.*, on the velocity which is already considered as a kinematical variable and on the acceleration of the particle. Rotation and translation invariance implies that L will be a function of only $\boldsymbol{u}^2$, $(d\boldsymbol{u}/dt)^2$ and $\boldsymbol{u}\cdot d\boldsymbol{u}/dt = d(u^2/2)/dt$, but this last term is a total time derivative and it will not be considered here.

Since from condition (2.57) $\boldsymbol{U} \sim \dot{\boldsymbol{u}}$, let us assume that our elementary system is represented by the following Lagrangian, which when written in terms of the three degrees of freedom and their derivatives is expressed as

$$L = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 - \frac{m}{2\omega^2}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)^2. \tag{2.60}$$

Parameter m is the mass of the particle because the first term is gauge variant in terms of the gauge function (2.41) defined by this constant m, while parameter ω of dimensions of time^{-1} represents an internal frequency. It is the frequency of the internal zitterbewegung.

In terms of the kinematical variables and their derivatives, and in terms of some group invariant evolution parameter τ, the Lagrangian can also be written as

$$\widetilde{L} = \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}} - \frac{m}{2\omega^2}\frac{\dot{\boldsymbol{u}}^2}{\dot{t}}, \tag{2.61}$$

where the dot means τ-derivative. If we consider that the evolution parameter is dimensionless, all terms in the Lagrangian have dimensions of action. Because the Lagrangian is a homogeneous function of first degree in terms of the derivatives of the kinematical variables, $\widetilde{L}$ can also be written as

$$\widetilde{L} = T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}} + \boldsymbol{U}\cdot\dot{\boldsymbol{u}}, \tag{2.62}$$

where the functions accompanying the derivatives of the kinematical variables are defined and explicitly given by

$$\begin{aligned} T &= \frac{\partial\widetilde{L}}{\partial\dot{t}} = -\frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 + \frac{m}{2\omega^2}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)^2, \\ \boldsymbol{R} &= \frac{\partial\widetilde{L}}{\partial\dot{\boldsymbol{r}}} = m\frac{d\boldsymbol{r}}{dt}, \qquad (2.63) \\ \boldsymbol{U} &= \frac{\partial\widetilde{L}}{\partial\dot{\boldsymbol{u}}} = -\frac{m}{\omega^2}\frac{d^2\boldsymbol{r}}{dt^2}. \qquad (2.64) \end{aligned}$$

Dynamical equations obtained from Lagrangian (2.60) are:

$$\frac{1}{\omega^2}\frac{d^4\boldsymbol{r}}{dt^4} + \frac{d^2\boldsymbol{r}}{dt^2} = 0, \tag{2.65}$$

whose general solution is:

$$\boldsymbol{r}(t) = \boldsymbol{A} + \boldsymbol{B}t + \boldsymbol{C}\cos\omega t + \boldsymbol{D}\sin\omega t, \tag{2.66}$$

in terms of the 12 integration constants $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$.

When applying Noether's theorem to the invariance of dynamical equations under the Galilei group, the corresponding constants of the motion can be written in terms of the above functions in the form:

$$\text{temporal momentum} \quad H = -T - \boldsymbol{u}\cdot\frac{d\boldsymbol{U}}{dt}, \tag{2.67}$$

$$\text{linear momentum} \quad \boldsymbol{P} = \boldsymbol{R} - \frac{d\boldsymbol{U}}{dt} = m\boldsymbol{u} - \frac{d\boldsymbol{U}}{dt}, \tag{2.68}$$

$$\text{kinematical momentum} \quad \boldsymbol{K} = m\boldsymbol{r} - \boldsymbol{P}t - \boldsymbol{U}, \tag{2.69}$$

$$\text{angular momentum} \quad \boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{u}\times\boldsymbol{U}. \tag{2.70}$$

It is the presence of the $\boldsymbol{U}$ function that distinguishes the features of this system with respect to the point particle case. We find that the total linear momentum is not lying along the direction of the velocity $\boldsymbol{u}$, and the spin structure is directly related to the function $\boldsymbol{U}$, i.e., to the dependence of the Lagrangian on the acceleration.

If we substitute the general solution (2.66) in (2.67-2.70) we see in fact that the integration constants are related to the above conserved quantities

$$H = \frac{m}{2}\boldsymbol{B}^2 - \frac{m\omega^2}{2}(\boldsymbol{C}^2 + \boldsymbol{D}^2), \tag{2.71}$$

$$\boldsymbol{P} = m\boldsymbol{B}, \tag{2.72}$$

$$\boldsymbol{K} = m\boldsymbol{A}, \tag{2.73}$$

$$\boldsymbol{J} = \boldsymbol{A}\times m\boldsymbol{B} - m\omega\boldsymbol{C}\times\boldsymbol{D}. \tag{2.74}$$

We see that the kinematical momentum $\boldsymbol{K}$ in (2.69) differs from the point particle case (2.14) in the term $-\boldsymbol{U}$, such that if we define the vector $\boldsymbol{k} = \boldsymbol{U}/m$, with dimensions of length, then $\dot{\boldsymbol{K}} = 0$ leads from (2.69) to the equation:

$$\boldsymbol{P} = m\frac{d(\boldsymbol{r}-\boldsymbol{k})}{dt},$$

and $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k}$, defines the position of the center of mass of the particle that is a different point than $\boldsymbol{r}$ and using (2.64) is given by

$$\boldsymbol{q} = \boldsymbol{r} - \frac{1}{m}\boldsymbol{U} = \boldsymbol{r} + \frac{1}{\omega^2}\,\frac{d^2\boldsymbol{r}}{dt^2}. \tag{2.75}$$

In terms of it, dynamical equations (2.65) can be separated into the form:

$$\frac{d^2\boldsymbol{q}}{dt^2} = 0, \tag{2.76}$$

$$\frac{d^2\boldsymbol{r}}{dt^2} + \omega^2(\boldsymbol{r}-\boldsymbol{q}) = 0, \tag{2.77}$$

where (2.76) is just eq. (2.65) after twice differentiating (2.75), and Equation (2.77) is (2.75) after collecting all terms on the left hand side.

From (2.76) we see that point $\boldsymbol{q}$ moves in a straight trajectory at constant velocity while the motion of point $\boldsymbol{r}$, given in (2.77), is an isotropic harmonic motion of angular frequency ω around the point $\boldsymbol{q}$.

The spin of the system with respect to the center of mass, $\boldsymbol{S}_{CM}$ is defined as

$$\boldsymbol{S}_{CM} = \boldsymbol{J} - \boldsymbol{q}\times\boldsymbol{P} = \boldsymbol{J} - \frac{1}{m}\boldsymbol{K}\times\boldsymbol{P}, \tag{2.78}$$

and since it is written in terms of constants of the motion it is clearly a constant of the motion, and its magnitude S^2_{CM} is also a Galilei invariant quantity that characterizes the system. In terms of the integration constants it is expressed as

$$\boldsymbol{S}_{CM} = -m\omega\, \boldsymbol{C} \times \boldsymbol{D}. \tag{2.79}$$

From its definition we get

$$\boldsymbol{S}_{CM} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{k} \times \boldsymbol{P} = -m(\boldsymbol{r} - \boldsymbol{q}) \times \frac{d}{dt}(\boldsymbol{r} - \boldsymbol{q}) = -\boldsymbol{k} \times m\frac{d\boldsymbol{k}}{dt}, \tag{2.80}$$

which appears as the (anti)orbital angular momentum of the relative motion of point $\boldsymbol{r}$ around the center of mass position $\boldsymbol{q}$ at rest, so that the total angular momentum can be written as

$$\boldsymbol{J} = \boldsymbol{q} \times \boldsymbol{P} + \boldsymbol{S}_{CM} = \boldsymbol{L} + \boldsymbol{S}_{CM}. \tag{2.81}$$

It is the sum of the orbital angular momentum $\boldsymbol{L}$ associated to the motion of the center of mass and the spin part $\boldsymbol{S}_{CM}$. For a free particle both $\boldsymbol{L}$ and $\boldsymbol{S}_{CM}$ are separately constants of the motion. We use the term (anti)orbital to suggest that if vector $\boldsymbol{k}$ represents the position of a point of mass m, the angular momentum of this motion is in the opposite direction as the obtained spin observable. But as we shall see in a moment, vector $\boldsymbol{k}$ does not represent the position of the mass m but rather the position of the charge e of the particle.

2.2.7 Interaction with an external electromagnetic field

But if $\boldsymbol{q}$ represents the center of mass position, then what position does point $\boldsymbol{r}$ represent? Point $\boldsymbol{r}$ represents the position of the charge of the particle. This can be seen by considering some interaction with an external field. The homogeneity condition of the Lagrangian in terms of the derivatives of the kinematical variables leads us to consider an interaction term of the form

$$\widetilde{L}_I = -e\phi(t, \boldsymbol{r})\dot{t} + e\boldsymbol{A}(t, \boldsymbol{r}) \cdot \dot{\boldsymbol{r}}, \tag{2.82}$$

which is linear in the derivatives of the kinematical variables t and $\boldsymbol{r}$ and where the external potentials are only functions of t and $\boldsymbol{r}$. We can also consider more general interaction terms of the form $\boldsymbol{N}(t, \boldsymbol{r}, \boldsymbol{u}) \cdot \dot{\boldsymbol{u}}$, and also more general terms in which functions ϕ and $\boldsymbol{A}$ also depend on $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$. If the interaction Lagrangian depends on $\dot{\boldsymbol{u}}$ this implies that the interaction modifies the definition of the observable $\boldsymbol{U} = m\boldsymbol{k}$ which defines the spin of the free system. But if the system is elementary the spin definition cannot be changed, so that (2.82) is the most general interaction term. See the discussion in section **2.1.3** about the independence of the potentials ϕ and A_i of the velocity variables.

Dynamical equations obtained from $L + L_I$ are

$$\frac{1}{\omega^2}\frac{d^4\boldsymbol{r}}{dt^4} + \frac{d^2\boldsymbol{r}}{dt^2} = \frac{e}{m}\left(\boldsymbol{E}(t, \boldsymbol{r}) + \boldsymbol{u} \times \boldsymbol{B}(t, \boldsymbol{r})\right), \tag{2.83}$$

where the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ are expressed in terms of the potentials in the usual form, $\boldsymbol{E} = -\nabla\phi - \partial\boldsymbol{A}/\partial t$, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Dynamical equations (2.83) can again be separated into the form

$$\frac{d^2\boldsymbol{q}}{dt^2} = \frac{e}{m}\left(\boldsymbol{E}(t, \boldsymbol{r}) + \boldsymbol{u} \times \boldsymbol{B}(t, \boldsymbol{r})\right), \tag{2.84}$$

$$\frac{d^2\boldsymbol{r}}{dt^2} + \omega^2(\boldsymbol{r} - \boldsymbol{q}) = 0. \tag{2.85}$$

The center of mass $\boldsymbol{q}$ satisfies Newton's equations under the action of the total external Lorentz force, while point $\boldsymbol{r}$ still satisfies the isotropic harmonic motion of angular frequency ω around

point $\boldsymbol{q}$. The external force modifies the motion of the CM but does not modify its internal relative motion according to the Atomic Principle. But the external force and the fields are defined at point $\boldsymbol{r}$ and not at point $\boldsymbol{q}$. It is the velocity $\boldsymbol{u}$ of point $\boldsymbol{r}$ that appears in the magnetic term of the Lorentz force. Point $\boldsymbol{r}$ clearly represents the position of the charge. In fact, this minimal coupling we have considered is the coupling of the electromagnetic potentials with the particle current, that in the relativistic case can be written as $j_\mu A^\mu$, but the current j_μ is associated to the motion of a charge e at point $\boldsymbol{r}$.

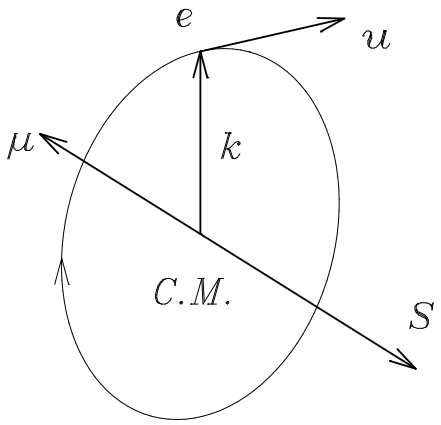


Figure 2.1: **Charge motion in the C.M. frame.**

This charge has an oscillatory motion of very high frequency ω that, in the case of the relativistic electron is $\omega = 2mc^2/\hbar \simeq 1.55 \times 10^{21}\mathrm{s}^{-1}$. The average position of the charge is the center of mass, but it is this internal orbital motion, usually known as the zitterbewegung, that gives rise to the spin structure for this model and also to the magnetic properties of the particle, as we shall see later.

When analyzed in the center of mass frame (see Fig. 2.1), $\boldsymbol{q} = 0$, $\boldsymbol{r} = \boldsymbol{k}$, the system reduces to a point charge whose motion is in general an ellipse, but if we choose $C = D$, and $\boldsymbol{C} \cdot \boldsymbol{D} = 0$, it reduces to a circle of radius $a = C = D$, orthogonal to the spin. Then if the particle has charge e, it has a magnetic moment that according to the usual classical definition is: [3]

$$\boldsymbol{\mu} = \frac{1}{2}\int \boldsymbol{r} \times \boldsymbol{j}\, d^3\boldsymbol{r} = \frac{e}{2}\,\boldsymbol{k} \times \frac{d\boldsymbol{k}}{dt} = -\frac{e}{2m}\boldsymbol{S}_{CM}, \tag{2.86}$$

where $\boldsymbol{j} = e\delta^3(\boldsymbol{r} - \boldsymbol{k})d\boldsymbol{k}/dt$ is the current associated to the motion of a charge e located at point $\boldsymbol{k}$. The magnetic moment is orthogonal to the zitterbewegung plane and opposite to the spin if $e > 0$. It also has a non-vanishing oscillating electric dipole $\boldsymbol{d} = e\boldsymbol{k}$, orthogonal to $\boldsymbol{\mu}$ and therefore to $\boldsymbol{S}_{CM}$ in the center of mass frame, such that its time average value vanishes for times larger than the natural period of this internal motion. Although this is a nonrelativistic example we see in (2.86) that its gyromangnetic ratio is $g = 1$. In order to obtain $g \neq 1$ it is necessary another contribution to the spin not related to this relative motion. It is interesting, nevertheless, to point out and compare with Dirac's relativistic analysis of the electron, [4] in which both momenta $\boldsymbol{\mu}$ and $\boldsymbol{d}$ appear, giving rise to two possible interacting terms in Dirac's Hamiltonian. We shall come back to this analysis later when we study the elementary relativistic particles.

[3] J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, NY (1998), 3rd. ed. p.186.
[4] P.A.M. Dirac, *The Principles of Quantum mechanics*, Oxford Univ. Press, 4th ed. (1967).

2.2.8 Spinning particle in a uniform magnetic field

Let us consider in detail the interaction of this model of particle with spin of orbital nature in an external uniform magnetic field $\boldsymbol{B}$. It is an exercise that can be solved explicitly. The advantage of a model defined in terms of a Lagrangian function is that we do not need to state any dynamical equation for spin, because the spin is a function of the independent degrees of freedom and therefore its dynamics can be obtained from them. The result is that we shall obtain as a first order approximation a torque equation of the usual form $d\boldsymbol{S}_{CM}/dt = \boldsymbol{\mu} \times \boldsymbol{B}$, when the magnetic moment $\boldsymbol{\mu}$ is properly interpreted in terms of the charge motion.

In this case, the system of equations (2.84-2.85) reduce to

$$\frac{d^2\boldsymbol{q}}{dt^2} = \frac{e}{m}\,\boldsymbol{u} \times \boldsymbol{B}, \quad \frac{d^2\boldsymbol{r}}{dt^2} + \omega^2(\boldsymbol{r} - \boldsymbol{q}) = 0.$$

With the definition of the variables $\boldsymbol{v} = d\boldsymbol{q}/dt$, it is equivalent to a linear system of twelve differential equations of first order for the components of $\boldsymbol{r}$, $\boldsymbol{u}$, $\boldsymbol{q}$ and $\boldsymbol{v}$. If we define a new dimensionless time variable $\tau = \omega t$, then the above system depends only on the dimensionless parameter $a = eB/m\omega$ which is the quotient between the cyclotron frequency $|\omega_c| = eB/m$ and ω, the natural frequency of the internal motion.

By taking the direction of the uniform magnetic field along the OZ axis, the external force is orthogonal to it. Then if we call q_3 and r_3 the corresponding coordinates along that axis of the centre of mass and center of charge, they satisfy

$$\frac{d^2q_3}{dt^2} = 0, \quad \frac{d^2q_3}{dt^2} + \omega^2(r_3 - q_3) = 0 \tag{2.87}$$

whose general solution in terms of the initial data $q_3(0)$, $r_3(0)$, $v_3(0)$ and $u_3(0)$ is

$$q_3(t) = q_3(0) + v_3(0)t, \tag{2.88}$$

$$r_3(t) = (r_3(0) - q_3(0))\cos\omega t + \frac{1}{\omega}(u_3(0) - v_3(0))\sin\omega t + q_3(0) + v_3(0)t. \tag{2.89}$$

Similarly, the other components of the center of mass in terms of the new time variable are

$$\frac{d^2q_1}{d\tau^2} = a\,\frac{dr_2}{d\tau}, \quad \frac{d^2q_2}{d\tau^2} = -a\,\frac{dr_1}{d\tau},$$

and once integrated we get

$$\frac{dq_1}{d\tau} = ar_2 + b_1, \quad \frac{dq_2}{d\tau} = -ar_1 + b_2, \tag{2.90}$$

where b_1 and b_2 are two integration constants with dimensions of length. Thus we are left with the integration of a first order system formed by these two last equations (2.90) and the equations for the other two components of the center of charge that can be written as

$$\frac{dr_1}{d\tau} = u_1, \quad \frac{dr_2}{d\tau} = u_2, \tag{2.91}$$

$$\frac{du_1}{d\tau} = q_1 - r_1, \quad \frac{du_2}{d\tau} = q_2 - r_2. \tag{2.92}$$

The matrix of this linear system in terms of the variables q_1, q_2, r_1, r_2, u_1 and u_2, taken in this order, is just

$$M = \begin{pmatrix} 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix},$$

whose characteristic equation is $\lambda^6 + 2\lambda^4 + \lambda^2 + a^2 = 0$. It is shown that it has six different roots, corresponding to the normal modes of the system. If we call $\lambda = iz$, these new variables verify $z^2(1-z^2)^2 = a^2$, and thus by solving the cubic equation $z(1-z^2) = a$, the six solutions of the form $\pm iz$ will be the six eigenvalues of the system. If we define

$$k = \frac{1}{3}\arcsin\left(\frac{3\sqrt{3}a}{2}\right), \tag{2.93}$$

then the six eigevalues are $\pm i\omega_j$, $j = 1, 2, 3$, where:

$$\omega_1 = \frac{2}{\sqrt{3}}\sin k, \quad \omega_2 = -\cos k - \frac{1}{\sqrt{3}}\sin k, \quad \omega_3 = \cos k - \frac{1}{\sqrt{3}}\sin k. \tag{2.94}$$

If $3\sqrt{3}|a|/2 \leq 1$ then the six roots are purely imaginary and the motion is three-periodic with these three frequencies. Otherwise, if there exist real roots, the corresponding solution will be exponential. In general, for the electron, as we shall see in the next chapter, the zitterbewegung frequency is $\omega = 2mc^2/\hbar$, and thus

$$a/B = e/m\omega = e\hbar/2m^2c^2 = 1.13 \times 10^{-10}\text{Tesla}^{-1},$$

so that even with very strong magnetic fields the parameter a is very small and the usual solution will be oscillatory.

The general solution of the complete system will be a linear combination of these three oscillations and it will depend on twelve integration constants that will be expressed in terms of the initial position and velocity of the center of mass and center of charge. The general form for the evolution of the center of charge is:

$$\begin{aligned}
r_1(\tau) &= A\cos\omega_1\tau + B\sin\omega_1\tau + C\cos\omega_2\tau + D\sin\omega_2\tau + E\cos\omega_3\tau \\
&+ F\sin\omega_3\tau + b_2/a, \\
r_2(\tau) &= B\cos\omega_1\tau - A\sin\omega_1\tau + D\cos\omega_2\tau - C\sin\omega_2\tau \\
&+ F\cos\omega_3\tau - E\sin\omega_3\tau - b_1/a, \\
r_3(t) &= (r_3(0) - q_3(0))\cos\omega t \\
&+ \frac{1}{\omega}(u_3(0) - v_3(0))\sin\omega t + q_3(0) + v_3(0)t,
\end{aligned}$$

where

$$b_1/a = v_1(0)/a\omega - r_2(0), \quad b_2/a = v_2(0)/a\omega + r_1(0).$$

For the center of mass coordinates we get

$$\begin{aligned}
q_1(\tau) &= (1-\omega_1^2)\,(A\cos\omega_1\tau + B\sin\omega_1\tau) \\
&+ (1-\omega_2^2)\,(C\cos\omega_2\tau + D\sin\omega_2\tau) \\
&+ (1-\omega_3^2)\,(E\cos\omega_3\tau + F\sin\omega_3\tau) + b_2/a, \\
q_2(\tau) &= (1-\omega_1^2)\,(B\cos\omega_1\tau - A\sin\omega_1\tau) \\
&+ (1-\omega_2^2)\,(D\cos\omega_2\tau - C\sin\omega_2\tau) \\
&+ (1-\omega_3^2)\,(F\cos\omega_3\tau - E\sin\omega_3\tau) - b_1/a, \\
q_3(t) &= q_3(0) + v_3(0)t.
\end{aligned}$$

The six unknown constants A, B, C, D, E, and F are of dimensions of length and satisfy the linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ \omega_1 & \omega_2 & \omega_3 \\ \omega_1^2 & \omega_2^2 & \omega_3^2 \end{pmatrix} \begin{pmatrix} A \\ C \\ E \end{pmatrix} = \begin{pmatrix} -v_2(0)/a\omega \\ -u_2(0)/\omega \\ r_1(0) - q_1(0) \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 1 & 1 \\ \omega_1 & \omega_2 & \omega_3 \\ \omega_1^2 & \omega_2^2 & \omega_3^2 \end{pmatrix} \begin{pmatrix} B \\ D \\ F \end{pmatrix} = \begin{pmatrix} v_1(0)/a\omega \\ u_1(0)/\omega \\ r_2(0) - q_2(0) \end{pmatrix},$$

where $\boldsymbol{q}(0)$, $\boldsymbol{v}(0)$ and $\boldsymbol{r}(0)$, $\boldsymbol{u}(0)$, are respectively the position and velocity of the center of mass and center of charge at time $t = 0$.

If we call N the inverse of the matrix containing the frequencies of the above equations, it is:

$$N = \frac{1}{\Delta} \begin{pmatrix} \omega_2\omega_3(\omega_3 - \omega_2) & \omega_2^2 - \omega_3^2 & \omega_3 - \omega_2 \\ \omega_1\omega_3(\omega_1 - \omega_3) & \omega_3^2 - \omega_1^2 & \omega_1 - \omega_3 \\ \omega_1\omega_2(\omega_2 - \omega_1) & \omega_1^2 - \omega_2^2 & \omega_2 - \omega_1 \end{pmatrix},$$

where $\Delta = (\omega_1 - \omega_2)(\omega_2 - \omega_3)(\omega_3 - \omega_1)$, in such a way that we can obtain the final expression of the integration constants in terms of the initial conditions.

To lowest order in a, since $k \approx \sqrt{3}a/2$, the normal modes are:

$$\omega_1 = a + O(a^3), \quad \omega_2 = -1 - \frac{a}{2} + \frac{3a^2}{8} + O(a^3), \quad \omega_3 = 1 - \frac{a}{2} - \frac{3a^2}{8} + O(a^3). \tag{2.95}$$

In terms of the physical parameters and in the time evolution description, these normal frequencies are to lowest order:

$$\omega_1 = \omega_c, \quad \omega_2 = \omega - \frac{\omega_c}{2} - \frac{3\omega_c^2}{8\omega}, \quad \omega_2 = \omega + \frac{\omega_c}{2} - \frac{3\omega_c^2}{8\omega}, \tag{2.96}$$

where $\omega_c = eB/m$ and ω are the cyclotron and zitterbewegung frequency, respectively.

To properly characterize these initial values in terms of physical parameters, like the radius of the internal motion R_0, the cyclotron radius R_c, the center of mass velocity v and the zitterbewegung frequency ω, let us consider an electron that is sent with a velocity $\boldsymbol{v}$ orthogonal to the external uniform magnetic field $\boldsymbol{B}$. We take the XOY plane such that the initial position of the center of mass is on the OX axis at the coordinate $R_c = -vm/eB$, and the initial velocity v along the positive direction of the OY axis. With this convention, the center of mass will have a precession around the OZ axis with cyclotron angular velocity $|\omega_c|$ in the positive direction while for a positive charged particle the initial position will be chosen as $-|R_c|$ on the OX axis and the angular velocity will point in the negative OZ axis.

The initial position of the center of charge is characterized by the three parameters ϕ, θ and ψ, where θ and ϕ represent the initial orientation of the internal angular velocity ω, and parameter ψ is the initial phase position of the center of charge as shown in Figure 2.2. If all these three parameters are zero, $\boldsymbol{\omega}$ is pointing along OZ and the initial position of the charge is at point $R_c + R_0$ on the OX axis.

We thus have as initial conditions for our system, written in column matrix form:

$$\boldsymbol{q}(0) = \begin{pmatrix} R_c \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{r}(0) = \begin{pmatrix} R_c \\ 0 \\ 0 \end{pmatrix} + \mathcal{R}_{\mathrm{OZ}}(\phi)\mathcal{R}_{\mathrm{OY}}(\theta)\mathcal{R}_{\mathrm{OZ}}(\psi) \begin{pmatrix} R_0 \\ 0 \\ 0 \end{pmatrix},$$

$$\boldsymbol{v}(0) = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}, \quad \boldsymbol{u}(0) = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} + \mathcal{R}_{\mathrm{OZ}}(\phi)\mathcal{R}_{\mathrm{OY}}(\theta)\mathcal{R}_{\mathrm{OZ}}(\psi) \begin{pmatrix} 0 \\ \omega R_0 \\ 0 \end{pmatrix},$$

where $\mathcal{R}_{\mathrm{OZ}}(\alpha)$ will represent a rotation in the active sense, of value α around the OZ axis. Since the spin is opposite to the internal angular velocity, its initial value is

$$\boldsymbol{S}_{CM}(0) = \mathcal{R}_{\mathrm{OZ}}(\phi)\mathcal{R}_{\mathrm{OY}}(\theta) \begin{pmatrix} 0 \\ 0 \\ -S \end{pmatrix}, \tag{2.97}$$

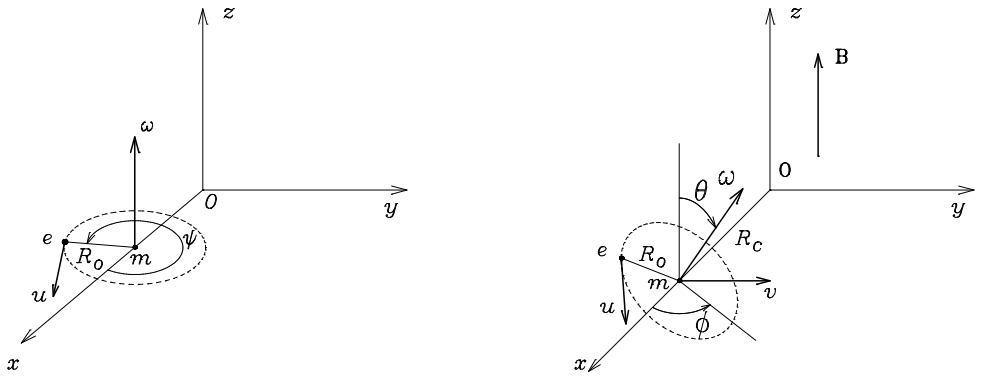


Figure 2.2: **Initial phase ψ of the charge and initial orientation (θ, ϕ) of angular velocity $\boldsymbol{\omega}$.**

where $S = m\omega R_0^2$. Thus the initial conditions to determine the coefficients of the general solution are:

$$\begin{pmatrix} -v_2(0)/a\omega \\ -u_2(0)/\omega \\ r_1(0) - q_1(0) \end{pmatrix} = \begin{pmatrix} R_c \\ aR_c - \alpha R_0 \\ \beta R_0 \end{pmatrix}, \quad \begin{pmatrix} v_1(0)/a\omega \\ u_1(0)/\omega \\ r_2(0) - q_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma R_0 \\ \delta R_0 \end{pmatrix},$$

where $R_c = -vm/eB$, $\omega_c = -eB/m = -a\omega$, as before and the constant parameters:

$$\begin{aligned} \alpha &= -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi, \\ \beta &= \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi, \\ \gamma &= -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi, \\ \delta &= \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi. \end{aligned}$$

To lowest order in a, the frequencies become:

$$\omega_1 - \omega_2 = 1 + \frac{3}{2}a, \quad \omega_2 - \omega_3 = -2, \quad \omega_3 - \omega_1 = 1 - \frac{3}{2}a,$$

$$\omega_1 + \omega_2 = -1 + \frac{a}{2}, \quad \omega_2 + \omega_3 = -a, \quad \omega_3 + \omega_1 = 1 + \frac{a}{2},$$

$$\omega_1\omega_2 = -a\left(1 + \frac{a}{2}\right), \quad \omega_2\omega_3 = -\left(1 - \frac{a^2}{4}\right), \quad \omega_3\omega_1 = a\left(1 - \frac{a}{2}\right),$$

and thus the inverse matrix N to order $O(a^2)$ is

$$N = \begin{pmatrix} 1 + 2a^2 & -a & -1 - 9a^2/4 \\ a/2 - a^2 & -1/2 + a/2 - 3a^2/4 & 1/2 - 3a/4 + 9a^2/8 \\ -a/2 - a^2 & 1/2 + a/2 + 3a^2/4 & 1/2 + 3a/4 + 9a^2/8 \end{pmatrix}.$$

In this way the coefficients of the general solution, to first order in a, are:

$$\begin{aligned} A &= R_c - \beta R_0 + aR_0\alpha, \\ B &= -R_0(a\gamma + \delta), \\ C &= \frac{R_0}{2}(\alpha + \beta) - \frac{aR_0}{4}(2\alpha + 3\beta), \\ D &= \frac{R_0}{2}(\delta - \gamma) + \frac{aR_0}{4}(2\gamma - 3\delta), \end{aligned}$$

$$
\begin{aligned}
E &= \frac{R_0}{2}(\beta-\alpha)+\frac{aR_0}{4}(3\beta-2\alpha),\\
F &= \frac{R_0}{2}(\delta+\gamma)+\frac{aR_0}{4}(2\gamma+3\delta),
\end{aligned}
$$

and the coefficients

$$
b_1/a=-\delta R_0,\quad b_2/a=\beta R_0.
$$

This motion depends on the cyclotron radius R_c, only through the parameter A, and the remaining terms depend on the internal radius R_0.

The general solution, neglecting terms of the order aR_0, can be written in a vector form as:

$$
\begin{aligned}
\boldsymbol{r}(t) &= \mathcal{R}_{\mathrm{OZ}}(\omega_c t)\begin{pmatrix}R_c\\0\\0\end{pmatrix}+\left(\mathbb{I}-\mathcal{R}_{\mathrm{OZ}}(\omega_c t)\right)\mathcal{R}(\phi,\theta,\psi)\begin{pmatrix}R_0\\0\\0\end{pmatrix}\\
&+ \mathcal{R}_{\mathrm{OZ}}\left(-\frac{\omega_c t}{2}\right)\mathcal{R}(\phi,\theta,\psi+\omega t)\begin{pmatrix}R_0\\0\\0\end{pmatrix}+O(aR_0),
\end{aligned}
$$

where $\mathbb{I}$ is the 3×3 unit matrix and $\mathcal{R}(\phi,\theta,\psi)\equiv\mathcal{R}_{\mathrm{OZ}}(\phi)\mathcal{R}_{\mathrm{oy}}(\theta)\mathcal{R}_{\mathrm{OZ}}(\psi)$. The first two terms represent the center of mass motion to this order of approximation, while the third is precisely the relative motion of the center of charge around the center of mass. The neglected contribution of order aR_0 can be written as

$$
\begin{aligned}
&O(aR_0)\\
=\;& -J_z\left[\mathcal{R}_{\mathrm{OZ}}(\omega_c t)\mathcal{R}(\phi,\theta,\psi)-\mathcal{R}_{\mathrm{OZ}}\left(-\frac{\omega_c t}{2}\right)\mathcal{R}(\phi,\theta,\psi+\omega t)\right]\begin{pmatrix}0\\aR_0\\0\end{pmatrix}\\
&-J_z\left[\frac{\sin(\omega t)}{2}\mathcal{R}_{\mathrm{OZ}}\left(-\frac{\omega_c t}{2}\right)\mathcal{R}(\phi,\theta,\psi)\begin{pmatrix}aR_0\\0\\0\end{pmatrix}\right],
\end{aligned}
$$

where

$$
J_z=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix},
$$

is the 3×3 generator of rotations around the OZ axis. The first two terms represent the correction to this order of the center of mass motion and the third is the correction of the internal relative motion. The presence of the generator J_z in this term means that this correction does not make any contribution to the motion along the OZ axis. The solution along OZ is exactly:

$$
q_3(t)=0,\quad r_3(t)=-R_0\sin\theta\,\cos(\omega t+\psi), \tag{2.98}
$$

i.e., a harmonic motion of amplitude $R_0\sin\theta$, and frequency ω.

The relative position of the center of charge with respect to the center of mass verifies:

$$
\begin{aligned}
\boldsymbol{k}(t) &= \mathcal{R}_{\mathrm{OZ}}\left(-\frac{\omega_c t}{2}\right)\mathcal{R}(\phi,\theta,\psi+\omega t)\begin{pmatrix}R_0\\0\\0\end{pmatrix}\\
&-J_z\left[\frac{\sin(\omega t)}{2}\mathcal{R}_{\mathrm{OZ}}\left(-\frac{\omega_c t}{2}\right)\mathcal{R}(\phi,\theta,\psi)\begin{pmatrix}aR_0\\0\\0\end{pmatrix}\right],
\end{aligned} \tag{2.99}
$$

and if we neglect contributions to order aR_0, it just reduces to the first term

$$\boldsymbol{k}(t) \approx \mathcal{R}_{\mathrm{OZ}}\left(-\frac{\omega_c t}{2}\right)\mathcal{R}(\phi,\theta,\psi+\omega t)\begin{pmatrix} R_0 \\ 0 \\ 0 \end{pmatrix}, \tag{2.100}$$

that represents an oscillation with the natural frequency ω of the zitterbewegung around the initial spin axis, with a backwards precession with an angular velocity $\omega_c/2$.

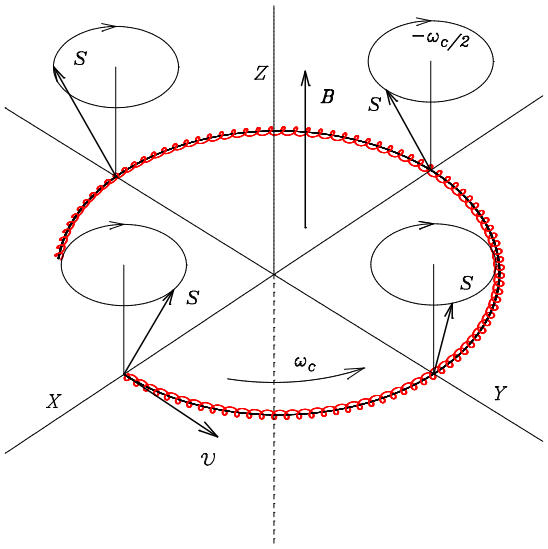


Figure 2.3: **Motion of the center of charge (red) and center of mass of a negative charged particle in a uniform magnetic field. The spin with respect to the center of mass precess in the opposite direction to the cyclotronic motion and with half the angular velocity. The velocity of the center of mass is orthogonal to the field.**

The center of charge and center of mass trajectory is depicted in the Figure 2.3, where the curly trajectory is the motion of the center of charge.

To study the spin dynamics, we just substitute the general solution in its analytical definition

$$\boldsymbol{S}_{CM}(t) = -m\boldsymbol{k}(t) \times \frac{d\boldsymbol{k}(t)}{dt}, \tag{2.101}$$

where we need to calculate the derivative of (2.100). To calculate this derivative, we have to take into account that

$$\mathcal{R}_{\mathrm{OZ}}(\omega t) = \exp(J_z \omega t),$$

and therefore

$$\dot{\mathcal{R}}_{\mathrm{OZ}}(\omega t) = \exp(J_z\omega t)J_z\omega = \mathcal{R}_{\mathrm{OZ}}(\omega t)J_z\omega = J_z\omega\,\mathcal{R}_{\mathrm{OZ}}(\omega t).$$

By taking the derivative of (2.100) we get the following terms:

$$\begin{aligned} \frac{d\boldsymbol{k}}{dt} &= \mathcal{R}_{\mathrm{OZ}}\left(-\frac{\omega_c t}{2}\right) J_z\,\mathcal{R}(\phi,\theta,\psi+\omega t)\begin{pmatrix} -\omega_c R_0/2 \\ 0 \\ 0 \end{pmatrix} \\ &+ \mathcal{R}_{\mathrm{OZ}}\left(-\frac{\omega_c t}{2}\right)\mathcal{R}(\phi,\theta,\psi+\omega t)\begin{pmatrix} 0 \\ \omega R_0 \\ 0 \end{pmatrix}, \end{aligned} \tag{2.102}$$

where

$$\begin{pmatrix} 0 \\ \omega R_0 \\ 0 \end{pmatrix} = \omega J_z \begin{pmatrix} R_0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.103}$$

Of these terms, the first is of order $\omega_c R_0 = vR_0/R_c = a\omega R_0 = ac$, and thus even with very high magnetic fields it can be neglected.

The dynamics of the spin with respect to the center of mass is reduced to

$$\boldsymbol{S}_{CM}(t) = \mathcal{R}_{\text{OZ}}\left(-\frac{\omega_c t}{2}\right)\mathcal{R}(\phi,\theta,\psi+\omega t)\begin{pmatrix}0\\0\\-m\omega R_0^2\end{pmatrix} = \mathcal{R}_{\text{OZ}}\left(-\frac{\omega_c t}{2}\right)\boldsymbol{S}(0), \tag{2.104}$$

where $\boldsymbol{S}_{CM}(0)$ is given in (2.97). The spin is precessing backwards with half the angular velocity of the cyclotron motion while its absolute value remains constant at first order. We represent in Figure 2.4 its evolution during the same time interval as the one depicted in Figure 2.3 with the initial orientation $\theta = 30°$ and $\phi = 90°$, where we can observe, in addition to the precession of constant absolute value, a tiny oscillation of the next order contribution.

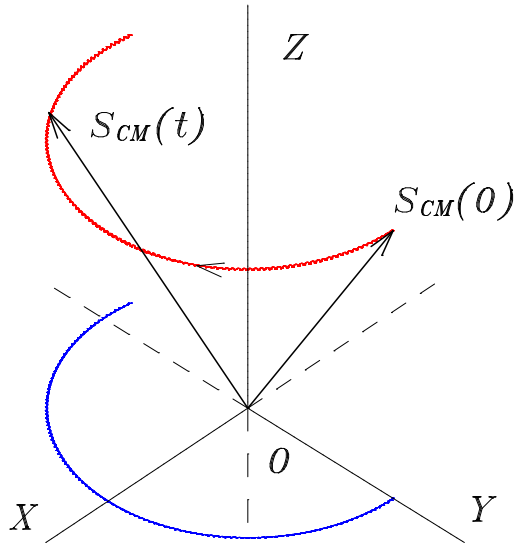


Figure 2.4: **Precession of spin with respect to the center of mass $\boldsymbol{S}_{CM}$ around the OZ axis, and its projection (in blue) on the XOY plane during the same time than in the figure 2.3.**

From another point of view, the relationship between tha spin and magnetic moment is given by (2.86), and the dynamics of the spin with respect to the center of mass depends on the torque of the magnetic field $\boldsymbol{B}$,

$$\frac{d\boldsymbol{S}_{CM}}{dt} = \boldsymbol{\mu}\times\boldsymbol{B} = -\frac{e}{2m}\boldsymbol{S}_{CM}\times\boldsymbol{B} = \boldsymbol{\Omega}\times\boldsymbol{S}_{CM}.$$

The constant angular velocity of precession of the spin, is Larmor's angular frequency

$$\boldsymbol{\Omega} = \frac{e\boldsymbol{B}}{2m} = -\frac{\boldsymbol{\omega}_c}{2},$$

since $\boldsymbol{\omega}_c = -e\boldsymbol{B}/m$, *i.e.*, half and of opposite direction to the cyclotronic angular velocity,as can be seen in the Figure 2.4. This produces the first order contribution, since at this order the absolute value of spin is conserved. This approach does not contain the additional terms of correction to the normal modes ω_i, which can be relevant in high energy processes, and that can be obtained using the exact general solution.

2.2.9 Dynamics of the spin with respect to the center of charge $\boldsymbol{S}$

It is interesting to compare the evolution of the center of mass spin $\boldsymbol{S}_{CM}$ with that of the center of charge spin $\boldsymbol{S} = \boldsymbol{u}\times\boldsymbol{U}$, which satisfies either in the free case and under interaction the same dynamcial equation

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{p}\times\boldsymbol{u}.$$

In the figure **2.5** we represent its evolution, as well as its projection on the XOY plane, during a complete turn of the electron.

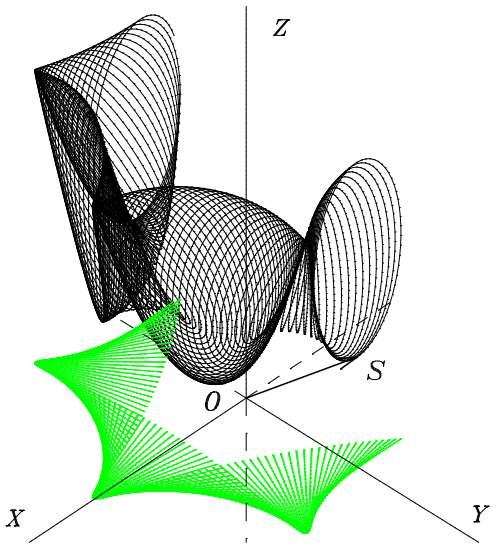


Figure 2.5: **Motion of the center of charge spin $\boldsymbol{S}$, and its projection (in green) onto the plane XOY, during a complete turn of the CM of the electron. Its evolution is always orthogonal to the linear momentum $\boldsymbol{p}$. The dynamics of the center of mass spin $\boldsymbol{S}$ represents the evolution, in the quantum case, of Dirac's spin operator, $\boldsymbol{S} = \hbar\boldsymbol{\sigma}/2$.**

Since the center of mass spin is written as

$$\boldsymbol{S}_{CM} = \boldsymbol{S} + \boldsymbol{k} \times \boldsymbol{p}, \tag{2.105}$$

where $\boldsymbol{k}$ is the relative position of the CC with respect to the CM. If we consider the average value of this expression during a complete turn of the center of charge, during this short time the linear momentum is almost constant and thus and the average value of $\boldsymbol{k}$ is zero, this implies that $< \boldsymbol{S}_{CM} >=< \boldsymbol{S} >$. We can show that by depicting the evolution of both spins $\boldsymbol{S}_{CM}$ and $\boldsymbol{S}$, of **2.4** and **2.5**, respectively. In the figure **2.6** we see this superposition.

Since Dirac's spin operator $\boldsymbol{S} = \frac{1}{2}\hbar\boldsymbol{\sigma}$ satisfies the dynamical equation $d\boldsymbol{S}/dt = \boldsymbol{p} \times \boldsymbol{u}$, it is the dynamics of the center of charge spin which represents the evolution od Dirac's spin operator.

Although this analysis of the average values of the spin has been done with a nonrelativistic model, this result is completely general since the relationship between both spins (2.105) is the same in the relativistic case and the average value the relative position $\boldsymbol{k}$, during a turn, is zero.

2.2.10 Energy of the particle

The energy of the system is

$$H = -T - \boldsymbol{u} \cdot \frac{d\boldsymbol{U}}{dt}, \tag{2.106}$$

that can be expressed as:

$$H = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 - \frac{m}{2\omega^2}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)^2 + \frac{m}{\omega^2}\frac{d\boldsymbol{r}}{dt} \cdot \frac{d^3\boldsymbol{r}}{dt^3} + eV(\boldsymbol{r}, t),$$

and, since the function $V(\boldsymbol{r}, t) = 0$ in the presence of a constant magnetic field, it becomes:

$$H = \frac{m}{2}\left(\frac{d\boldsymbol{q}}{dt}\right)^2 - \frac{m}{2}\left(\frac{d\boldsymbol{k}}{dt}\right)^2 - \frac{m\omega^2}{2}\boldsymbol{k}^2 = \frac{(\boldsymbol{P} - e\boldsymbol{A})^2}{2m} + H_0. \tag{2.107}$$

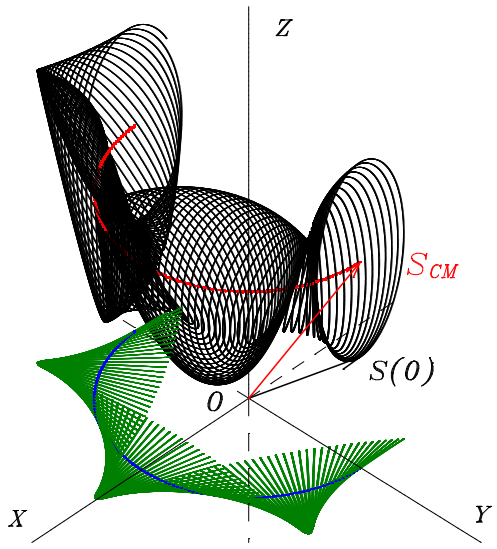


Figure 2.6: **Superposition of the dynamics of both spins. The dynamics of the center of mass spin $\boldsymbol{S}_{CM}$ (in red), is the average value during a turn of the center of charge of the dynamics of the center of charge spin $\boldsymbol{S}$. The center of mass spin is the average value during a turn of the center of charge of Dirac's spin.**

To lowest order the contribution comes from

$$\boldsymbol{q}(t) = \mathcal{R}_{\mathrm{OZ}}(\omega_c t) \begin{pmatrix} R_c \\ 0 \\ 0 \end{pmatrix} + (\mathbb{I} - \mathcal{R}_{\mathrm{OZ}}(\omega_c t))\, \mathcal{R}(\phi,\theta,\psi) \begin{pmatrix} R_0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus

$$\frac{d\boldsymbol{q}}{dt} = \mathcal{R}_{\mathrm{OZ}}(\omega_c t) \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} - \mathcal{R}_{\mathrm{OZ}}(\omega_c t)\, J_z\, \mathcal{R}(\phi,\theta,\psi) \begin{pmatrix} \omega_c R_0 \\ 0 \\ 0 \end{pmatrix},$$

in such a way that taking into account (2.99) and (2.102)

$$\begin{aligned} \left(\frac{d\boldsymbol{q}}{dt}\right)^2 &= v^2 + \left[J_z\, \mathcal{R}(\phi,\theta,\psi) \begin{pmatrix} \omega_c R_0 \\ 0 \\ 0 \end{pmatrix} \right]^2 \\ &- 2 \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} \cdot \left[J_z\, \mathcal{R}(\phi,\theta,\psi) \begin{pmatrix} \omega_c R_0 \\ 0 \\ 0 \end{pmatrix} \right], \end{aligned}$$

$$\left(\frac{d\boldsymbol{k}}{dt}\right)^2 = \omega^2 R_0^2 + \left[J_z\, \mathcal{R}(\phi,\theta,\psi+\omega t) \begin{pmatrix} -\omega_c R_0/2 \\ 0 \\ 0 \end{pmatrix} \right]^2$$

$$+2 \left[J_z\, \mathcal{R}(\phi,\theta,\psi+\omega t) \begin{pmatrix} -\omega_c R_0/2 \\ 0 \\ 0 \end{pmatrix} \right] \cdot \left[\mathcal{R}(\phi,\theta,\psi+\omega t) \begin{pmatrix} 0 \\ \omega R_0 \\ 0 \end{pmatrix} \right].$$

Since

$$\mathcal{R}(\phi,\theta,\psi+\omega t) = \begin{pmatrix} \beta(t) & \gamma(t) & \cos\phi\sin\theta \\ \delta(t) & \alpha(t) & \sin\phi\sin\theta \\ -\sin\theta\cos(\omega t+\psi) & \sin\theta\sin(\omega t+\psi) & \cos\theta \end{pmatrix},$$

$$J_z\,\mathcal{R}(\phi,\theta,\psi+\omega t)=\begin{pmatrix}-\delta(t) & -\alpha(t) & -\sin\phi\sin\theta\\ \beta(t) & \gamma(t) & \cos\phi\sin\theta\\ 0 & 0 & 0\end{pmatrix},$$

where

$$\begin{aligned}\alpha(t) &= -\sin\phi\cos\theta\sin(\psi+\omega t)+\cos\phi\cos(\psi+\omega t),\\ \beta(t) &= \cos\phi\cos\theta\cos(\psi+\omega t)-\sin\phi\sin(\psi+\omega t),\\ \gamma(t) &= -\cos\phi\cos\theta\sin(\psi+\omega t)-\sin\phi\cos(\psi+\omega t),\\ \delta(t) &= \sin\phi\cos\theta\cos(\psi+\omega t)+\cos\phi\sin(\psi+\omega t)\end{aligned}$$

then

$$J_z\,\mathcal{R}(\phi,\theta,\psi+\omega t)\begin{pmatrix}\omega_c R_0\\ 0\\ 0\end{pmatrix}=\omega_c R_0\begin{pmatrix}-\delta(t)\\ \beta(t)\\ 0\end{pmatrix}.$$

Consequently

$$\left(\frac{d\boldsymbol{q}}{dt}\right)^2=v^2+\omega_c^2R_0^2(\delta(0)^2+\beta(0)^2)-2v\omega_cR_0\beta(0),$$

$$\left(\frac{d\boldsymbol{k}}{dt}\right)^2=\omega^2R_0^2+\frac{\omega_c^2R_0^2}{4}(\delta(t)^2+\beta(t)^2)+\omega\omega_cR_0^2(\delta(t)\gamma(t)-\beta(t)\alpha(t)).$$

Because

$$\delta(t)\gamma(t)-\beta(t)\alpha(t)=-\cos\theta,$$
$$\delta(0)^2+\beta(0)^2=1-\sin^2\theta\cos^2\psi,$$
$$\delta(t)^2+\beta(t)^2=1-\sin^2\theta\cos^2(\psi+\omega t),$$

if we write ω_c in terms of the parameter a, $\omega_c=-a\omega$, in the case of the electron $\omega R_0=c$, the energy of this system to lower order of approximation in a is:

$$H=H_0-a\left(\frac{mc^2\cos\theta}{2}-mvc\beta(0)\right)$$

$$+a^2\frac{mc^2}{2}\left(\delta(0)^2+\beta(0)^2-\frac{1}{4}(\delta(t)^2+\beta(t)^2)\right).$$

The lowest order of the interaction energy can be expressed as:

$$H_I=-\frac{1}{2}amc^2\cos\theta=-\frac{eB}{2m}\frac{mc^2}{\omega}\cos\theta=-\boldsymbol{\mu}\cdot\boldsymbol{B},\tag{2.108}$$

and since $S=m\omega R_0^2=mc^2/\omega$, $S_z=-S\cos\theta$, it implies

$$\mu_z=\frac{eS\cos\theta}{2m}=-\frac{eS_z}{2m},\tag{2.109}$$

or

$$\boldsymbol{\mu}=-\frac{e}{2m}\boldsymbol{S}_{CM}.\tag{2.110}$$

The interaction energy can also be written as

$$H_I=-\frac{eB}{2m}S\cos\theta=\frac{e}{2m}\boldsymbol{B}\cdot\boldsymbol{S}_{CM}=\frac{-\boldsymbol{\omega}_c}{2}\cdot\boldsymbol{S}_{CM},\tag{2.111}$$

i.e., as the scalar product of the spin with respect to the center of mass and the angular velocity of precession of this spin.

2.2.11 Spinning Galilei particle with orientation

Another simple example of a spinning particle is the one in which the spin is related only to the angular variables which describe orientation.

Let us assume now a dynamical system whose kinematical space is $X = \mathcal{G}/\mathbb{R}^3_v$, where $\mathbb{R}^3_v \equiv \{\mathbb{R}^3, +\}$ is the 3-parameter Abelian subgroup of pure Galilei transformations. Then, the kinematical variables are $x \equiv (t, \boldsymbol{r}, \boldsymbol{\rho})$, which are interpreted as the time, position and orientation respectively.

The Lagrangian for this model takes the general form

$$\widetilde{L} = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{W} \cdot \boldsymbol{\omega}.$$

Because of the structure of the exponent (2.247), the gauge function for this system can be taken the same as before. The general relationship (2.57) leads to $\boldsymbol{W} \times \boldsymbol{\omega} = 0$, because the Lagrangian is independent of $\dot{\boldsymbol{u}}$, and therefore $\boldsymbol{W}$ and $\boldsymbol{\omega}$ must be collinear. According to the transformation properties of the Lagrangian, the third term $\boldsymbol{W} \cdot \boldsymbol{\omega}$ is Galilei invariant and since $\boldsymbol{W}$ and $\boldsymbol{\omega}$ are collinear, we can take $\boldsymbol{W} \sim \boldsymbol{\omega}$ and one possible Lagrangian that describes this model is of the form:

$$\widetilde{L} = \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}} + \frac{I}{2}\frac{\boldsymbol{\omega}^2}{\dot{t}}. \tag{2.112}$$

The different Noether's constants are

$$H = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 + \frac{I}{2}\boldsymbol{\Omega}^2, \quad \boldsymbol{P} = m\boldsymbol{u},$$

$$\boldsymbol{K} = m\boldsymbol{r} - \boldsymbol{P}t, \quad \boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{W},$$

where $\boldsymbol{u} = d\boldsymbol{r}/dt$ is the velocity of point $\boldsymbol{r}$, and $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$ is the time evolution angular velocity. Point $\boldsymbol{r}$ is moving at a constant speed and it also represents the position of the center of mass. The spin is just the observable $\boldsymbol{S} \equiv \boldsymbol{W}$ that satisfies the dynamical equation $d\boldsymbol{S}/dt = \boldsymbol{\omega} \times \boldsymbol{S} = 0$, and thus the frame linked to the body rotates with a constant angular velocity $\boldsymbol{\Omega}$.

The spin takes the constant value $\boldsymbol{S} = I\boldsymbol{\Omega}$, whose absolute value is independent of the inertial observer and also the angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$ is constant. The parameter I plays the role of a principal moment of inertia, suggesting a linear relationship between the spin and the angular velocity, which corresponds to a particle with spherical symmetry. The particle can also be considered as an extended object of gyration radius R_0, related to the other particle parameters by $I = mR_0^2$.

This system corresponds classically to a rigid body with spherical symmetry where the orientation variables $\boldsymbol{\rho}$ can describe for instance, the orientation of its principal axes of inertia in a suitable parameterization of the rotation group. This is a system of six degrees of freedom. Three represent the position of the center of charge $\boldsymbol{r}$ and the other three $\boldsymbol{\rho}$, represent the orientation of a Cartesian frame linked to that point $\boldsymbol{r}$. Since for this system there is no dependence on the acceleration, the center of mass and the center of charge will be represented by the same point.

In the center of mass frame there is no current associated to this particle and therefore it has neither magnetic nor electric dipole structure. As seen in previous examples, all magnetic properties seem therefore to be related to the zitterbewegung part of the spin and are absent in this rigid body-like model.

RELATIVISTIC PARTICLES

2.3 Relativistic point particle

See the Appendix about the Poincaré group at the end of this chapter for the group notation used throughout this section.

The kinematical space is the quotient structure $X = \mathcal{P}/\mathcal{L}$, where $\mathcal{P}$ is the Poincaré group and the subgroup $\mathcal{L}$ is the Lorentz group. Then every point $x \in X$ is characterized by the variables $x \equiv (t(\tau), \boldsymbol{r}(\tau))$, with domains $t \in \mathbb{R}$, $\boldsymbol{r} \in \mathbb{R}^3$ as the corresponding group parameters, b and $\boldsymbol{a}$, respectively, in such a way that under the action of a group element $g \equiv (b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})$ of $\mathcal{P}$ they transform as:

$$t'(\tau) = \gamma t(\tau) + \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r}(\tau))/c^2 + b, \tag{2.113}$$

$$\boldsymbol{r}'(\tau) = R(\boldsymbol{\mu})\boldsymbol{r}(\tau) + \gamma \boldsymbol{v} t(\tau) + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r}(\tau))\boldsymbol{v} + \boldsymbol{a}, \tag{2.114}$$

and are interpreted as the time and position of the system. If, as usual, we assume that the evolution parameter τ is invariant under the group, taking the τ-derivative of (2.113) and (2.114) we get

$$\dot{t}'(\tau) = \gamma \dot{t}(\tau) + \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\dot{\boldsymbol{r}}(\tau))/c^2, \tag{2.115}$$

$$\dot{\boldsymbol{r}}'(\tau) = R(\boldsymbol{\mu})\dot{\boldsymbol{r}}(\tau) + \gamma \boldsymbol{v} \dot{t}(\tau) + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\dot{\boldsymbol{r}}(\tau))\boldsymbol{v}. \tag{2.116}$$

The velociof the point $\boldsymbol{u} = d\boldsymbol{r}/dt$ transforms between inertial observers as

$$\boldsymbol{u}'(\tau) = \frac{\dot{\boldsymbol{r}}'}{\dot{t}'} = \frac{R(\boldsymbol{\mu})\boldsymbol{u}(\tau) + \gamma \boldsymbol{v} + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{u}(\tau))\boldsymbol{v}}{\gamma(1 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\,\boldsymbol{u}(\tau)/c^2)}. \tag{2.117}$$

In this way we should obtain the transformation laws of the different time derivatives, like acceleration and so on.

If from (2.117) we calculate the absolute value of the velocity, this gives rise to:

$$u'^2 = \frac{u^2 - c^2}{\gamma^2\,(1 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\,\boldsymbol{u}(\tau)/c^2)^2} + c^2. \tag{2.118}$$

In principle, the value of the velocity of a point is unrestricted, but if $u < c$ then $u' < c$ for every inertial observer, and the same if $u > c$, $u' > c$ and also if $u = c$ it implies that $u' = c$. The relativistic description produces three different kinds of pointlike particles, according to the value of its velocity, whether the velocity u will be $u < c$, $u = c$ or $u > c$, for every inertial observer. For the cases $u = c$ and $u > c$ it is not possible to find an observer at rest with respecto to the particles, because the group parameter $\boldsymbol{v}$ of the Poincaré grup is restricted to $v < c$.

The homogeneity condition of the Lagrangian, in terms of the derivatives of the kinematical variables, reduces to three the number of degrees of freedom of the system. This leads to the general expression

$$\widetilde{L} = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}}, \tag{2.119}$$

where $T = \partial\widetilde{L}/\partial\dot{t}$ and $R_i = \partial\widetilde{L}/\partial\dot{r}_i$, will be functions of t and $\boldsymbol{r}$ and homogeneous functions of zero degree of $\dot{t}(\tau)$ and $\dot{\boldsymbol{r}}(\tau)$.

2.3.1 Free point particle

If the particle is free, the dynamical equations will be invariant under $\mathcal{P}$, and the Lagrangian will also be invariant because the Poincaré group has no non trivial exponents and the possible gauge functions associated to this group can be reduced to zero.

From the infinitesimal point of view, since $\widetilde{L}_0(t,\boldsymbol{r},\dot{t},\dot{\boldsymbol{r}})$ depends on these variables which transform according to (2.113-2.116), the different generators of the Poincaré group, when acting on functions of these variables, are:

$$H=\frac{\partial}{\partial t},\quad \boldsymbol{P}=\nabla,\quad \boldsymbol{J}=\boldsymbol{r}\times\nabla+\dot{\boldsymbol{r}}\times\nabla_{\dot{r}},\quad \boldsymbol{K}=\frac{\boldsymbol{r}}{c^2}\frac{\partial}{\partial t}+t\nabla+\frac{\dot{\boldsymbol{r}}}{c^2}\frac{\partial}{\partial \dot{t}}+\dot{t}\nabla_{\dot{r}},$$

like the Galilei case, except the generator of the Lorentz boosts $\boldsymbol{K}$, which has a different structure because the infinitesimal transformation of velocity $\delta\boldsymbol{v}$ affects, not only to the space variables $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ like the Galilei case, but also to the time variables t and $\dot{t}$.

If $H\widetilde{L}_0=0$ and $\boldsymbol{P}\widetilde{L}_0=0$, implies that $\widetilde{L}_0$ is not a function of t and $\boldsymbol{r}$. If $\boldsymbol{J}\widetilde{L}_0=0$ implies that it is a function $\dot{\boldsymbol{r}}^2$ and also of $\dot{t}$ and has to be homogeneous of degree 1 in these derivatives. Finally, if it is invariant under the Lorentz boosts $\boldsymbol{K}\widetilde{L}_0=0$, and therefore

$$\left(\frac{\dot{\boldsymbol{r}}}{c^2}\frac{\partial}{\partial \dot{t}}+\dot{t}\nabla_{\dot{r}}\right)\widetilde{L}_0=0,$$

which implies that $\widetilde{L}_0$ is an arbitrary function of $c^2\dot{t}^2-\dot{\boldsymbol{r}}^2$. The condition of homogeneity of degree 1 in these derivatives and that it has dimension of action implies that a possibility is $s\sqrt{c^2\dot{t}^2-\dot{\boldsymbol{r}}^2}$, with s a parameter of dimensions of mass×velocity, for instance mc.

Because the Lagrangian is invariant under $\mathcal{P}$, the functions T and $\boldsymbol{R}$ transform under the group $\mathcal{P}$ in the form:

$$T' = \gamma T-\gamma(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{R}), \tag{2.120}$$

$$\boldsymbol{R}' = R(\boldsymbol{\mu})\boldsymbol{R}-\gamma\boldsymbol{v}T/c^2+\frac{\gamma^2}{1+\gamma}(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{R})\boldsymbol{v}/c^2. \tag{2.121}$$

We thus see that T and $\boldsymbol{R}$ are invariant under translations and therefore they must be functions independent of t and $\boldsymbol{r}$.

The conjugate momenta of the generalized variables $q_i=r_i$ are $p_i=\partial\widetilde{L}/\partial\dot{r}_i$, and consequently Noether's theorem leads to the following constants of the motion, that are calculated similarly as in the Galilei case except for the invariance under pure Lorentz transformations. We have now no gauge function and the variations are $\delta t=\boldsymbol{r}\cdot\delta\boldsymbol{v}/c^2$, $M_i=r_i/c^2$ and $\delta\boldsymbol{r}=t\delta\boldsymbol{v}$, $M_{ij}=t\delta_{ij}$ and thus we get:

$$\text{temporal momentum}\quad H = -T, \tag{2.122}$$

$$\text{linear momentum}\quad \boldsymbol{P} = \boldsymbol{R}=\boldsymbol{p}, \tag{2.123}$$

$$\text{kinematical momentum}\quad \boldsymbol{K} = H\boldsymbol{r}/c^2-\boldsymbol{P}t, \tag{2.124}$$

$$\text{angular momentum}\quad \boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P}. \tag{2.125}$$

The energy (temporal momentum) and the linear momentum transform as:

$$H'(\tau) = \gamma H(\tau)+\gamma(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{P}(\tau)), \tag{2.126}$$

$$\boldsymbol{P}'(\tau) = R(\boldsymbol{\mu})\boldsymbol{P}(\tau)+\frac{\gamma\boldsymbol{v}}{c^2}H(\tau)+\frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{P}(\tau))\boldsymbol{v}. \tag{2.127}$$

They transform like the contravariant components of a four-vector $P^\mu\equiv(H/c,\boldsymbol{P})$. The observables $c\boldsymbol{K}$ and $\boldsymbol{J}$ are the essential components of the antisymmetric tensor $J^{\mu\nu}=-J^{\nu\mu}=x^\mu P^\nu-x^\nu P^\mu$, $cK_i=J^{i0}$ and $J_k=\epsilon_{kil}J^{il}/2$.

Taking the τ derivative of the kinematical momentum, $\dot{\boldsymbol{K}} = 0$, we get $\boldsymbol{P} = H\dot{\boldsymbol{r}}/c^2\dot{t} = H\boldsymbol{u}/c^2$, where $\boldsymbol{u} = \dot{\boldsymbol{r}}/\dot{t}$ is the velocity of the particle and the point $\boldsymbol{r}$ represents both the center of mass and center of charge position of the particle.

The six conditions $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$, imply $\boldsymbol{u} = 0$ and $\boldsymbol{r} = 0$, so that the system is at rest and placed at the origin of the reference frame, similarly as in the nonrelativistic case. We again call this class of observers the center of mass observer.

From (2.126) and (2.127) we see that the magnitude $(H/c)^2 - \boldsymbol{P}^2 = (H'/c)^2 - \boldsymbol{P}'^2$ is a Poincaré invariant and a constant of the motion of dimensions (mass×velocity)^2. Since $P^2 = (H/c)^2u^2/c^2 < (H/c)^2$, if $u < c$, and it is definite positive. We write this magnitude as m^2c^2 in terms of a positive number m, the rest mass of the particle. By using the expression of $\boldsymbol{P} = H\boldsymbol{u}/c^2$, we get

$$H = \pm mc^2(1 - u^2/c^2)^{-1/2} = \pm\gamma(u)mc^2.$$

We are going to see that the sign of H, is another Poincaré invariant property of the particle

> For the center of mass observer, $\boldsymbol{P} = 0$, and thus $H = \pm mc^2$. If $H > 0$ for the center of mass observer, then from (2.126) we get that for any other observer, $H' = \gamma H \geq H > 0$, since $\gamma \geq 1$. If $H < 0$, also in this case $H' = \gamma H \leq H < 0$. The sign of H is another invariant between observers and therefore an intrinsic property of the particle. If $H > 0$ the system is called a **particle**, and **antiparticle** if $H < 0$.

The velocity $u < c$, otherwise H will be imaginary. If $u > c$ the invariant $(H/c)^2 - P^2 < 0$ and it is not possible to define the rest mass of the system. By substitution of the found expressions for T and $\boldsymbol{R}$ in (2.119), there are two possible Lagrangians for a point particle of mass m, characterized by the sign of H

$$\widetilde{L} = \mp mc\sqrt{c^2\dot{t}^2 - \dot{\boldsymbol{r}}^2}. \tag{2.128}$$

The system described by the Lagrangian (2.128) with the sign $+$, has a temporal momentum $H < 0$, and represents an antiparticle, while that of sign $-$, $H > 0$. Particles and antiparticles appear more symmetrically in the relativistic formulation.

Expansion of this Lagrangian to lowest order in u/c, in the case of positive H, we get

$$\widetilde{L} = -mc^2\dot{t} + \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}},$$

where the first term $-mc^2\dot{t}$ that can be withdrawn is just the equivalent to the Galilei internal energy term $-H_0\dot{t}$ of (2.18). The Lagrangian with $H < 0$ has as nonrelativistic limit $-(m/2)\dot{\boldsymbol{r}}^2/\dot{t}$ which is not obtained in the Galilei case. See section **2.7** to analyze the difference between particles and antiparticles, as far as its interaction properties are concerned.

The spin of this system, defined as the angular momentum with respect to the point $\boldsymbol{r}$, is

$$\boldsymbol{S} \equiv \boldsymbol{J} - \boldsymbol{q} \times \boldsymbol{P} = \boldsymbol{J} - \frac{c^2}{H}\boldsymbol{K} \times \boldsymbol{P} = 0, \tag{2.129}$$

vanishes, so that the relativistic point particle is also a spinless system.

2.4 Relativistic spinning particles

There are three maximal homogeneous spaces of $\mathcal{P}$, all of them at first parameterized by the variables $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho})$, where the velocity variable $\boldsymbol{u}$ can be either $u < c$, $u = c$ or $u > c$. We shall call these kinds of particles by the following names: The first one, since the motion of the position of the charge $\boldsymbol{r}$ satisfies $u < c$, we call a **Bradyon**, from the Greek term $\beta\rho\alpha\delta\upsilon\varsigma \equiv$ slow. Bradyons are thus particles for which point $\boldsymbol{r}$ never reaches the speed of light. The second class

of particles ($u = c$) will be called **Luxons** because point $\boldsymbol{r}$ is always moving at the speed of light for every observer, and finally those of the third group, because $u > c$, are called **Tachyons**, from the Greek $\tau\alpha\chi\upsilon\varsigma \equiv$ fast.

For the second class we use the Latin denomination Luxons in spite of the Greek one of photons, because this class of particles will supply the description not only of classical photons but also a classical model of the electron. This class of models is very important and it has no nonrelativistic limit. Therefore the models this manifold produce have no nonrelativistic equivalent.

The first class corresponds to a kinematical space that is the Poincaré group itself and produces models equivalent to the ones analyzed in the non-relativistic case. Readers interested on these models should go through the book by the author. To describe the classical electron and the photon we shall consider next the case of luxons.

2.5 Luxons

Let us consider those elementary particles whose kinematical space is the manifold X generated by the variables $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho})$ with domains $t \in \mathbb{R},\ \boldsymbol{r} \in \mathbb{R}^3,\ \boldsymbol{\rho} \in \mathbb{R}^3_c$ as in the previous case, and $\boldsymbol{u} \in \mathbb{R}^3$ but now with $u = c$. Since $u = c$ we shall call this kind of particles **Luxons**. This manifold is in fact a homogeneous space of the Poincaré group $\mathcal{P}$, and therefore, according to our definition of elementary particle has to be considered as a possible candidate for describing the kinematical space of an elementary system. In fact, if we consider the point in this manifold $x \equiv (0, 0, \boldsymbol{u}, 0)$, the little group that leaves x invariant is the one-parameter subgroup $\mathcal{V}_u$ of pure Lorentz transformations in the direction of the vector $\boldsymbol{u}$. Then $X \sim \mathcal{P}/\mathcal{V}_u$, is a nine-dimensional homogeneous space.

For this kind of systems the variables t, $\boldsymbol{r}$ transform according to (2.113) and (2.114), respectively and the derivatives as in (2.115) and (2.116). For the velocity $\boldsymbol{u}$ the transformation is obtained from the quotient of (2.116) by (2.115) and is

$$\boldsymbol{u}'(\tau) = \frac{R(\boldsymbol{\mu})\boldsymbol{u}(\tau) + \gamma\boldsymbol{v} + \dfrac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{u}(\tau))\boldsymbol{v}}{\gamma(1 + \boldsymbol{v}\cdot R(\boldsymbol{\mu})\,\boldsymbol{u}(\tau)/c^2)}. \tag{2.130}$$

From here we obtain that

$$u'^2 = \frac{u^2 - c^2}{\gamma^2\left(1 + \boldsymbol{v}\cdot R\boldsymbol{u}/c^2\right)^2} + c^2,$$

and thus if $u = c$ for some observer, this implies $u' = c$, for any other one, so that the manifold is a homogeneous space of $\mathcal{P}$.

The general transformation of the orientation variables $\boldsymbol{\rho}$ are obtained from (2.259) but now the functions $\boldsymbol{F}$ and G, which involve some $\gamma(u)$ factors, become infinite and in the limit $u \to c$ they take the form

$$\boldsymbol{\rho}'(\tau) = \frac{\boldsymbol{\mu} + \boldsymbol{\rho}(\tau) + \boldsymbol{\mu}\times\boldsymbol{\rho}(\tau) + \boldsymbol{F}_c(\boldsymbol{v},\boldsymbol{\mu};\boldsymbol{u}(\tau),\boldsymbol{\rho}(\tau))}{1 - \boldsymbol{\mu}\cdot\boldsymbol{\rho}(\tau) + G_c(\boldsymbol{v},\boldsymbol{\mu};\boldsymbol{u}(\tau),\boldsymbol{\rho}(\tau))}, \tag{2.131}$$

where the functions $\boldsymbol{F}_c$ and G_c are given now by:

$$\begin{aligned}\boldsymbol{F}_c(\boldsymbol{v},\boldsymbol{\mu};\boldsymbol{u},\boldsymbol{\rho}) &= \frac{\gamma(v)}{(1+\gamma(v))c^2}\left[\boldsymbol{u}\times\boldsymbol{v} + \boldsymbol{u}(\boldsymbol{v}\cdot\boldsymbol{\mu}) + \boldsymbol{v}(\boldsymbol{u}\cdot\boldsymbol{\rho})\right.\\ &+ \boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{\mu}) + (\boldsymbol{u}\times\boldsymbol{\rho})\times\boldsymbol{v}\ + (\boldsymbol{u}\cdot\boldsymbol{\rho})(\boldsymbol{v}\times\boldsymbol{\mu})\\ &+ \left.(\boldsymbol{u}\times\boldsymbol{\rho})(\boldsymbol{v}\cdot\boldsymbol{\mu}) + (\boldsymbol{u}\times\boldsymbol{\rho})\times(\boldsymbol{v}\times\boldsymbol{\mu})\right],\end{aligned} \tag{2.132}$$

$$G_c(\boldsymbol{v},\boldsymbol{\mu};\boldsymbol{u},\boldsymbol{\rho}) = \frac{\gamma(v)}{(1+\gamma(v))c^2}\left[\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{\mu})+\boldsymbol{v}\cdot(\boldsymbol{u}\times\boldsymbol{\rho})\right.$$
$$\left. - (\boldsymbol{u}\cdot\boldsymbol{\rho})(\boldsymbol{v}\cdot\boldsymbol{\mu})+(\boldsymbol{u}\times\boldsymbol{\rho})\cdot(\boldsymbol{v}\times\boldsymbol{\mu})\right]. \tag{2.133}$$

Since $u' = u = c$, the absolute value of the velocity vector is conserved and it means that $\boldsymbol{u}'$ can be obtained from $\boldsymbol{u}$ by an orthogonal transformation, so that the transformation equations of the velocity under $\mathcal{P}$, (2.130) can be expressed as:

$$\boldsymbol{u}' = R(\boldsymbol{\phi})\boldsymbol{u}, \tag{2.134}$$

where the kinematical rotation of parameter $\boldsymbol{\phi}$ is

$$\boldsymbol{\phi} = \frac{\boldsymbol{\mu}+\boldsymbol{F}_c(\boldsymbol{v},\boldsymbol{\mu};\boldsymbol{u}(\tau),0)}{1+G_c(\boldsymbol{v},\boldsymbol{\mu};\boldsymbol{u}(\tau),0)} = \frac{\boldsymbol{\mu}+\frac{\gamma}{(1+\gamma)c^2}[\boldsymbol{u}\times\boldsymbol{v}+\boldsymbol{u}(\boldsymbol{v}\cdot\boldsymbol{\mu})+\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{\mu})]}{1+\frac{\gamma}{(1+\gamma)c^2}[\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{\mu})]}. \tag{2.135}$$

In this case there also exist among the kinematical variables the constraints $\boldsymbol{u} = \dot{\boldsymbol{r}}/\dot{t}$.

Equation (2.131) also corresponds to

$$R(\boldsymbol{\rho}') = R(\boldsymbol{\phi})R(\boldsymbol{\rho}), \quad \boldsymbol{\rho}' = \frac{\boldsymbol{\phi}+\boldsymbol{\rho}-\boldsymbol{\phi}\times\boldsymbol{\rho}}{1-\boldsymbol{\phi}\cdot\boldsymbol{\rho}}, \tag{2.136}$$

with so that the three unit vectors $\boldsymbol{e}_i'$ which define by columns the rotation matrix $R(\boldsymbol{\rho}')$, and $\boldsymbol{e}_i$ those which corresponds to the rotation matrix $R(\boldsymbol{\rho})$, transform with the same rotation as the velocity $\boldsymbol{u}$,

$$\boldsymbol{e}_i' = R(\boldsymbol{\phi})\boldsymbol{e}_i, \quad i = 1,2,3,$$

with the same $\boldsymbol{\phi}$ in both cases, as in (2.135).
Since the variable $u(\tau) = c$, during the whole evolution, we can distinguish two different kinds of systems, because, by taking the derivative with respect to τ of this expression $\dot{\boldsymbol{u}}(\tau)\cdot\boldsymbol{u}(\tau) = 0$, *i.e.*, systems for which $\dot{\boldsymbol{u}} = 0$ or massless systems as we shall see, and systems where $\dot{\boldsymbol{u}} \neq 0$ but always orthogonal to $\boldsymbol{u}$. These systems will correspond to massive particles whose charge internal motion occurs at the constant velocity c, although their center of mass moves with velocity below c. This kind of particles are consistent with the analysis performed in the preamble for elementary objects whose center of charge and center of mass are two different points.

2.5.1 Massless particles. (The photon)

If $\dot{\boldsymbol{u}} = 0$, $\boldsymbol{u}$ is constant and the system follows a straight trajectory with constant velocity, and therefore the kinematical variables reduce simply to $(t,\boldsymbol{r},\boldsymbol{\rho})$ with domains and physical meaning as usual as, time, position and orientation, respectively. The derivatives $\dot{t}$ and $\dot{\boldsymbol{r}}$ transform like (2.115) and (2.116) and instead of the variable $\dot{\boldsymbol{\rho}}$ we shall consider the linear function $\boldsymbol{\omega}$ defined in (2.35) that transforms under $\mathcal{P}$:

$$\boldsymbol{\omega}'(\tau) = R(\boldsymbol{\phi})\boldsymbol{\omega}(\tau), \tag{2.137}$$

where, again, $\boldsymbol{\phi}$ is given by (2.135).

In fact, from (2.136), since $\dot{\boldsymbol{u}} = 0$, taking the τ-derivative,

$$\dot{R}(\boldsymbol{\rho}') = R(\boldsymbol{\phi})\dot{R}(\boldsymbol{\rho}),$$

the antisymmetric matrix $\Omega = \dot{R}(\boldsymbol{\rho})R^T(\boldsymbol{\rho})$ has as essential components the angular velocity $\boldsymbol{\omega}$,

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}. \tag{2.138}$$

It transforms as

$$\Omega' = \dot{R}(\boldsymbol{\rho}')R^T(\boldsymbol{\rho}') = R(\boldsymbol{\phi})\dot{R}(\boldsymbol{\rho})R^T(\boldsymbol{\rho})R^T(\phi) = R(\boldsymbol{\phi})\Omega R^T(\phi),$$

and this matrix transformation leads for its essential components to (2.137).

For this system there are no constraints among the kinematical variables, and, since $\dot{\boldsymbol{u}} = 0$, the general form of its Lagrangian is

$$\widetilde{L} = T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}} + \boldsymbol{W}\cdot\boldsymbol{\omega}. \tag{2.139}$$

Funtions $T = \partial\widetilde{L}/\partial\dot{t}$, $R_i = \partial\widetilde{L}/\partial\dot{r}^i$, $W_i = \partial\widetilde{L}/\partial\omega^i$, will depend on the variables $(t, \boldsymbol{r}, \boldsymbol{\rho})$ and are homogeneous functions of zero degree in terms of the derivatives of the kinematical variables $(\dot{t}, \dot{\boldsymbol{r}}, \boldsymbol{\omega})$. Since $\dot{t} \neq 0$ they will be expressed in terms of $\boldsymbol{u} = \dot{\boldsymbol{r}}/\dot{t}$ and $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$, which are the true velocity and angular velocity of the particle respectively.

Invariance of the Lagrangian under $\mathcal{P}$ leads to the following transformation form of these functions under the group $\mathcal{P}$:

$$T' = \gamma T - \gamma(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{R}), \tag{2.140}$$

$$\boldsymbol{R}' = R(\boldsymbol{\mu})\boldsymbol{R} - \gamma\boldsymbol{v}T/c^2 + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{R})\boldsymbol{v}, \tag{2.141}$$

$$\boldsymbol{W}' = R(\boldsymbol{\phi})\boldsymbol{W}. \tag{2.142}$$

They are translation invariant and therefore independent of t and $\boldsymbol{r}$. They will be functions of only $(\boldsymbol{\rho}, \boldsymbol{u}, \boldsymbol{\Omega})$, with the constraint $u = c$. Invariance under rotations forbids the explicit dependence on $\boldsymbol{\rho}$, so that the dependence of these functions on $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ variables is only through the angular velocity $\boldsymbol{\omega}$.

Noether's theorem gives rise, as before, to the following constants of the motion:

$$\begin{aligned}
\text{temporal momentum} \quad & H &=& -T, & (2.143)\\
\text{linear momentum} \quad & \boldsymbol{P} &=& \boldsymbol{R}, & (2.144)\\
\text{kinematical momentum} \quad & \boldsymbol{K} &=& H\boldsymbol{r}/c^2 - \boldsymbol{P}\,t - \boldsymbol{W}\times\boldsymbol{u}/c^2, & (2.145)\\
\text{angular momentum} \quad & \boldsymbol{J} &=& \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{W}. & (2.146)
\end{aligned}$$

In this case the system has no zitterbewegung term $\boldsymbol{u}\times\boldsymbol{U}$, because the Lagrangian does not depend on $\dot{\boldsymbol{u}}$ and $\boldsymbol{U}$ vanishes. The particle, located at point $\boldsymbol{r}$, is moving in a straight trajectory at the speed of light and therefore it is not possible to find an inertial rest frame observer. Although we have no center of mass observer, we define the spin as the angular momentum with respect to the point $\boldsymbol{r}$ by $\boldsymbol{S} = \boldsymbol{J} - \boldsymbol{r}\times\boldsymbol{P} = \boldsymbol{W}$.

If we take in (2.146) the τ-derivative we get $d\boldsymbol{S}/dt = \boldsymbol{P}\times\boldsymbol{u}$. Since $\boldsymbol{P}$ and $\boldsymbol{u}$ are two non-vanishing constant vectors, then the spin has a constant time derivative. It represents a particle with a continuously increasing angular momentum. This is not what we understand by an elementary particle except if this constant $d\boldsymbol{S}/dt = 0$. Therefore for this system the spin is a constant of the motion and $\boldsymbol{P}$ and $\boldsymbol{u}$ must be collinear vectors.

Energy (temporal momentum) and linear momentum are in fact the components of a four-vector and with the spin they transform as

$$H' = \gamma H + \gamma(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{P}), \tag{2.147}$$

$$\boldsymbol{P}' = R(\boldsymbol{\mu})\boldsymbol{P} + \gamma\boldsymbol{v}H/c^2 + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{P})\boldsymbol{v}, \tag{2.148}$$

$$\boldsymbol{S}' = R(\boldsymbol{\phi})\boldsymbol{S}. \tag{2.149}$$

The relation between $\boldsymbol{P}$ and $\boldsymbol{u}$ can be obtained from (2.145), taking the τ-derivative and the condition that the spin $\boldsymbol{W}$ is constant, $\dot{\boldsymbol{K}} = 0 = -H\dot{\boldsymbol{r}}/c^2 + \boldsymbol{P}\dot{t}$, *i.e.*, $\boldsymbol{P} = H\boldsymbol{u}/c^2$. If we take the scalar product of this expression with $\boldsymbol{u}$ we also get $H = \boldsymbol{P}\cdot\boldsymbol{u}$.

Then, from (2.147) and (2.148), an invariant and constant of the motion, which vanishes, is $(H/c)^2 - \boldsymbol{P}^2$. The mass of this system is zero. It turns out that for this particle both H and $\boldsymbol{P}$ are non-vanishing for every inertial observer. Otherwise, if one of them vanishes for a single observer they vanish for all of them. By (2.149), S^2 is another Poincaré invariant property of the system that is also a constant of the motion.

The first part of the Lagrangian $T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}} = -H\dot{t} + \boldsymbol{P}\cdot\dot{\boldsymbol{r}}$, which can be written as $-(H - \boldsymbol{P}\cdot\boldsymbol{u})\dot{t} = 0$, also vanishes. Then the Lagrangian is reduced to the third term $\boldsymbol{S}\cdot\boldsymbol{\omega}$. A massles particle moving along a straight line at the speed of light, necessarily has to depend on extra orientation variables, otherwise $\widetilde{L} = 0$, i.e., **photons necessarily rotate**. The relativistic formulation forbids the existence of massive point particles moving along a straight line at the speed of light.

We see from (2.134) and (2.149) that the dimensionless magnitude $\epsilon = \boldsymbol{S}\cdot\boldsymbol{u}/Sc$ is another invariant and constant of the motion, and we thus expect that the Lagrangian will be explicitly dependent on both constant parameters S and ϵ. Taking into account the transformation properties under $\mathcal{P}$ of $\boldsymbol{u}$, $\boldsymbol{\omega}$ and $\boldsymbol{S}$, given in (2.134), (2.137) and (2.149) respectively, it turns out that the spin must necessarily be a vector function of $\boldsymbol{u}$ and $\boldsymbol{\omega}$.

If the spin is not transversal, as it happens for real photons, then $\boldsymbol{S} = \epsilon\, S\boldsymbol{u}/c$ where $\epsilon = \pm 1$, and thus the free Lagrangian finally becomes:

$$\widetilde{L} = \left(\frac{\epsilon\, S}{c}\right)\frac{\dot{\boldsymbol{r}}\cdot\boldsymbol{\omega}}{\dot{t}}. \tag{2.150}$$

From this Lagrangian the temporal momentum is $H = -\partial\widetilde{L}/\partial\dot{t} = \boldsymbol{S}\cdot\boldsymbol{\Omega}$, where $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$ is the angular velocity of the particle. The linear momentum is $\boldsymbol{P} = \partial\widetilde{L}/\partial\dot{\boldsymbol{r}} = \epsilon\, S\,\boldsymbol{\Omega}/c$, and, since $\boldsymbol{P}$ and $\boldsymbol{u}$ are parallel vectors, $\boldsymbol{\Omega}$ and $\boldsymbol{u}$ must also be parallel, and if the energy is definite positive, then $\boldsymbol{\Omega} = \epsilon\Omega\boldsymbol{u}/c$.

This means that the energy $H = S\Omega$. For photons we know that $S = \hbar$, and thus $H = \hbar\Omega = h\nu$. In this way the frequency of a photon is the frequency of its rotational motion around the direction of its trajectory. We thus see that the spin and angular velocity for $H > 0$ particles have the same direction, although they are not analytically related, because S is invariant under $\mathcal{P}$ while Ω is not. When we change of inertial observer the spin remains the same while the frequency experiences the Doppler effect.

> If the laboratory observer O_L sees another inertial observer O moving with velocity $\boldsymbol{v}$ both with the axes parallel, the relationship between them os given by a boost $L(\boldsymbol{v})$ and perhaps some space and time translation. For the relationship between the energy and linear momentum measurements of a photon only involves the boost $L(\boldsymbol{v})$, which is the Jacobian of the transformation. If observer O emits photons of frequency ν, the measurements performed by O_L are given by
>
> $$H_L = \gamma H + \gamma\boldsymbol{v}\cdot\boldsymbol{p}, \quad \boldsymbol{p}_L = \boldsymbol{p} + \gamma H\boldsymbol{v}/c^2 + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot\boldsymbol{p})\boldsymbol{v}.$$
>
> Since $H = h\nu$, $\boldsymbol{p} = H\boldsymbol{u}/c^2$, for the frequency mesaured in O_L we get
>
> $$\nu_L = \gamma\nu(1 + \boldsymbol{v}\cdot\boldsymbol{u}/c^2)$$
>
> where $\boldsymbol{u}$ is the velocity of the del photon measured by O. Let us assume that the origin of O departs from O_L. The photons which arrive to O_L are those such that $\boldsymbol{v}\cdot\boldsymbol{u} = -vc$, and the detected frequency is
>
> $$\nu_L = \gamma\nu(1 - v/c) = \nu\sqrt{\frac{1-v/c}{1+v/c}}, \quad \nu_L < \nu.$$

However if O_L sees O approaching, the photons which measures are those for which $\boldsymbol{v}\cdot\boldsymbol{u} = vc$, and the detected frequency is larger:

$$\nu_L = \gamma\nu(1+v/c) = \nu\sqrt{\frac{1+v/c}{1-v/c}}, \quad \nu_L > \nu.$$

We say that the Lagrangian (2.150) represents a photon of spin S and polarization ϵ. A set of photons of this kind, all with the same polarization, corresponds to circularly polarized light, as has been shown by direct measurement of the angular momentum carried by these photons.

Beth's experiment

Beth's experiment [5] performed in 1936, consists in producing a beam of monocromatic circularly polarized light of frequency ν, which is sent into a plate attached to a torsion pendulum (see figure **2.7**). Photons are absorbed by the plate, and therefore energy and linear momentum are transfered, and also angular momentum. If the power of the beam is P the beam contains $n = P/h\nu$ photons per second, all of them with the spin in the same direction. When absorbed, the angular momentum of the plate J, with respect to the point O, changes with time as $dJ/dt = n\hbar$ per second, so that the torque M of the external forces produced by the torsion of the pendulum must equilibrate this variation.

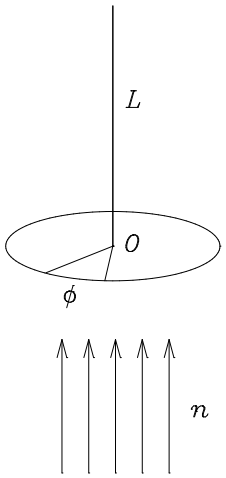


Figure 2.7: **A monocromatic circularly polarized light beam of intensity n photons per second, are absorbed by a plate which is in equilibrium by means of a torsion pendulum. When photons are absorbed, the plate rotates an angle ϕ, to the left or right, according to the left or right polarization of the beam. This experiment verifies that all photons of the beam have the same spin orientation**

For a thread of length L, radius a and torsion modulus μ, the torque of the external forces which produce a rotation of angle ϕ, is:

$$M = \mu\frac{\pi a^4}{2L}\phi.$$

By measuring the angle ϕ rotated by the plate, one checks in this experiment that $M = n\hbar$, because each absorbed photon contributes with an angular momentum of value $\hbar$.

[5] R. A. Beth, *Mechanical detection and measurement of the angular momentum of light*, *Phys. Rev.* **50**, 115 (1936).

But this Beth's device is also an analyzer of the photon polarization. If the circularly polarized light beam turns the plate to the right, the beam contains photons with their spins pointing forward. If the disc rotates left, all the photons of the beam have their spins pointing backward.

Left and right polarized photons correspond to $\epsilon = 1$ and $\epsilon = -1$, respectively. Energy is related to the angular frequency $H = \hbar\Omega$, and linear momentum to the wave number $\boldsymbol{P} = \hbar\boldsymbol{k}$, that therefore is related to the angular velocity vector by $\boldsymbol{k} = \epsilon\boldsymbol{\Omega}/c$. If it is possible to talk about the 'wave-length' of a single photon this will be the distance run by the particle during a complete turn.

The antiphotons, i.e., those particles for which $H < 0$, they satisfy $H = \boldsymbol{S}\cdot\boldsymbol{\Omega} = \boldsymbol{p}\cdot\boldsymbol{u} < 0$ and therefore the spin and and the angular velocity have opposite direction and the same happens for the velocity and linear momentum. In any case they have the same energy than the photons with $H > 0$. To determine whether a material system absorbs a photon or an antiphoton we have to measure separately the velocity of the photon and the linear momentum, which have to be opposite to each other. It seems that the radiation of normal matter produces photons, because the radiation preasure has the direction of the motion, and thus linear momentum and velicity are parallel. In the electron-positron interaction, in order to the particles approach to each other by means of an interchange of a photonic particle, this has to be an antiphoton. However in the electron-electron interaction the particles separate from each other and they interchange a photon. See the section **2.7** for the analysis of particles and antiparticles.

The relationship between the different observables for the photon ($H > 0$) and the antiphoton ($H < 0$) is represented in the figure 2.8

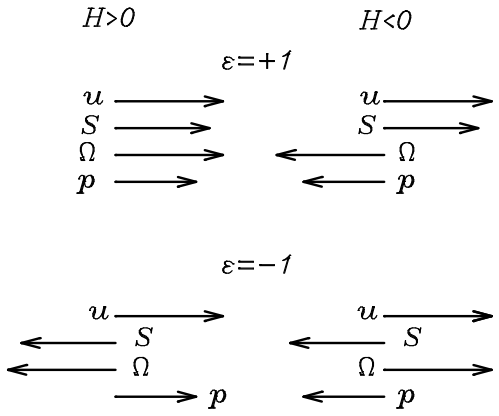


Figure 2.8: **Relative orientation between the different observables u, S, Ω y p, for the photon $H > 0$ on the left hand side column and for the antiphoton $H < 0$ on the right hand side, for the two possible helicities $\epsilon = \pm 1$. $\boldsymbol{S} = \epsilon S\boldsymbol{u}/c$, $\boldsymbol{p} = H\boldsymbol{u}/c^2 = \epsilon S\boldsymbol{\Omega}/c$.**

If the possible states of a photon are represented in vector form like $|\,\mathrm{sign}(H), \epsilon >$, the states represented on the left of the figure are $|+,+>$ and $|+,->$, and those of the right by $|-,+>$ and $|-,->$, respectively. They are independent and orthogonal states. If the radiation field is only composed of photons ($H > 0$), then the classical description of the vector states of the monochromatic light is given by complex vectors of the two-dimensional complex space $\mathbb{C}^2$ and the different polarized states by the Poincaré sphere, as a convex linear combination of pure

states (See Appendix **3.4**). See the section **2.7** for the analysis and detection of antiphotons.

2.5.2 Massive particles. (The electron)

If we consider now the other possibility, $\dot{\boldsymbol{u}} \neq 0$ but orthogonal to $\boldsymbol{u}$, then variables $\dot{t}$ and $\dot{\boldsymbol{r}}$ transform as in the previous case (2.115) and (2.116), but for $\dot{\boldsymbol{u}}$ and $\boldsymbol{\omega}$ we have:

$$\dot{\boldsymbol{u}}' = R(\boldsymbol{\phi})\dot{\boldsymbol{u}} + \dot{R}(\boldsymbol{\phi})\boldsymbol{u}, \tag{2.151}$$

$$\boldsymbol{\omega}' = R(\boldsymbol{\phi})\boldsymbol{\omega} + \boldsymbol{\omega}_\phi, \tag{2.152}$$

where the rotation of parameter $\boldsymbol{\phi}$ is again given by (2.135) and vector $\boldsymbol{\omega}_\phi$ is:

$$\boldsymbol{\omega}_\phi = \frac{\gamma R\boldsymbol{u} \times \boldsymbol{v} - (\gamma - 1)R(\boldsymbol{u} \times \dot{\boldsymbol{u}}) + 2\gamma^2(\boldsymbol{v} \cdot R(\boldsymbol{u} \times \dot{\boldsymbol{u}}))\boldsymbol{v}/(1+\gamma)c^2}{\gamma(c^2 + \boldsymbol{v} \cdot R\boldsymbol{u})}. \tag{2.153}$$

Expression (2.151) is the τ-derivative of (2.134) and can also be written in the form:

$$\dot{\boldsymbol{u}}' = \frac{R(\boldsymbol{\phi})\dot{\boldsymbol{u}}}{\gamma(1 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{u}/c^2)}. \tag{2.154}$$

Expression (2.152) comes from $R(\boldsymbol{\rho}') = R(\boldsymbol{\phi})R(\boldsymbol{\rho})$ and taking the τ-derivative of this expression $\dot{R}(\boldsymbol{\rho}') = \dot{R}(\boldsymbol{\phi})R(\boldsymbol{\rho}) + R(\boldsymbol{\phi})\dot{R}(\boldsymbol{\rho})$, because parameter $\boldsymbol{\phi}$ depends on τ through the velocity $\boldsymbol{u}(\tau)$, and therefore

$$\Omega' = \dot{R}(\boldsymbol{\rho}')R^T(\boldsymbol{\rho}') = R(\boldsymbol{\phi})\Omega R^T(\boldsymbol{\phi}) + \dot{R}(\boldsymbol{\phi})R^T(\boldsymbol{\phi}).$$

$R(\boldsymbol{\phi})\Omega R^T(\boldsymbol{\phi})$ corresponds to $R(\boldsymbol{\phi})\boldsymbol{\omega}$ and the antisymmetric matrix $\Omega_\phi = \dot{R}(\boldsymbol{\phi})R^T(\boldsymbol{\phi})$ has as essential components the $\boldsymbol{\omega}_\phi$ vector, *i.e.*, equation (2.153).

The homogeneity condition of the Lagrangian leads to the general form

$$\widetilde{L} = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{U} \cdot \dot{\boldsymbol{u}} + \boldsymbol{W} \cdot \boldsymbol{\omega}, \tag{2.155}$$

where $T = \partial\widetilde{L}/\partial\dot{t}$, $R_i = \partial\widetilde{L}/\partial\dot{r}^i$, $U_i = \partial\widetilde{L}/\partial\dot{u}^i$ and $W_i = \partial\widetilde{L}/\partial\omega^i$, and Noether's theorem provides the following constants of the motion:

$$\text{temporal momentum} \quad H = -T - (d\boldsymbol{U}/dt) \cdot \boldsymbol{u}, \tag{2.156}$$

$$\text{linear momentum} \quad \boldsymbol{P} = \boldsymbol{R} - (d\boldsymbol{U}/dt), \tag{2.157}$$

$$\text{kinematical momentum} \quad \boldsymbol{K} = H\boldsymbol{r}/c^2 - \boldsymbol{P}\,t - \boldsymbol{S} \times \boldsymbol{u}/c^2, \tag{2.158}$$

$$\text{angular momentum} \quad \boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{S}. \tag{2.159}$$

In this case the spin $\boldsymbol{S}$, i.e. the angular momentum with respecto to the point $\boldsymbol{r}$, is defined as in the Galilei case, by

$$\boldsymbol{S} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W} = \boldsymbol{Z} + \boldsymbol{W}. \tag{2.160}$$

Like in the Galilei case, we also have invariance of dynamical equations under the local rotations group $SO(3)_L$ which affect only to the orientation variables without modification of the angular velocity, like in (2.51). We obtain another three constants of the motion

$$\text{Local angular Momentum} \quad T_i = \boldsymbol{W} \cdot \boldsymbol{e}_i, \tag{2.161}$$

which are the projection on the body axes of the rotative part of the spin $\boldsymbol{W}$.

Expressions (2.156, 2.157) imply that H/c and $\boldsymbol{P}$ transform like the components of a four-vector, similarly as in (2.126-2.127), thus defining the invariant and constant of the motion $(H/c)^2 - \boldsymbol{P}^2 = m^2c^2$, in terms of the positive parameter m which is interpreted as the mass of the particle.

Observable $\boldsymbol{S}$ transforms as:

$$\boldsymbol{S}'(\tau) = \gamma R(\boldsymbol{\mu})\boldsymbol{S}(\tau) - \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{S}(\tau))\boldsymbol{v} + \frac{\gamma}{c^2}(\boldsymbol{v}\times R(\boldsymbol{\mu})(\boldsymbol{S}(\tau)\times\boldsymbol{u})), \tag{2.162}$$

an expression that corresponds to the transformation of an antisymmetric tensor $S^{\mu\nu}$ with strict components $S^{0i} = (\boldsymbol{S}\times\boldsymbol{u})^i/c$, and $S^{ij} = \epsilon^{ijk}S_k$, which transform under the Poincaré group as,

$$S'^{\mu\nu} = \Lambda^\mu_\sigma(\boldsymbol{v},\boldsymbol{\mu})\Lambda^\nu_\lambda(\boldsymbol{v},\boldsymbol{\mu})S^{\sigma\lambda}, \quad \Lambda(\boldsymbol{v},\boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu})$$

If we define the vector $\boldsymbol{k} = \boldsymbol{S}\times\boldsymbol{u}/H$, with dimensions of length, the kinematical momentum (2.158) can be rewritten as

$$\boldsymbol{K} = H\boldsymbol{q}/c^2 - \boldsymbol{P}\,t,$$

where $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k}$, represents the center of mass position of the particle. The time derivative of this expression we obtain a linear relationship between H and $\boldsymbol{P}$ as in the case of the point particle:

$$\boldsymbol{P} = \frac{H}{c^2}\boldsymbol{v}, \quad \boldsymbol{v} = \frac{d\boldsymbol{q}}{dt}$$

in terms of the velocity of the center of mass $\boldsymbol{v}$. This implies that again H and $\boldsymbol{P}$ are expressed in terms of the center of mass velocity and the invariant m, like for the point particle as

$$H = \gamma(v)mc^2, \quad \boldsymbol{P} = \gamma(v)m\boldsymbol{v}.$$

If we call $\boldsymbol{v}_{CM}$ to the center of mass velocity, it transforms among inertial observers like:

$$\boldsymbol{v}'_{CM}(\tau) = \frac{R(\boldsymbol{\mu})\boldsymbol{v}_{CM}(\tau) + \gamma\boldsymbol{v} + \dfrac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{v}_{CM}(\tau))\boldsymbol{v}}{\gamma(1+\boldsymbol{v}\cdot R(\boldsymbol{\mu})\,\boldsymbol{v}_{CM}(\tau)/c^2)}. \tag{2.163}$$

Although the Poincaré transformation of the position of the center of mass does not correspond with the center of mass of the electron in the new reference system, the center of mass velocity is efectively the transformed of that velocity. If $\boldsymbol{v}_{CM} = 0$ is the velocity of the center of mass in the center of mass frame, then the velocity in another frame is $\boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity of the center of mass frame for that arbitrary observer.

The spin with respect to the center of mass, is defined as usual by

$$\boldsymbol{S}_{CM} = \boldsymbol{J} - \boldsymbol{q}\times\boldsymbol{P} = \boldsymbol{J} - \frac{c^2}{H}\boldsymbol{K}\times\boldsymbol{P}, \tag{2.164}$$

and is a constant of the motion. It takes the form

$$\boldsymbol{S}_{CM} = \boldsymbol{S} + \boldsymbol{k}\times\boldsymbol{P} = \boldsymbol{S} + \frac{1}{H}(\boldsymbol{S}\times\boldsymbol{u})\times\boldsymbol{P}. \tag{2.165}$$

The helicity $\boldsymbol{S}_{CM}\cdot\boldsymbol{P} = \boldsymbol{S}\cdot\boldsymbol{P} = \boldsymbol{J}\cdot\boldsymbol{P}$, is also a constant of the motion. We can construct the constant Pauli-Lubanski four-vector

$$w^\mu \equiv (\boldsymbol{P}\cdot\boldsymbol{S}_{CM}, H\boldsymbol{S}_{CM}/c) = (\boldsymbol{P}\cdot\boldsymbol{S}, H\boldsymbol{S}/c + (\boldsymbol{S}\times\boldsymbol{u})\times\boldsymbol{P}/c) = \frac{H}{c^2}(\boldsymbol{v}\cdot\boldsymbol{S}_{CM}, c\boldsymbol{S}_{CM}), \tag{2.166}$$

which is expressed for the free particle in terms of constants of the motion and therefore it is another constant of the motion. Its absolute value $-w^\mu w_\mu = m^2c^2S^2_{CM}$, where S_{CM} is the spin in the center of mass frame, is another constant of the motion for the free particle. If we accept the atomic principle it has to take the same value even under some external interaction. It is expressed in terms of the invariant properties, and therefore intrinsic values, m and S_{CM} of the particle, where S^2_{CM} is the absolute value squared of the $\boldsymbol{S}_{CM}$, which in the quantum case is $3\hbar^2/4$. The absolute value of this spin is not invariant and for an observer which sees the center of mass moving with velocity $\boldsymbol{v}$, takes the value:

$$S'^2_{CM} = \left(\frac{c^2 - v^2}{c^2 - v^2\cos^2\phi}\right)S^2_{CM},$$

where ϕ is the angle between $\boldsymbol{v}$ and $\boldsymbol{S}_{CM}$.

2.5.3 Motion of the electron in the CM frame

The center of mass frame is defined by the conditions $\boldsymbol{P} = \boldsymbol{K} = 0$. For this class of observers $\boldsymbol{q} = 0$ and $\boldsymbol{v} = 0$, the center of mass is at rest and located at the origin of the reference frame. The spin $\boldsymbol{S} = \boldsymbol{S}_{CM}$ is constant, $H = \pm mc^2$ and from (2.158) we obtain

$$\boldsymbol{r} = \pm \frac{1}{mc^2}\, \boldsymbol{S} \times \boldsymbol{u}. \tag{2.167}$$

This is the dynamical equation of the point $\boldsymbol{r}$ for the center of mass observer and this internal motion takes place on a plane orthogonal to the constant spin $\boldsymbol{S}$. The scalar product with $\boldsymbol{u}$ gives $\boldsymbol{r} \cdot d\boldsymbol{r}/dt = 0$, and therefore the radius of this motion (the zitterbewegung) is constant. Taking the time derivative of both sides of (2.167), we get $mc^2\boldsymbol{u} = \pm(\boldsymbol{S} \times d\boldsymbol{u}/dt)$, because the spin is constant in this frame, which implies that $\boldsymbol{u}$ and $\boldsymbol{S}$ are orthogonal. If we derivate again this expression we conclude that $d\boldsymbol{u}/dt$ and $\boldsymbol{S}$ are also orthogonal. If we introduce in (2.167) this expression of $\boldsymbol{u}$ and taking into account the orthogonality between the spin and the acceleration, we get for the particle and antiparticle,

$$\frac{d^2\boldsymbol{r}}{dt^2} + \omega^2 \boldsymbol{r} = 0, \quad \omega = \frac{mc^2}{S}. \tag{2.168}$$

which is exactly the same equation of the Preamble (2) and of the nonrelativistic particle (2.75) when the center of mass is at rest. Taking in (2.167) the cross product with $\boldsymbol{u}$ and using the orthogonality of the spin with the velocity we arrive to

$$\boldsymbol{S} = \pm m\boldsymbol{u} \times \boldsymbol{r}. \tag{2.169}$$

Since S and $u = c$ are constant, the motion is a circle of radius $R_0 = S/mc$. For the electron we take in the quantum case $S = \hbar/2$, and the radius is $\hbar/2m_ec = 1.93 \times 10^{-13}$ m., half Compton's wave length of the electron. The frequency of this motion in the C.M. frame is $\nu = 2m_ec^2/h = 2.47 \times 10^{20}$ s^{-1}, and $\omega = 2\pi\nu = 1.55 \times 10^{21}$ rad s^{-1}. The ratio of this radius to the so-called classical radius $R_{cl} = e^2/8\pi\varepsilon_0 m_e c^2 = 1.409 \times 10^{-15}$ m, is precisely $R_{cl}/R_0 = e^2/2\varepsilon_0 hc = 1/136.97 = \alpha$, the fine structure constant. The radius of the electron, estimated from high energy $e-e$ sccatering in the experiments performed at the LEP in CERN, give the value $R_e < 10^{-19}$ m. The analysis of the measurement of the gyromagnetic ratio g in a Penning trap gives a smaller value $R_e < 10^{-22}$m.[6] If we compare with the Bohr radius, the estimated radius of a circular trajectory of an electron of the Hydrogen atom, with an orbital angular momentum $L = 1$, is $R_B = 4\pi\epsilon_0\hbar^2/m_e e^2$, $R_B/R_0 = 2/\alpha \approx 274$.

There are two different types of particles, as far as the sign of H is concerned. In both the energy is mc^2. It is called particle the object with $H > 0$ and antiparticle with $H < 0$. The kinetics of this is opposite to the other once the spin direction is fixed. Particle and antiparticle have the time reversed motion of each other. Motions of this sort, in which the particle is moving at the speed of light, can be found in early literature, but the distinction between the motion of center of charge and center of mass is not sufficiently clarified. [7, 8]

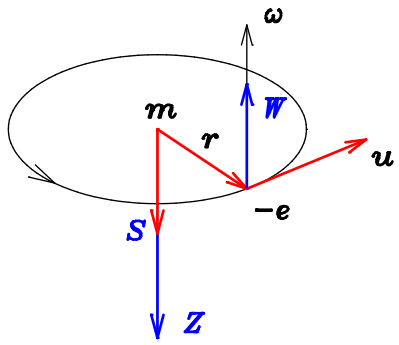
Nevertheless, in the model we are analyzing, the idea that the electron has a size of the order of the zitterbewegung radius is a plausible macroscopic vision but it is not necessary to maintain any longer, because the only important point from the dynamical point of view is the center of charge position, whose motion completely determines the dynamics of the particle.

[6] H. Dehmelt, *A single atomic particle forever floating at rest in free space: New value for Electron radius*, Physica Scripta **T22**, 102-110 (1988)

[7] M. Mathisson, *Acta Phys. Pol.* **6**, 163 (1937); **6**, 218 (1937)

[8] M.H.L. Weyssenhof, *Acta Phys. Pol.* **9**, 46 (1947). M.H.L. Weyssenhof and A. Raabe, *Acta Phys. Pol.* **9**, 7 (1947); **9**, 19 (1947).

In this form, elementary particles, the kind of objects we are describing, look like extended objects. Nevertheless, although some kind of related length can be defined, they are dealt with as point particles with orientation because the physical attributes are all located at the single point $\boldsymbol{r}$. The dynamics of equation (2.167) for the particle, can be represented in figure **2.9**, and for the antiparticle in figure **2.10**, where we have separated the two contributions to the total spin $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$, related respectively to the orbital and rotational motion.



$$S=Z+W=-\gamma(v)m(r-q)\times u$$

Figure 2.9: **Motion of the center of charge of the electron in the center of mass frame. The (anti)orbital part $\boldsymbol{Z} = \boldsymbol{u} \times \boldsymbol{U}$ of the spin has the direction of $\boldsymbol{S}$ while the part $\boldsymbol{W}$, in the direction of the angular velocity $\boldsymbol{\omega}$, has the opposite direction. The spin is orthogonal to the velocity of the center of charge and to the separation between CC and CM.**

2.5.4 The spin and the center of mass position for an arbitrary observer

If we take in (2.158) the τ-derivative and the scalar product with the velocity $\boldsymbol{u}$ we get the Poincaré invariant relation:

$$H = \boldsymbol{P} \cdot \boldsymbol{u} + \frac{1}{c^2} \boldsymbol{S} \cdot \left(\frac{d\boldsymbol{u}}{dt} \times \boldsymbol{u} \right). \tag{2.170}$$

In this way, the temporal momentum or Dirac's Hamiltonian, is the sum of two terms, one translational, related to $\boldsymbol{P}$, which vanishes for the center of mass observer, and another rotational and related to $\boldsymbol{S}$, which never vanishes. In the quantum case it will be related to $H = c\boldsymbol{P} \cdot \boldsymbol{\alpha} + \beta mc^2$, in terms of the $\boldsymbol{\alpha}$ and β Dirac matrices. Since $c\boldsymbol{\alpha}$ is usually interpreted as the local velocity operator $\boldsymbol{u}$ of the electron, [9] we have $H = \boldsymbol{P} \cdot \boldsymbol{u} + \beta mc^2$ and this relation suggests the identification

$$\beta = \frac{1}{mc^4} \boldsymbol{S} \cdot \left(\frac{d\boldsymbol{u}}{dt} \times \boldsymbol{u} \right).$$

Here all magnitudes on the right-hand side are measured in the center of mass frame. We shall come back to this relation after quantization of this system.

We are going to express the general form of the spin observable and the position of the center of mass in terms of the kinematics of the center of charge position. The transformation equation for the function $\boldsymbol{S}$, (2.162) can also be written as

$$\boldsymbol{S}' = \gamma(1 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{u}/c^2) R(\boldsymbol{\phi})\boldsymbol{S}, \tag{2.171}$$

[9] J.J. Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley Reading, MA (1967).

and therefore, from this expresion and (2.154), $\boldsymbol{S}\cdot\dot{\boldsymbol{u}}=\boldsymbol{S}'\cdot\dot{\boldsymbol{u}}'$ which vanish in any reference frame, and also from (2.134), $\boldsymbol{S}'\cdot\boldsymbol{u}'=\gamma(1+\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{u}/c^2)\boldsymbol{S}\cdot\boldsymbol{u}$, which also vanishes. Since the center of charge spin is orthogonal to $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$, for the center of mass observer, it is also orthogonal to $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ for any other inertial observer.

An alternative method of verifying this is to take the time derivative in (2.158) and (2.159), and thus

$$H\boldsymbol{u}-c^2\boldsymbol{P}-\frac{d\boldsymbol{S}}{dt}\times\boldsymbol{u}-\boldsymbol{S}\times\frac{d\boldsymbol{u}}{dt}=0,$$

$$\frac{d\boldsymbol{S}}{dt}=\boldsymbol{P}\times\boldsymbol{u},$$

i.e.,

$$\boldsymbol{S}\times\frac{d\boldsymbol{u}}{dt}=(H-\boldsymbol{u}\cdot\boldsymbol{P})\boldsymbol{u}.$$

and a final scalar product with $\boldsymbol{S}$, leads to $(H-\boldsymbol{u}\cdot\boldsymbol{P})\boldsymbol{u}\cdot\boldsymbol{S}=0$. The first factor does not vanish since the invariant $H^2/c^2-P^2=m^2c^2$ is positive definite and if $H=\boldsymbol{u}\cdot\boldsymbol{P}$, then $(\boldsymbol{u}\cdot\boldsymbol{P})^2/c^2-P^2$ with $u\leq c$ is always negative, then $\boldsymbol{S}\cdot\boldsymbol{u}=0$. If we take the time derivative of this last expression, with the condition that $d\boldsymbol{S}/dt$ is orthogonal to $\boldsymbol{u}$, we obtain $\boldsymbol{S}\cdot\dot{\boldsymbol{u}}=0$. The observable $\boldsymbol{S}$ has always the direction of the non-vanishing vector $\dot{\boldsymbol{u}}\times\boldsymbol{u}$ for positive temporal momentum particles and the opposite direction for antiparticles of negative temporal momentum.

If we take the time derivative of the kinematical momentum (2.158) for the free particle, we get

$$H\boldsymbol{u}-c^2\boldsymbol{P}+\frac{d\boldsymbol{S}}{dt}\times\boldsymbol{u}+\boldsymbol{S}\times\frac{d\boldsymbol{u}}{dt}=0.$$

Taking into account that $d\boldsymbol{S}/dt=\boldsymbol{P}\times\boldsymbol{u}$ and making a cross product with $d\boldsymbol{u}/dt$ we get

$$\boldsymbol{S}=\left(\frac{H-\boldsymbol{u}\cdot\boldsymbol{P}}{(d\boldsymbol{u}/dt)^2}\right)\frac{d\boldsymbol{u}}{dt}\times\boldsymbol{u}, \tag{2.172}$$

and $\boldsymbol{q}=\boldsymbol{r}-\boldsymbol{S}\times\boldsymbol{u}/H$ leads for the center of mass position to

$$\boldsymbol{q}=\boldsymbol{r}+\frac{c^2}{H}\left(\frac{H-\boldsymbol{u}\cdot\boldsymbol{P}}{(d\boldsymbol{u}/dt)^2}\right)\frac{d\boldsymbol{u}}{dt}. \tag{2.173}$$

The expression (2.172) of the spin with respect to the center of charge can be rewritten as

$$\boldsymbol{S}=\frac{(H\dot{t}-\boldsymbol{P}\cdot\dot{\boldsymbol{r}})}{\dot{\boldsymbol{u}}^2}\,\dot{\boldsymbol{u}}\times\boldsymbol{u}.$$

Because from (2.134) we know that $\boldsymbol{u}'=R(\boldsymbol{\phi})\boldsymbol{u}$ and from (2.154) that $\dot{\boldsymbol{u}}'=R(\boldsymbol{\phi})\dot{\boldsymbol{u}}/\gamma(1+\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{u}/c^2)$, it implies that $\dot{\boldsymbol{u}}'^2=\dot{\boldsymbol{u}}^2/\gamma^2(1+\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{u}/c^2)^2$ and the numerator $(H\dot{t}-\boldsymbol{P}\cdot\dot{\boldsymbol{r}})$ is Poincaré invariant, and from this we obtain the previous expression for the transformation of $\boldsymbol{S}$, (2.171).

From the geometrical point of view, since the vector $\boldsymbol{u}$ is tangent to the trajectory of the center of charge and its derivative is orthogonal to it, the spin with respect to the center of charge (2.172) has the direction opposite to the binormal and in the same direction for the antiparticle.

The center of mass, with respect to the center of charge, is in the direction of the acceleration for the particle and antiparticle. The point $\boldsymbol{r}$ makes a central motion around the center of mass. If from (2.173) we express the acceleration in terms of $\boldsymbol{r}-\boldsymbol{q}$, the spin with respect to the point $\boldsymbol{r}$ can also be written as

$$\boldsymbol{S}=-\frac{H}{c^2}(\boldsymbol{r}-\boldsymbol{q})\times\boldsymbol{u}=\mp\gamma(v)m(\boldsymbol{r}-\boldsymbol{q})\times\boldsymbol{u}, \tag{2.174}$$

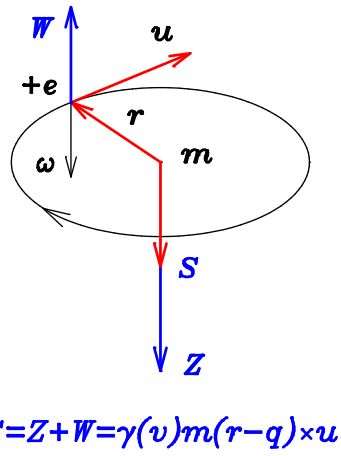


Figure 2.10: **Motion of the center of charge of the positron in the center of mass frame. The part W of the spin of the positron is oriented opposite to the angular velocity. The part Z has the usual orbital direction and therefore the same direction as S.**

which enhances its antiorbital character for the particle $H > 0$ and of orbital orientation for the antiparticle. It is expressed in terms of the two characteristic points $\boldsymbol{r}$ and $\boldsymbol{q}$ and their corresponding velocities $\boldsymbol{u}$ and v and of the positive parameter m, the mass of the particle. Because the total spin has two parts $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$, this means that for the antiparticle the part $\boldsymbol{Z}$ has the direction of $\boldsymbol{S}$ while the rotational part $\boldsymbol{W}$ has the opposite orientation, and therefore this part $\boldsymbol{W}$ is opposite to the angular velocity as depicted in the figure **2.10**. This feature is the same than for photons and antiphotons. For photons the spin is of rotational nature like $\boldsymbol{W}$, and has the same direction than the angular velocity while for antiphotons has the opposite orientation.

The spin with respect to the center of mass can be obtained from $\boldsymbol{S}$, from the mechanical linear momentum for the particle $\boldsymbol{P}_p = \gamma(v)m\boldsymbol{v}$, in the form,

$$\boldsymbol{S}_{CM_p} = \boldsymbol{S}_p + (\boldsymbol{r} - \boldsymbol{q}) \times \boldsymbol{P}_p = -\gamma(v)m(\boldsymbol{r} - \boldsymbol{q}) \times (\boldsymbol{u} - \boldsymbol{v}),$$

while for the antiparticle $\boldsymbol{P}_a = -\gamma(v)m\boldsymbol{v}$

$$\boldsymbol{S}_{CM_a} = \boldsymbol{S}_a + (\boldsymbol{r} - \boldsymbol{q}) \times \boldsymbol{P}_a = \gamma(v)m(\boldsymbol{r} - \boldsymbol{q}) \times (\boldsymbol{u} - \boldsymbol{v}).$$

From these expressions can be checked that for the free particle the spin S_{CM} is a conserved magnitude. In fact, for the free particle $\boldsymbol{v} = \text{const.}$, the time derivative of of $\boldsymbol{r} - \boldsymbol{q}$ is $\boldsymbol{u} - \boldsymbol{v}$ and the derivative of $\boldsymbol{u}$ has the direction of the vector $\boldsymbol{r} - \boldsymbol{q}$. The spin $\boldsymbol{S}$, in the case of the free particle is not conserved and its time derivative leads us to $d\boldsymbol{S}/dt = \boldsymbol{P} \times \boldsymbol{u}$, for the particle and antiparticle.

Under interaction we have,

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{P} \times \boldsymbol{u} - \frac{1}{c^2}\gamma(v)^2 \left(\boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt}\right) \boldsymbol{S},$$

$$\frac{d\boldsymbol{S}_{CM}}{dt} = (\boldsymbol{r} - \boldsymbol{q}) \times \boldsymbol{F} - \frac{1}{c^2}\gamma(v)^2 \left(\boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt}\right) \boldsymbol{S},$$

where the last term is of the order S/c^2, and can be neglected, the dynamical equation of $\boldsymbol{S}$ is the same than in the free case and the time variation of $\boldsymbol{S}_{CM}$ is the torque with respect to the CM of the external forces defined at the CC.

The final expressions of both spins are

$$\boldsymbol{S} = \mp\gamma(v)m(\boldsymbol{r}-\boldsymbol{q})\times\boldsymbol{u}, \quad \boldsymbol{S}_{CM} = \mp\gamma(v)m(\boldsymbol{r}-\boldsymbol{q})\times(\boldsymbol{u}-\boldsymbol{v}), \tag{2.175}$$

where the expression of $\boldsymbol{S}_{CM}$ has the same structure, with the sign included and without the factor $\gamma(v)$, than the non-relativistic spin.

Since $\boldsymbol{P} = H\boldsymbol{v}/c^2$ for both particle and antiparticle, the expression of the center of mass position (2.173) is

$$\boldsymbol{q} = \boldsymbol{r} + \left(\frac{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}}{(d\boldsymbol{u}/dt)^2}\right)\frac{d\boldsymbol{u}}{dt}. \tag{2.176}$$

We see that the particle has mass and spin, and the center of charge moves in circles at the speed of light in a plane orthogonal to the spin, for the center of mass observer. All these features are independent of the particular Lagrangian of the type (2.155) we can consider.

2.5.5 Poincaré invariance of Dirac's Hamiltonian

The expression which gives rise to Dirac's equation is Poincaré invariant and it takes the same form in any reference frame. In fact, if from (2.170) we take all terms to the left hand side and multiply by $\dot{t}$, we get

$$\dot{t}H - \dot{\boldsymbol{r}}\cdot\boldsymbol{P} - \frac{1}{c^2}\boldsymbol{S}\cdot(\dot{\boldsymbol{u}}\times\boldsymbol{u}) = 0.$$

The first part $\dot{t}H - \dot{\boldsymbol{r}}\cdot\boldsymbol{P} = \dot{x}^\mu P_\mu = \dot{x}'^\mu P'_\mu$, with $\dot{x}^\mu \equiv (c\dot{t}, \dot{\boldsymbol{r}})$ and $P^\mu \equiv (H/c, \boldsymbol{P})$, is Poincaré invariant. The term which contains the spin we see from (2.134), (2.154) and (2.171), that the velocity $\boldsymbol{u}$, acceleration $\dot{\boldsymbol{u}}$ and spin $\boldsymbol{S}$, respectively, transform:

$$\boldsymbol{u}' = R(\boldsymbol{\phi})\boldsymbol{u}, \quad \dot{\boldsymbol{u}}' = \frac{R(\boldsymbol{\phi})\dot{\boldsymbol{u}}}{\gamma(1+\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{u}/c^2)}, \quad \boldsymbol{S}' = \gamma\left(1+\frac{\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{u}}{c^2}\right)R(\boldsymbol{\phi})\boldsymbol{S},$$

and we deduce that

$$\boldsymbol{S}'\cdot(\dot{\boldsymbol{u}}'\times\boldsymbol{u}') = \boldsymbol{S}\cdot(\dot{\boldsymbol{u}}\times\boldsymbol{u}).$$

2.5.6 Dirac analysis

To end this section and with the above model of the electron in mind, it is convenient to remember some of the features that Dirac [10] obtained for the motion of a free electron. Let point $\boldsymbol{r}$ be the position vector on which Dirac's spinor $\psi(t,\boldsymbol{r})$ is defined. When computing the velocity of point $\boldsymbol{r}$, Dirac arrives at:

a) The velocity $\boldsymbol{u} = i/\hbar[H,\boldsymbol{r}] = c\boldsymbol{\alpha}$, is expressed in terms of $\boldsymbol{\alpha}$ matrices and writes, '*... a measurement of a component of the velocity of a free electron is certain to lead to the result* $\pm c$'.

b) The linear momentum does not have the direction of this velocity $\boldsymbol{u}$, but must be related to some average value of it: ... '*the* x_1 *component of the velocity,* $c\alpha_1$*, consists of two parts, a constant part* $c^2p_1H^{-1}$*, connected with the momentum by the classical relativistic formula, and an oscillatory part, whose frequency is at least* $2mc^2/h$*, ...*'.

c) About the position $\boldsymbol{r}$: '*The oscillatory part of* x_1 *is small, ... , which is of order of magnitude* $\hbar/mc$*, ...*'.

d) When analyzing, in his original 1928 paper, [11] the interaction of the electron with an external electromagnetic field, after performing the square of Dirac's operator, he obtains two new interaction terms:

$$\frac{e\hbar}{2mc}\boldsymbol{\Sigma}\cdot\boldsymbol{B} + \frac{ie\hbar}{2mc}\boldsymbol{\alpha}\cdot\boldsymbol{E}, \tag{2.177}$$

[10] P.A.M. Dirac, *The Principles of Quantum mechanics*, Oxford Univ. Press, 4th ed. Oxford (1967).
[11] P.A.M. Dirac, *Proc. Roy. Soc. Lon.* **A117**, 610 (1928).

where the electron spin is written as $\boldsymbol{S} = \hbar\boldsymbol{\Sigma}/2$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix},$$

in terms of σ-Pauli matrices and $\boldsymbol{E}$ and $\boldsymbol{B}$ are the external electric and magnetic fields, respectively. He says, *'The electron will therefore behave as though it has a magnetic moment* $(e\hbar/2mc)\,\boldsymbol{\Sigma}$ *and an electric moment* $(ie\hbar/2mc)\,\boldsymbol{\alpha}$*. The magnetic moment is just that assumed in the spinning electron model'* (Pauli model). *'The electric moment, being a pure imaginary, we should not expect to appear in the model.'*

However, if we look at our classical model, we see that for the center of mass observer, there is a non-vanishing electric and magnetic dipole moment

$$\boldsymbol{d} = e\boldsymbol{k} = \frac{e}{mc^2}\boldsymbol{S}\times\boldsymbol{u}, \quad \boldsymbol{\mu} = \frac{e}{2}\boldsymbol{k}\times\frac{d\boldsymbol{k}}{dt} = -\frac{e}{2m}\boldsymbol{Z}, \tag{2.178}$$

where $\boldsymbol{S}$ is the total spin and $\boldsymbol{Z} = -m\boldsymbol{k}\times d\boldsymbol{k}/dt$ is the zitterbewegung part of spin. The time average value of $\boldsymbol{d}$ is zero, and the average value of $\boldsymbol{\mu}$ is the constant vector $\boldsymbol{\mu}$.

e) In his book [12] analyzes the dynamics of the spin $\boldsymbol{S} = \hbar\boldsymbol{\sigma}/2$ and arrives to the conclusion that

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{p}\times c\boldsymbol{\alpha} = \boldsymbol{p}\times\boldsymbol{u},$$

for the free and interacting electron. This differential equation is the same as the dynamical equation of the spin with respect to the center of charge. Dirac's spin operator represents the angular momentum of the electron with respect to the center of charge.

This classical model gives rise to the same kinematical prediction as the nonrelativistic model described in Sec.**2.2.7**. If the charge of the particle is negative, the current of Fig.2.9 produces a magnetic moment that necessarily has the same direction as the spin. If the electron spin and magnetic moments are antiparallel, then we need another contribution to the total spin, different from the zitterbewegung. All real experiments to determine very accurately the gyromagnetic ratio are based on the determination of precession frequencies, but these precession frequencies are independent of the spin orientation. However, the difficulty to separate electrons in a Stern-Gerlach type experiment, suggests to perform polarization experiments in order to determine in a direct way whether spin and magnetic moment for elementary particles are either parallel or antiparallel. We have suggested a couple of plausible experiments to determine the relative orientation between the spin and magnetic moment of free electrons and also for electrons in the outer shell of atoms[13], which are considered in section **4.2.7**.

Another consequence of the classical model is that it enhances the role of the so-called minimal coupling interaction $j_\mu A^\mu$. The magnetic properties of the electron are produced by the current of its internal motion and not by some possible distribution of magnetic dipoles, so that the only possible interaction of a point charge at $\boldsymbol{r}$ with the external electromagnetic field is that of the current j^μ, associated to the motion of point $\boldsymbol{r}$, with the external potentials.

2.6 The dynamical equation of the spinning electron

We have seen that for relativistic particles with $u = c$ and $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ orthogonal vectors, the position vector $\boldsymbol{r}$ moves in circles according to the dynamical equation (2.167) in the center of mass frame, as depicted in figure 2.9. But this solution is independent of the particular Lagrangian we choose as an invariant function of the kinematical variables and their derivatives,

[12] P.A.M. Dirac, *The principles of Quantum Mechanics*, Oxford 4th edition 1958, p. 266

[13] M. Rivas, *Are the electron spin parallel or antiparallel vectors?, ArXiv:physics/0112057.*

which accomplish with this orthogonality $\boldsymbol{u}\cdot\dot{\boldsymbol{u}}=0$, requirement. We are going to analyze this dynamical equation for any arbitrary inertial observer.[14]

As mentioned in the Preamble, let us consider the trajectory $\boldsymbol{r}(t)$, $t\in[t_1,t_2]$ followed by a point for an arbitrary inertial observer O. Any other inertial observer O' is related to the previous one by a transformation of a kinematical group such that their relative space-time measurements of any space-time event are given by

$$t'=T(t,\boldsymbol{r};g_1,\ldots,g_r),\quad \boldsymbol{r}'=\boldsymbol{R}(t,\boldsymbol{r};g_1,\ldots,g_r),$$

where the functions T and $\boldsymbol{R}$ define the action of the kinematical group G, of parameters $(g_1,\ldots,g_r)$, on space-time. Then the description of the trajectory of that point for observer O' is obtained from

$$t'(t)=T(t,\boldsymbol{r}(t);g_1,\ldots,g_r),\quad \boldsymbol{r}'(t)=\boldsymbol{R}(t,\boldsymbol{r}(t);g_1,\ldots,g_r),\quad \forall t\in[t_1,t_2].$$

If we eliminate t as a function of t' from the first equation and substitute into the second we shall get

$$\boldsymbol{r}'(t')=\boldsymbol{r}'(t';g_1,\ldots,g_r). \tag{2.179}$$

Since observer O' is arbitrary, equation (2.179) represents the complete family of trajectories of the point for all inertial observers. Elimination of the r group parameters among the function $\boldsymbol{r}'(t')$ and their time derivatives will give us the differential equation satisfied by the trajectory of the point. This differential equation is invariant by construction because it is independent of the group parameters and therefore independent of the inertial observer. If G is the Poincaré group, it is a ten-parameter group so that we have to work out in general up to the fourth derivative to obtain sufficient equations to eliminate the ten group parameters. Therefore the order of the differential equation is dictated by the number of parameters and the structure of the kinematical group.

2.6.1 The relativistic spinning electron

Let us assume the above electron model. For the center of mass observer O^*, the trajectory of the center of charge of the electron is contained on the XOY plane and if we write in vector form, and with units $R_0=\hbar/2mc$, $\omega_0=2mc^2/\hbar$

$$\boldsymbol{r}^*(t^*)=R_0\begin{pmatrix}\cos\omega_0 t^*\\ \sin\omega_0 t^*\\ 0\end{pmatrix},\quad \frac{d\boldsymbol{r}^*}{dt^*}=c\begin{pmatrix}-\sin\omega_0 t^*\\ \cos\omega_0 t^*\\ 0\end{pmatrix},$$

For the center of mass observer O^* this point satisfies the differential equation

$$\frac{d^2\boldsymbol{r}^*(t^*)}{dt^{*2}}=-\omega_0^2\boldsymbol{r}^*(t^*). \tag{2.180}$$

Since the center of charge is moving at the speed of light for the center of mass observer O^* it is moving at this speed for every other inertial observer O. Now, the relationship of space-time measurements between the center of mass observer O^* and any arbitrary inertial observer O, is given by:

$$\begin{aligned}t(t^*;g) &= \gamma\left(t^*+\boldsymbol{v}\cdot R(\boldsymbol{\alpha})\boldsymbol{r}^*(t^*)\right)+b,\\ \boldsymbol{r}(t^*;g) &= R(\boldsymbol{\alpha})\boldsymbol{r}^*(t^*)+\gamma\boldsymbol{v}t^*+\frac{\gamma^2}{1+\gamma}\left(\boldsymbol{v}\cdot R(\boldsymbol{\alpha})\boldsymbol{r}^*(t^*)\right)\boldsymbol{v}+\boldsymbol{a}.\end{aligned}$$

[14] M. Rivas, *The dynamical equation of the spinning electron*, J. Phys. A, **36**, 4703, (2003), ArXiv:physics/0112005.

The velocity of the point for the observer O

$$\boldsymbol{r}^{(1)} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}/dt^*}{dt/dt^*},$$

and the same method for the remaining derivatives.

With the shorthand notation for the following expressions:

$$\boldsymbol{K}(t^*) = R(\boldsymbol{\alpha})\boldsymbol{r}^*(t^*), \quad \boldsymbol{V}(t^*) = R(\boldsymbol{\alpha})\frac{d\boldsymbol{r}^*(t^*)}{dt^*} = \frac{d\boldsymbol{K}}{dt^*}, \quad \frac{d\boldsymbol{V}}{dt^*} = -\omega_0^2\boldsymbol{K},$$

$$B(t^*) = \boldsymbol{v}\cdot\boldsymbol{K}/c^2, \quad A(t^*) = \boldsymbol{v}\cdot\boldsymbol{V}/c^2 = \frac{dB}{dt^*}, \quad \frac{dA}{dt^*} = -\omega_0^2 B,$$

where A is dimensionless and B of dimension of time. $\boldsymbol{K}$ has dimension of length and $\boldsymbol{V}$ of velocity. In particular

$$\frac{dt}{dt^*} = \gamma(1 + \boldsymbol{v}\cdot\boldsymbol{V}/c^2) = \gamma(1+A),$$

and

$$\boldsymbol{K}^2 = R_0^2, \quad \boldsymbol{V}^2 = c^2, \quad \boldsymbol{K}\cdot\boldsymbol{V} = 0, \quad \boldsymbol{K}\cdot\boldsymbol{v} = c^2 B, \quad \boldsymbol{V}\cdot\boldsymbol{v} = c^2 A.$$

By making use of the relation (2.180) and its derivatives, we get the following expressions for the subsequent time derivatives of the point $\boldsymbol{r}$ in the arbitrary reference frame O:

$$\boldsymbol{r}^{(1)} = \frac{1}{\gamma(1+A)}\left(\boldsymbol{V} + \frac{\gamma}{1+\gamma}\left(1+\gamma+\gamma A\right)\boldsymbol{v}\right) \tag{2.181}$$

$$\boldsymbol{r}^{(2)} = \frac{\omega_0^2}{\gamma^2(1+A)^3}\left(-(1+A)\boldsymbol{K} + B\boldsymbol{V} + \frac{\gamma}{1+\gamma}B\boldsymbol{v}\right), \tag{2.182}$$

$$\boldsymbol{r}^{(3)} = \frac{\omega_0^2}{\gamma^3(1+A)^5}\left(-3\omega_0^2 B(1+A)\boldsymbol{K} - (1+A-3\omega_0^2B^2)\boldsymbol{V} + \right.$$
$$\left. \frac{\gamma}{1+\gamma}\left(A(1+A) + 3\omega_0^2B^2\right)\boldsymbol{v}\right) \tag{2.183}$$

$$\boldsymbol{r}^{(4)} = \frac{\omega_0^4}{\gamma^4(1+A)^7}\left((1+A)(1-2A-3A^2-15\omega_0^2B^2)\boldsymbol{K} - \right.$$
$$(7+4A-3A^2-15\omega_0^2B^2)B\boldsymbol{V} -$$
$$\left. \frac{\gamma}{1+\gamma}\left(1-8A-9A^2-15\omega_0^2B^2\right)B\boldsymbol{v}\right). \tag{2.184}$$

From these derivatives we obtain

$$\left(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(1)}\right) = \omega_0^2R_0^2 = c^2, \quad \left(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(2)}\right) = 0, \tag{2.185}$$

$$\left(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)}\right) = -\left(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(3)}\right) = \frac{\omega_0^4R_0^2}{\gamma^4(1+A)^4}, \tag{2.186}$$

$$\left(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)}\right) = -\frac{1}{3}\left(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(4)}\right) = \frac{\omega_0^5R_0^2}{\gamma^5(1+A)^6}(2\omega_0 B), \tag{2.187}$$

$$\left(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}\right) = \frac{\omega_0^6R_0^2}{\gamma^6(1+A)^8}\left(1-A^2+3\omega_0^2B^2\right), \tag{2.188}$$

$$\left(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(4)}\right) = \frac{\omega_0^6R_0^2}{\gamma^6(1+A)^8}\left(-1+2A+3A^2+9\omega_0^2B^2\right), \tag{2.189}$$

$$\left(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(4)}\right) = \frac{4\omega_0^7R_0^2}{\gamma^7(1+A)^{10}}\left(1+A+3\omega_0^2B^2\right)(\omega_0 B). \tag{2.190}$$

Since $\omega_0 B$ is dimensionless, the dimensionality of these terms is contained in the coefficients $\omega_0^k R_0^2$, i.e., L^2T^{-k}.

By inspection of equations (2.181-2.184) we see that the four time derivatives of the position vector can be expressed as a linear combination of the three vectors $\boldsymbol{V}$, $\boldsymbol{K}$ y $\boldsymbol{v}$, where the first two vectors are orthogonal and the third is a constant vector which, in general, it is not a linear combination of the other two:

$$\boldsymbol{v} = \boldsymbol{r}^{(1)} - 3B\gamma(1+A)\boldsymbol{r}^{(2)} + \frac{\gamma^2(1+A)^3}{\omega_0^2}\boldsymbol{r}^{(3)}, \tag{2.191}$$

$$\boldsymbol{V} = \frac{\gamma A}{1+\gamma}\boldsymbol{r}^{(1)} + \frac{3\gamma^2 B}{1+\gamma}(1+\gamma+\gamma A)(1+A)\boldsymbol{r}^{(2)} - \frac{\gamma^3(1+A)^3}{(1+\gamma)\omega_0^2}(1+\gamma+\gamma A)\,\boldsymbol{r}^{(3)} \tag{2.192}$$

$$\boldsymbol{K} = \frac{\gamma B}{1+\gamma}\boldsymbol{r}^{(1)} + \frac{\gamma^2(1+A)}{(1+\gamma)\omega_0^2}[3\omega_0^2B^2\gamma - (1+\gamma)(1+A)]\boldsymbol{r}^{(2)} - \frac{\gamma^4 B(1+A)^3}{(1+\gamma)\omega_0^2}\boldsymbol{r}^{(3)}. \tag{2.193}$$

It is clear that the three derivatives $\boldsymbol{r}^{(i)}$, $i = 2, 3, 4$, can be expressed as a linear combination of the three vectors $\boldsymbol{V}$, $\boldsymbol{K}$ and $\boldsymbol{v}$. If we define

$$\boldsymbol{d}_1 = \gamma(1+A)\boldsymbol{r}^{(1)}, \quad \boldsymbol{d}_2 = \frac{\gamma^2(1+A)^3}{\omega_0^2}\boldsymbol{r}^{(2)}, \quad \boldsymbol{d}_3 = \frac{\gamma^3(1+A)^5}{\omega_0^2}\boldsymbol{r}^{(3)}, \quad \boldsymbol{d}_4 = \frac{\gamma^4(1+A)^7}{\omega_0^4}\boldsymbol{r}^{(4)},$$

we get the relation:

$$(1 - 2A - 3A^2 + 3\omega_0^2B^2)\boldsymbol{d}_2 - 6B\boldsymbol{d}_3 + \boldsymbol{d}_4 = 0,$$

which in terms of the derivatives $\boldsymbol{r}^{(i)}$, becomes

$$(1 - 2A - 3A^2 + 3\omega_0^2B^2)\boldsymbol{r}^{(2)} - 6B\gamma(1+A)^2\boldsymbol{r}^{(3)} + \frac{1}{\omega_0^2}\gamma^2(1+A)^4\boldsymbol{r}^{(4)} = 0, \tag{2.194}$$

and represents the Poincaré invariant differential ecuation which satisfies the position of the center of charge, in any inertial reference frame.

From equations (2.186)-(2.188) we can express the magnitudes A, B and γ in terms of these scalar products between the different time derivatives $(\boldsymbol{r}^{(i)} \cdot \boldsymbol{r}^{(j)})$, $i, j = 2, 3$. The constraint that the velocity is c implies that all these and further scalar products for higher derivatives can be expressed in terms of only three of them. The expression of the coefficients A, B and γ, in terms of the different scalar products of these time derivatives is:

$$1 + A = \frac{8(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})^{5/2}/R_0}{4(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})^{5/2}/R_0 + 4(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})(\boldsymbol{r}^{(3)} \cdot \boldsymbol{r}^{(3)}) - 3(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(3)})^2}, \tag{2.195}$$

$$\omega_0 B = \frac{4(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})^{5/4}(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(3)})/R_0^{1/2}}{4(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})^{5/2}/R_0 + 4(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})(\boldsymbol{r}^{(3)} \cdot \boldsymbol{r}^{(3)}) - 3(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(3)})^2}, \tag{2.196}$$

$$\gamma = \frac{4(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})^{5/2}/R_0 + 4(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})(\boldsymbol{r}^{(3)} \cdot \boldsymbol{r}^{(3)}) - 3(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(3)})^2}{8(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})^{11/4}/(cR_0^{1/2})}. \tag{2.197}$$

with $R_0 = c/\omega_0$ and therefore all terms in the numerator and denominator have the same spacetime dimensions.

We thus arrive to:

$$\boldsymbol{r}^{(4)} - \frac{3(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})}\boldsymbol{r}^{(3)} + \left(\frac{2(\boldsymbol{r}^{(3)} \cdot \boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})} - \frac{3(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})^2} - (\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})^{1/2}\right)\boldsymbol{r}^{(2)} = 0. \tag{2.198}$$

It is a fourth order ordinary differential equation which contains as solutions motions at the speed of light. In fact, if $(\boldsymbol{r}^{(1)} \cdot \boldsymbol{r}^{(1)}) = c^2$, then by derivation we have $(\boldsymbol{r}^{(1)} \cdot \boldsymbol{r}^{(2)}) = 0$ and the

next derivative leads to $(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})+(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(3)})=0$. If we take this into account and make the scalar product of (2.198) with $\boldsymbol{r}^{(1)}$, we get $(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(4)})+3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})=0$, which is another relationship between the derivatives as a consequence of $|\boldsymbol{r}^{(1)}|=c$.

Let us go to compare with the most general differential equation of a point in three dimensional space given in the Preamble (6),

$$\boldsymbol{r}^{(4)}-\left(\frac{2\dot\kappa}{\kappa}+\frac{\dot\tau}{\tau}\right)\boldsymbol{r}^{(3)}+\left(\kappa^2+\tau^2+\frac{\dot\kappa\dot\tau}{\kappa\tau}+\frac{2\dot\kappa^2-\kappa\ddot\kappa}{\kappa^2}\right)\boldsymbol{r}^{(2)}+\kappa^2\left(\frac{\dot\kappa}{\kappa}-\frac{\dot\tau}{\tau}\right)\boldsymbol{r}^{(1)}=0,$$

where the dots over κ and τ represent time derivatives. The differential equation for the center of charge of the electron describes a helical motion helicoidal because the term in the first derivative $\boldsymbol{r}^{(1)}$, is lacking. This implies, according to the mentioned result in the Preamble, that there exists a constant relation between curvature and torsion. In fact, if the coefficient of $\boldsymbol{r}^{(1)}$ is zero, this implies that $\dot\kappa/\kappa=\dot\tau/\tau$, and therefore the coefficient of $\boldsymbol{r}^{(3)}$ has to be $-3\dot\kappa/\kappa$. Since the curvature $\kappa=|d^2\boldsymbol{r}/ds^2|=(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{1/2}/c^2$, in terms of the time derivatives and taking another time derivative, we get,

$$\dot\kappa=\frac{1}{c^2}\frac{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{1/2}},\quad \frac{3\dot\kappa}{\kappa}=\frac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})},$$

which is in fact the coefficient of $\boldsymbol{r}^{(3)}$ in (2.198).

Because in terms of the arc length used as a parameter, and in terms of the three Frenet-Serret unit vectors,

$$\boldsymbol{r}^{(1)}=\boldsymbol{t},\quad \boldsymbol{r}^{(2)}=\kappa\boldsymbol{n},\quad \boldsymbol{r}^{(3)}=-\kappa^2\boldsymbol{t}+\dot\kappa\boldsymbol{n}+\kappa\tau\boldsymbol{b},$$

it implies that

$$\frac{(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}=\kappa^2+\tau^2+\left(\frac{\dot\kappa}{\kappa}\right)^2,$$

and the coefficient of $\boldsymbol{r}^{(2)}$, if we consider the relationship between curvature and torsion is

$$\kappa^2+\tau^2+3\left(\frac{\dot\kappa}{\kappa}\right)^2-\frac{\ddot\kappa}{\kappa}=\frac{(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}+2\left(\frac{\dot\kappa}{\kappa}\right)^2-\frac{\ddot\kappa}{\kappa},$$

and since

$$\frac{\ddot\kappa}{\kappa}=\frac{(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)})+(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(4)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}-\frac{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^2},$$

where the scalar product $(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(4)})$ can be expressed in terms of the other three. If we add the terms (2.186)-(2.188), this coefficient is

$$(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(4)})=\frac{1}{R_0}(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{3/2}-2(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)})+\frac{15(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}$$

$$2\left(\frac{\dot\kappa}{\kappa}\right)^2-\frac{\ddot\kappa}{\kappa}=\frac{(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}-\frac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^2}-\frac{1}{R_0}(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{1/2},$$

and we finally obtain the coefficient of the derivative $\boldsymbol{r}^{(2)}$.

If we select as a boundary condition a velocity $|\boldsymbol{r}^{(1)}(0)|\neq c$, this differential equation contains solutions in which the point is not moving at the constant velocity c. But if $|\boldsymbol{r}^{(1)}(0)|=c$, Then the solution satisfies $|\boldsymbol{r}^{(1)}(t)|=c$, for any time t.

2.6.2 The center of mass

The center of mass position is defined by

$$\boldsymbol{q} = \boldsymbol{r} + \frac{1}{\omega_0^2}\gamma^2(1+A)^3\boldsymbol{r}^{(2)} = \boldsymbol{r} + \frac{2(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})\,\boldsymbol{r}^{(2)}}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{3/2}/R_0 + (\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}) - \dfrac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}}. \tag{2.199}$$

in such a way that its time derivative represents the velocity $\boldsymbol{v}$ of the origin of the observer frame O^* with respect to O. In fact, its time derivative is

$$\boldsymbol{q}^{(1)} = \boldsymbol{r}^{(1)} + \frac{1}{\omega_0^2}\gamma^2(1+A)^3\boldsymbol{r}^{(3)} + \frac{1}{\omega_0^2}\frac{3\gamma^2(1+A)^2(-\omega_0^2 B)}{\gamma(1+A)}\boldsymbol{r}^{(2)} = \boldsymbol{v},$$

i.e., expression (2.191) because $dA/dt^* = -\omega_0^2 B$ and we have to divide by $dt/dt^* = \gamma(1+A)$. We can check that $\boldsymbol{q}$ and $\boldsymbol{q}^{(1)}$ vanish for the center of mass observer.

If we take the next time derivative

$$\boldsymbol{q}^{(2)} = \frac{1}{1+A}\left(1 - 2A - 3A^2 + 3\omega_0^2 B^2\right)\boldsymbol{r}^{(2)} - 6B\gamma(1+A)\boldsymbol{r}^{(3)} + \frac{1}{\omega_0^2}\gamma^2(1+A)^3\boldsymbol{r}^{(4)} = 0,$$

which is another form of the dynamical equation of the free particle (2.194) y (2.198).

Because

$$(\boldsymbol{q}-\boldsymbol{r})^2 = \frac{1}{\omega_0^4}\gamma^4(1+A)^6(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)}),$$

$$\boldsymbol{q}^{(1)}\cdot\boldsymbol{r}^{(1)} = c^2 + \frac{1}{\omega_0^2}\gamma^2(1+A)^3(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(1)}) = c^2 - \frac{1}{\omega_0^2}\gamma^2(1+A)^3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})$$

by (2.186) and thus

$$\frac{c^2 - \boldsymbol{q}^{(1)}\cdot\boldsymbol{r}^{(1)}}{(\boldsymbol{q}-\boldsymbol{r})^2} = \frac{\omega_0^2}{\gamma^2(1+A)^3}.$$

which is the inverse of the coefficient of $\boldsymbol{r}^{(2)}$ in the definition of $\boldsymbol{q}$. Then, the fourth order dynamical equation (2.198) for the position of the charge can also be rewritten here as a system of two second order differential equations for the positions $\boldsymbol{q}$ and $\boldsymbol{r}$

$$\frac{d^2\boldsymbol{q}}{dt^2} = 0, \quad \frac{d^2\boldsymbol{r}}{dt^2} = \frac{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}}{(\boldsymbol{q}-\boldsymbol{r})^2}(\boldsymbol{q}-\boldsymbol{r}), \tag{2.200}$$

with $\boldsymbol{v} = \boldsymbol{q}^{(1)}$ and $\boldsymbol{u} = \boldsymbol{r}^{(1)}$, i.e., a free motion for the center of mass and a kind of central motion for the charge around the center of mass.

If we consider the general expression for the center of mass obtained in (2.173), because $\boldsymbol{P} = H\boldsymbol{v}/c^2$, it can also be written as

$$\boldsymbol{q} = \boldsymbol{r} + \left(\frac{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}}{(d\boldsymbol{u}/dt)^2}\right)\frac{d\boldsymbol{u}}{dt}, \qquad \Rightarrow \qquad \frac{d\boldsymbol{u}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2} = \frac{(d\boldsymbol{u}/dt)^2}{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}}(\boldsymbol{q}-\boldsymbol{r})$$

which when compared with (2.200) we obtain the relation

$$c^2 - \boldsymbol{v}\cdot\boldsymbol{u} = \left|\frac{d\boldsymbol{u}}{dt}\right||\boldsymbol{q}-\boldsymbol{r}|, \qquad \text{or} \qquad \left|\frac{d\boldsymbol{u}}{dt}\right| = \frac{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}}{|\boldsymbol{q}-\boldsymbol{r}|} = \frac{c^2}{R},$$

because the acceleration is always normal, and where R is the curvature radius of the trajectory of the center of charge. Thus, the separation between the center of mass and center of charge satisfies

$$|\boldsymbol{q}-\boldsymbol{r}| = R\left(1 - \frac{\boldsymbol{v}\cdot\boldsymbol{u}}{c^2}\right).$$

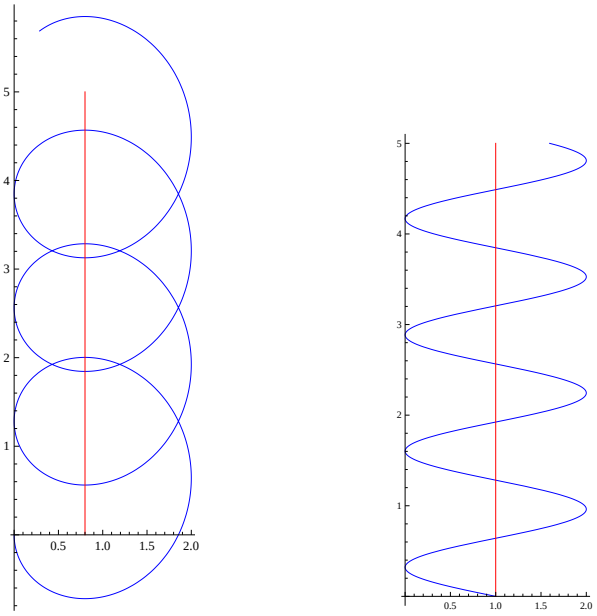


Figure 2.11: **Projection on the plane XOY of the motion of the center of charge (blue) and center of mass (red) of a free electron with $v/c = 0.2$. The trajectory on the left, the electron is boosted on the zitterbewegung plane and the spin is orthogonal to this trajectory and the separation between CC and CM is not constant. The trajectory on the right corresponds to an electron polarized with the spin pointing in the forward direction. Here the separation between both points is constant. Both motions have a spatial period of value $\boldsymbol{d} = v\gamma(v)T_0$ =1.28255, in these units.**

This separation is not constant. If we start with the electron at rest and boost it in the direction orthogonal to the zitterbewegung plane, then $\boldsymbol{v}\cdot\boldsymbol{u} = v^2$ and in this case the trajectory is a helix of constant curvature and torsion and the separation is constant, R_0, which is related to the constant curvature radius by

$$R = R_0\gamma(v)^2.$$

In any other situation this $\boldsymbol{v}\cdot\boldsymbol{u}$ is not constant and the separation oscillates. For instance, if we boost the electron with a velocity $\boldsymbol{v}$ contained on the zitterbewegung plane, the trajectory of the center of charge is flat and in units $R_0 = 1$, and $v/c = 0.2$, we get the picture on the left of the figure **2.11**. We see that the separation oscillates between $|\boldsymbol{q}-\boldsymbol{r}| = 0.8R_0$ and $|\boldsymbol{q}-\boldsymbol{r}| = 1.2R_0$. In fact, in these units the internal period is $T_0 = 2\pi R_0/c = 2\pi$, for the center of mass observer. For the laboratory observer this period is $T = \gamma(v)T_0$, during this time the center of mass moves a distance $d = v\gamma(v)T_0 = 1.28255$ in these units. We see this is the spatial period of the above figure. The trajectory on the right, is produced if the electron is boosted in the direction orthogonal to the zitterbewegung plane and the spin is pointing forward. The spatial periodicity is exactly the same and the separation between the center of mass and center of charge remains constant.

For the non-relativistic electron we get in the low velocity case $\boldsymbol{v}/c \to 0$ and $|\boldsymbol{q}-\boldsymbol{r}| = R_0$, the equations of the Galilei case

$$\frac{d^2\boldsymbol{q}}{dt^2} = 0, \quad \frac{d^2\boldsymbol{r}}{dt^2} = \omega_0^2(\boldsymbol{q}-\boldsymbol{r}). \tag{2.201}$$

a free motion for the center of mass and a harmonic motion around $\boldsymbol{q}$ for the position of the charge, of constant frequency $\omega_0 = c/R_0$.

In the figure **2.12** is represented the motion of the CC and CM on the plane XOY for four different velocities, in which we appreciate that the relative separation between these centers

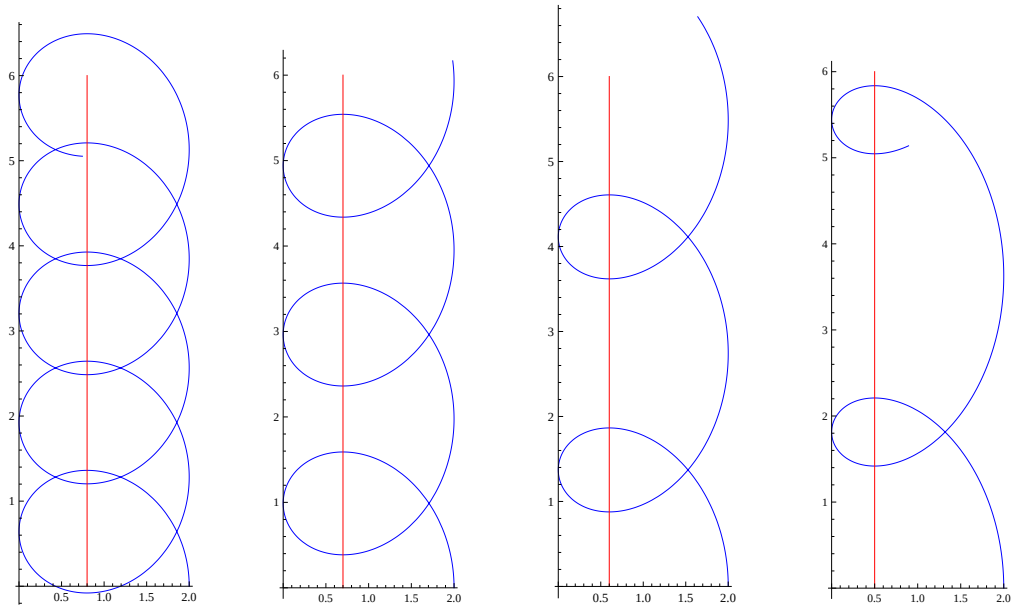


Figure 2.12: **Projection on the plane XOY, of the motion of the CC and CM, with velocities v/c=0.2,0.3,0.4 y 0.5. Remark that the relative separation of the CM oscillates between $(1-v/c)R_0$ and $(1+v/c)R_0$, for this spin orientation.**

oscillates in the interval between $(1-v/c)R_0$ and $(1+v/c)R_0$. The wavelength (the distance followed by the CM during a complete turn of the CC) of these motions are, respectively 1.28255, 1.97597, 2.74221 and 3.6276. In the figure **2.13** the same motions as before but with the CM velocity perpendicular to the zitterbewegung plane. In this case the separation between these centers is constant.

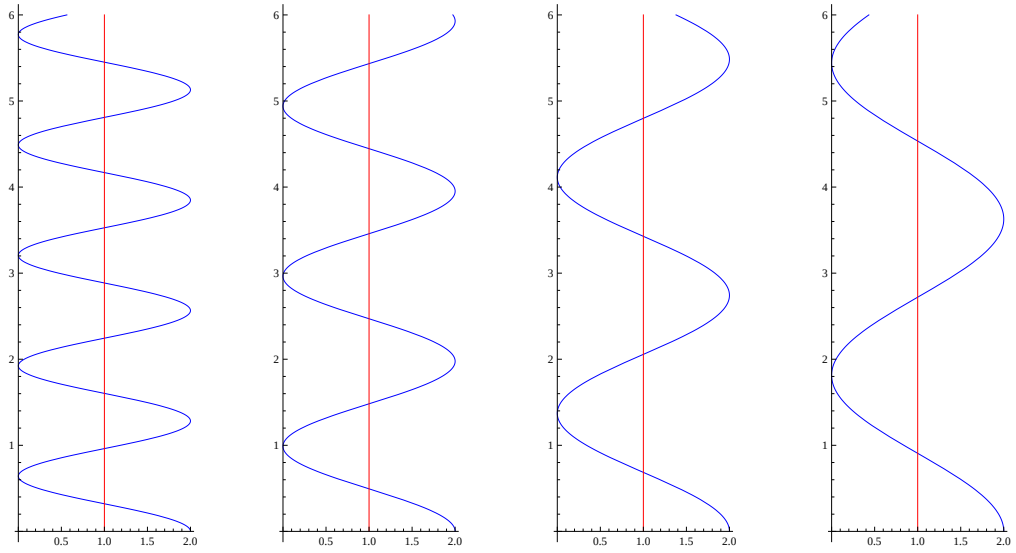


Figure 2.13: **Projection on the plane XOZ, of the motion of the CC and CM, with velocities v/c=0.2,0.3,0.4 and 0.5. Remark that the relative separation between the CC and the CM is constant, but the length followed by the CM during a turn of the CC is the same as in the previous figure.**

The figure **2.14** represents a three-dimensional picture, of the motion of the CC and CM when the CM velocity $\boldsymbol{v}$ is oriented an angle of 30° with respect to the zitterbewegung plane.

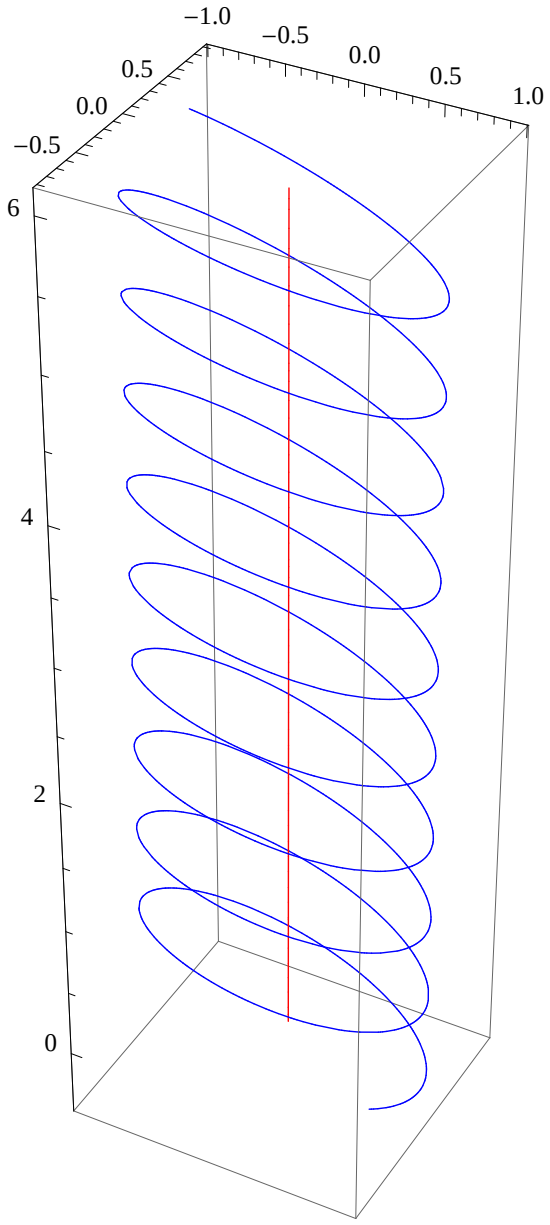


Figure 2.14: **Motion of the CC and CM where the velocity of the CM v/c =0.1 and it is oriented 30°, with respect to the zitterbewegung plane. The separation $|\boldsymbol{q}-\boldsymbol{r}|$ is not constant, but this vector $\boldsymbol{q}-\boldsymbol{r}$ is always orthogonal to the velocity vector of the center of charge $\boldsymbol{u}$.**

2.6.3 Interaction with some external field

In the most general situation, the Lagrangian which describes an interacting electron is of the form

$$\widetilde{L} = \widetilde{L}_m(\boldsymbol{u}, \dot{t}, \dot{\boldsymbol{r}}, \dot{\boldsymbol{u}}, \boldsymbol{\omega}) + \widetilde{L}_I(t, \boldsymbol{r}, \dot{t}, \dot{\boldsymbol{r}}), \tag{2.202}$$

where the free Lagrangian $\widetilde{L}_m$ is related to the mechanical properties of the electron, its mass m and spin S_{CM}, is independent of t and $\boldsymbol{r}$ and the dependence of the orientation $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ is through the dependence on $\boldsymbol{\omega}$. For the interaction Lagrangian $\widetilde{L}_I$, there is no dependence on $\dot{u}$ and $\boldsymbol{\omega}$, because according to the atomic principle m and S_{CM} cannot be modified by any interaction, and therefore the functions which define the spin $\boldsymbol{U} = \partial\widetilde{L}/\partial\dot{\boldsymbol{u}}$, and $\boldsymbol{W} = \partial\widetilde{L}/\partial\boldsymbol{\omega}$, must come from only of the mechanical part $\widetilde{L}_m$. It is in the part $\widetilde{L}_I$ where the interactive properties of the particle are contained. In the time evolution description $L_m = L_m(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{\Omega})$ where $\boldsymbol{a}$ is the acceleration of the point $\boldsymbol{r}$, $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$ and $L_I = -e\phi(t, \boldsymbol{r}) + e\boldsymbol{A}(t, \boldsymbol{r}) \cdot \boldsymbol{u}$, which only depends on the charge of the particle e and the external potentials. The dynamical equations of the three degrees of freedom $\boldsymbol{r}$, are

$$-\frac{d}{dt}\left(\frac{\partial L_m}{\partial \boldsymbol{u}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_m}{\partial \boldsymbol{a}}\right) + \frac{\partial L_I}{\partial \boldsymbol{r}} - \frac{d}{dt}\left(\frac{\partial L_I}{\partial \boldsymbol{u}}\right) = 0.$$

The part related to L_m is reduced to $-d\boldsymbol{P}_m/dt$, where $\boldsymbol{P}_m$ is the mechanical linear momentum while the part related to L_I supplies the Lorentz force, defined at the charge position $\boldsymbol{r}$. Separating the time derivative of the linear momentum to the left hand side, we get

$$\frac{d\boldsymbol{P}_m}{dt} = \boldsymbol{F}, \tag{2.203}$$

while the definition of the center of mass remains the same,

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}}{(\boldsymbol{q}-\boldsymbol{r})^2}(\boldsymbol{q}-\boldsymbol{r}). \tag{2.204}$$

This equation also comes from the definition of the spin with respect to the center of charge (2.175)

$$\boldsymbol{S} = -\frac{H}{c^2}(\boldsymbol{r}-\boldsymbol{q}) \times \boldsymbol{u}.$$

In the defintion of $\boldsymbol{S} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W}$ only the functions $\boldsymbol{U}$ and $\boldsymbol{W}$ appear, and these mechanical properties are not modified by the interaction because $\widetilde{L}_I$ does not depend on $\dot{\boldsymbol{u}}$ and on $\boldsymbol{\omega}$. If the spin $\boldsymbol{S}$ is not modified by the interaction, this means that in the above definition, the function H represents the mechanical temporal momentum H_m, which is also unmodified by the interaction, because its definition comes from the Lagrangian $\widetilde{L}_m$, and therefore the definition of the center of mass

$$\boldsymbol{q} - \boldsymbol{r} = \frac{\boldsymbol{u} \times \boldsymbol{S}}{H_m},$$

which leads to the equation (2.204), remains the same under interaction.

But the mechanical linear momentum is written in terms of the center of mass velocity as $\boldsymbol{P} = m\gamma(v)\boldsymbol{v}$, so that the free dynamical equations (2.200) in the presence of an external field have been replaced by (2.203) and (2.204). We are going to modify (2.203),

$$\frac{d\boldsymbol{P}_m}{dt} = m\gamma(v)\frac{d\boldsymbol{v}}{dt} + m\gamma(v)^3\left(\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt}\right)\frac{\boldsymbol{v}}{c^2} = \boldsymbol{F},$$

and taking the scalar product with $\boldsymbol{v}$, it gives

$$m\gamma(v)^3\left(\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt}\right) = \boldsymbol{F}\cdot\boldsymbol{v}$$

and leaving the highest order derivative $d\boldsymbol{v}/dt = d^2\boldsymbol{q}/dt^2$ on the left hand side, we get the differential equations which describe the evolution of the center of mass and center of charge of a spinning electron, under the action of an external electromagnetic field and in any inertial reference system:

$$\frac{d^2\boldsymbol{q}}{dt} = \frac{e}{m\gamma(v)}\left[\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} - \frac{1}{c^2}\boldsymbol{v}\left([\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}]\cdot\boldsymbol{v}\right)\right], \tag{2.205}$$

$$\frac{d^2\boldsymbol{r}}{dt} = \frac{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}}{(\boldsymbol{q}-\boldsymbol{r})^2}(\boldsymbol{q}-\boldsymbol{r}). \tag{2.206}$$

where

$$\boldsymbol{v} = \frac{d\boldsymbol{q}}{dt}, \quad \boldsymbol{u} = \frac{d\boldsymbol{r}}{dt},$$

with the constraint $|\boldsymbol{u}| = c$. We can compare these relativistic equations with the non-relativistic ones of the Galilei particle (2.84) and (2.85). For the other three degrees of freedom the same dynamical equation holds (2.52)

$$\frac{\partial L_m}{\partial \boldsymbol{\rho}} - \frac{d}{dt}\left(\frac{\partial L_m}{\partial(d\boldsymbol{\rho}/dt)}\right) = 0, \quad \frac{d\boldsymbol{W}}{dt} = \boldsymbol{\Omega}\times\boldsymbol{W}. \tag{2.207}$$

because the dependence on these orientation variables is through $\boldsymbol{\Omega}$ and the interaction Lagrangian L_I is independent of them.

From equation (2.206), if we take the scalar product of both members with $(\boldsymbol{q}-\boldsymbol{r})$, it gives:

$$(\boldsymbol{q}-\boldsymbol{r})\cdot\frac{d\boldsymbol{u}}{dt} = c^2 - \boldsymbol{v}\cdot\boldsymbol{u}.$$

On the other side

$$\frac{d}{dt}(\boldsymbol{q}\cdot\boldsymbol{u} - \boldsymbol{r}\cdot\boldsymbol{u}) = \boldsymbol{v}\cdot\boldsymbol{u} + \boldsymbol{q}\cdot\frac{d\boldsymbol{u}}{dt} - c^2 - \boldsymbol{r}\cdot\frac{d\boldsymbol{u}}{dt} = 0,$$

and the function $(\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{u}$ is a first integral of the system. If we take again the scalar product with $\boldsymbol{u}$ in both sides of (2.206), we see that this first integral vanishes because $\boldsymbol{u}\cdot d\boldsymbol{u}/dt = 0$. The velocity vector $\boldsymbol{u}$ is always orthogonal to the vector $\boldsymbol{q}-\boldsymbol{r}$.

2.6.4 Boundary conditions

The differential equations for the center of charge $\boldsymbol{r}$ of a spinning particle (2.198) are ordinary differential equations of fourth order. To single out a solution we have two possibilities. From the variational point of view we have to supply the values of the kinematical variables $\boldsymbol{r}(t_1)$ and $\boldsymbol{u}(t_1)$ at the initial instant t_1 and also the values of $\boldsymbol{r}(t_2)$ and $\boldsymbol{u}(t_2)$ at the final time t_2. If what we want is to single out a unique solution by giving boundary values at the initial time t_1, we have to give the values of $\boldsymbol{r}(t_1)$, $\boldsymbol{u}(t_1)$, $\boldsymbol{a}(t_1)$ and $\boldsymbol{w}(t_1)$, of the position, velocity, acceleration and jerk, respectively, at the initial time t_1, i.e., the values of the variables r_i up to third order derivatives at that time. It seems that we need to provide 12 initial values, but they are a smaller number because $u(t_1) = c$ and $\boldsymbol{u}(t_1)\cdot\boldsymbol{a}(t_1) = 0$, and these two constraints reduce to 10 the number of initial independent values.

However, according to (2.205) and (2.206), the fourth order differential equations for the variable $\boldsymbol{r}$ have been separated into a system of second order differential equations for the variables $\boldsymbol{q}$ and $\boldsymbol{r}$, of the center of mass and center of charge, respectively. Therefore, to obtain a solution in terms of the boundary conditions at the initial time we have to supply the values of the $\boldsymbol{r}(t_1)$, $\boldsymbol{u}(t_1)$, $\boldsymbol{q}(t_1)$ and $\boldsymbol{v}(t_1)$, of the positions and velocities of both points, evaluated at t_1. This has the advantage that we shall use as a boundary condition the center of mass velocity

of the electron. When preparing the experiment we can estimate the value of the electron linear momentum and therefore its center of mass velocity. We shall also give the initial location of the center of mass $\boldsymbol{q}$ and therefore the initial values of $\boldsymbol{r}$ and $\boldsymbol{u}$ should be expressed in terms of the initial spin orientation. This initial spin orientation could be controlled by means of some external magnetic field.

Let us assume that for the center of mass observer the spin of the electron is along OZ axis as is depicted in the figure **2.15** with the center of mass at the origin in this frame. The center of charge is located at a point of coordinates $(R_0, 0, 0)$, along the OX axis. For this observer the values at the initial time of position, velocity, acceleration and jerk are:

$$\boldsymbol{r}(0) = R_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}(0) = c\epsilon \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{a}(0) = c\omega_0 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{w}(0) = c\omega^2\epsilon \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon = \pm 1,$$

with $\epsilon = +1$ for the particle and $\epsilon = -1$ for the antiparticle.

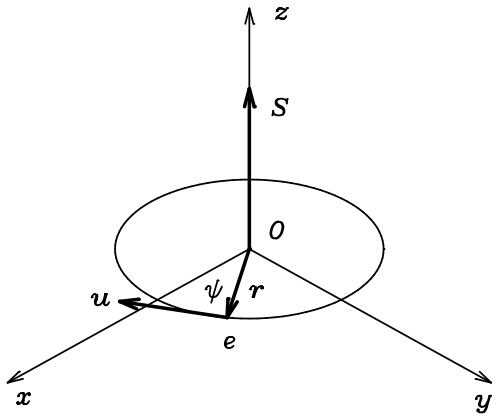


Figure 2.15: **The electron in the CM reference system, with the spin along OZ axis. The position of the CC on the XOY plane is fixed by the phase ψ.**

If at the initial time the center of charge is at a phase ψ and the spin orientation is changed by the zenithal angle θ and azimuthal angle ϕ, since the rotation matrices are

$$R_{OZ}(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_{OY}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad R_{OZ}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the rotated initial variables are $\boldsymbol{r}_0 = R_{OZ}(\phi)R_{OY}(\theta)R_{OZ}(\psi)\boldsymbol{r}(0)$, and the same for the remaining ones, and thus

$$\boldsymbol{r}_0 = R_0 \begin{pmatrix} \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi \\ \cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi \\ -\sin\theta\cos\psi \end{pmatrix}, \quad \boldsymbol{u}_0 = c\epsilon \begin{pmatrix} \cos\theta\cos\phi\sin\psi + \sin\phi\cos\psi \\ \cos\theta\sin\phi\sin\psi - \cos\phi\cos\psi \\ -\sin\theta\sin\psi \end{pmatrix}, \tag{2.208}$$

$$\boldsymbol{a}_0 = -c\omega_0 \begin{pmatrix} \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi \\ \cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi \\ -\sin\theta\cos\psi \end{pmatrix}, \quad \boldsymbol{w}_0 = -c\omega_0^2\epsilon \begin{pmatrix} \cos\theta\cos\phi\sin\psi + \sin\phi\cos\psi \\ \cos\theta\sin\phi\sin\psi - \cos\phi\cos\psi \\ -\sin\theta\sin\psi \end{pmatrix}.$$

The relation of these values with those measured by the inertial observer O' which sees at observer O moving with velocity $\boldsymbol{v}$, is given by

$$t_0' = \gamma(t_0 + \frac{\boldsymbol{v}\cdot\boldsymbol{r}_0}{c^2}), \quad \boldsymbol{r}_0' = \boldsymbol{r}_0 + \gamma\boldsymbol{v}t_0 + \frac{\gamma^2}{1+\gamma}\frac{(\boldsymbol{v}\cdot\boldsymbol{r}_0)\boldsymbol{v}}{c^2}.$$

If we consider that the initial time to integrate the system, corresponds to time $t_0' = 0$ in this reference frame, this corresponds to $t_0 = -\boldsymbol{v}\cdot\boldsymbol{r}_0/c^2$, for the center of mass observer, so that the initial position of the center of charge at the laboratory frame is at $t_0' = 0$,

$$\boldsymbol{r}_0' = \boldsymbol{r}_0 - \frac{\gamma}{1+\gamma}\frac{(\boldsymbol{v}\cdot\boldsymbol{r}_0)\boldsymbol{v}}{c^2}.$$

For the other variables is

$$\boldsymbol{u}_0' = \frac{\boldsymbol{u}_0 + \gamma\boldsymbol{v} + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot\boldsymbol{u}_0)\boldsymbol{v}}{\gamma(1+\boldsymbol{v}\cdot\boldsymbol{u}_0/c^2)}, \tag{2.209}$$

$$\boldsymbol{a}_0' = \frac{(1+\boldsymbol{v}\cdot\boldsymbol{u}_0/c^2)\boldsymbol{a}_0 - (\boldsymbol{v}\cdot\boldsymbol{a}_0/c^2)\boldsymbol{u}_0 - \frac{\gamma}{(1+\gamma)c^2}(\boldsymbol{v}\cdot\boldsymbol{a}_0)\boldsymbol{v}}{\gamma^2(1+\boldsymbol{v}\cdot\boldsymbol{u}_0/c^2)^3}, \tag{2.210}$$

These boundary conditions contain information of the velocity $\boldsymbol{v}$ of the center of mass of the electron. If they are interpreted as boundary conditions at initial time $t_0' = 0$, of the system of equations (2.205) and (2.206), it means that we know $\boldsymbol{r}_0'$, $\boldsymbol{u}_0'$, $\boldsymbol{v}_0' = \boldsymbol{v}$ and we need still to know $\boldsymbol{q}_0'$, the initial position of the center of mass.

If we consider that the center of mass position given in (2.173) with $H = \gamma mc^2$, and $\boldsymbol{P} = \gamma m\boldsymbol{v}$, we get

$$\boldsymbol{q} = \boldsymbol{r} + \left(\frac{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}}{(d\boldsymbol{u}/dt)^2}\right)\frac{d\boldsymbol{u}}{dt},$$

and therefore the boundary condition for the center of mass in the laboratory frame will be

$$\boldsymbol{q}_0' = \boldsymbol{r}_0' + \frac{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}_0'}{\boldsymbol{a}_0'^{\,2}}\,\boldsymbol{a}_0',$$

in terms of the position, velocity and acceleration of the center of charge. Please remark that in the expression of the center of mass velocity it is contained the dependence up to the third derivative of the center of charge $\boldsymbol{r}$.

As a summary, to characterize the boundary conditions at the initial time $t_0' = 0$, in the laboratory reference system, we have to give the values of

$$\boldsymbol{r}_0' = \boldsymbol{r}_0 - \frac{\gamma}{1+\gamma}\frac{(\boldsymbol{v}\cdot\boldsymbol{r}_0)\boldsymbol{v}}{c^2}, \quad \boldsymbol{u}_0' = \frac{\boldsymbol{u}_0 + \gamma\boldsymbol{v} + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot\boldsymbol{u}_0)\boldsymbol{v}}{\gamma(1+\boldsymbol{v}\cdot\boldsymbol{u}_0/c^2)}, \tag{2.211}$$

$$\boldsymbol{q}_0' = \boldsymbol{r}_0' + \frac{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}_0'}{\boldsymbol{a}_0'^{\,2}}\,\boldsymbol{a}_0', \quad \boldsymbol{v}_0' = \boldsymbol{v}, \tag{2.212}$$

with $\boldsymbol{a}_0'$ given in (2.210), and where the magnitudes $\boldsymbol{r}_0$, $\boldsymbol{u}_0$ and $\boldsymbol{a}_0$, are those magnitudes measured in the center of mass reference frame (2.208). We do not need to produce the values of the jerk $\boldsymbol{w}_0$, because these values are already contained in the definition of the center of mass velocity $\boldsymbol{v}$.

If the initial time is $t'_0 = 0$, and CM location is determined, we only need another 6 boundary conditions: the 3 components of the CM velocity in the laboratory frame and the initial phase ψ and spin orientation θ, ϕ for the CM observer. For particles coming from a different initial point we have to add the other 3 extra desplacement variables for the initial CM position.

In natural units, $c = R_0 = \omega_0 = 1$ and $\boldsymbol{a}_0 = -\boldsymbol{r}_0$. In this case, taking the square of (2.210)

$$\boldsymbol{a}_0'^{\,2} = \frac{1}{\gamma^4(1+\boldsymbol{v}\cdot\boldsymbol{u}_0)^4},$$

and from (2.209), by multiplying with $\boldsymbol{v}$, we get the term

$$1 - \boldsymbol{v}\cdot\boldsymbol{u}'_0 = \frac{1}{\gamma^2(1+\boldsymbol{v}\cdot\boldsymbol{u}_0)},$$

and the boundary condition for the CM position, instead of (2.212) becomes:

$$\boldsymbol{q}'_0 = (\boldsymbol{v}\cdot\boldsymbol{r}_0)\boldsymbol{u}_0 - (\boldsymbol{v}\cdot\boldsymbol{u}_0)\boldsymbol{r}_0 = \boldsymbol{v}\times(\boldsymbol{u}_0\times\boldsymbol{r}_0), \quad \boldsymbol{v}'_0 = \boldsymbol{v}. \tag{2.213}$$

In general, the initial CM position given in (2.213) is contained in the zitterbewegung plane, and is depicted in the figure **2.16** where the distance to the center is $v\sin\theta$ in dimensionless units. From here we see that if $\boldsymbol{v}$ is orthogonal to the ziterbewegung plane, $\boldsymbol{v}\cdot\boldsymbol{r}_0 = \boldsymbol{v}\cdot\boldsymbol{u}_0 = 0$, and the CM is at the origin and the separation with the CC is constant.

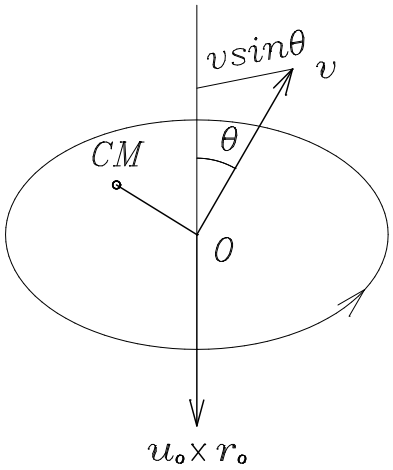


Figure 2.16: **Initial position of the CM when the velocity $\boldsymbol{v}$ is at an angle θ with the direction orthogonal to the zitterbewegung plane. The separation is perpendicular to the vectors $\boldsymbol{v}$ and $\boldsymbol{u}_0\times\boldsymbol{r}_0$ and of value $O-CM = v\sin\theta$ in these dimensionless units. It is independent of the intial CC position.**

In the figure **2.12**, since $\theta = 90°$, we see that the CM is desplaced to the left of the central position, in a direction perpendicular to the velocity $\boldsymbol{v}$, at the distances $v/c = 0.2, 0.3, 0.4$ and 0.5.

In the pictures **2.17**, we represent the plane motion on the plane XOZ of the CC and CM when the spin is orthogonal to the velocity $\boldsymbol{v}$, $\theta = \phi = 90°$, with various initial positions of the CC phase. In all cases the CM initial position is the same, but not the CC position.

2.6.5 Natural units

We have a natural unit of velocity c and also a spatial scale factor R_0, the radius of the zitterbewegung for the center of mass observer. If we define the dimensionless magnitudes

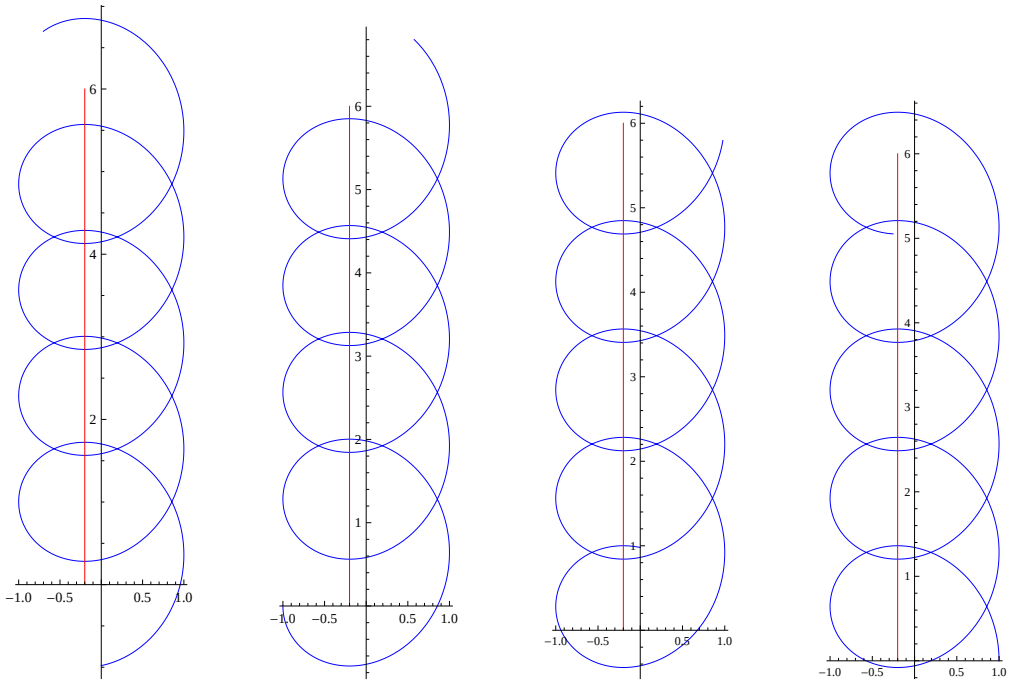


Figure 2.17: **Projection on the XOY plane, of the CC and CM motion, with velocity $v/c = 0.2$, $\theta = \phi = 90°$ and the initial phase $\psi = 0, 90°, 180°, 270°$. In all cases the initial CC position is different, but that of the CM is the same, at a distance $v/c \sin\theta =$-0.2 from the origin.**

$\boldsymbol{r} = R_0\widetilde{\boldsymbol{r}}$, $\widetilde{t} = \omega_0 t$, with $\omega_0 = c/R_0$, this means that

$$\boldsymbol{u} = \frac{d\boldsymbol{r}}{dt} = R_0\frac{d\widetilde{\boldsymbol{r}}}{d\widetilde{t}}\omega_0 = c\frac{d\widetilde{\boldsymbol{r}}}{d\widetilde{t}},$$

$$\frac{d\boldsymbol{u}}{dt} = c\frac{d^2\widetilde{\boldsymbol{r}}}{d\widetilde{t}^2}\omega_0 = \frac{c^2}{R_0}\frac{d^2\widetilde{\boldsymbol{r}}}{d\widetilde{t}^2}.$$

and the differential equation (2.206) in natural units becomes

$$\frac{d^2\widetilde{\boldsymbol{r}}}{d\widetilde{t}^2} = \frac{1 - \widetilde{\boldsymbol{v}}\cdot\widetilde{\boldsymbol{u}}}{(\widetilde{\boldsymbol{q}} - \widetilde{\boldsymbol{r}})^2}(\widetilde{\boldsymbol{q}} - \widetilde{\boldsymbol{r}}),$$

where now $0 \leq \widetilde{v} < 1$ And $\widetilde{u} = 1$. For the equation (2.205) in natural units, we arrive to

$$\frac{c^2}{R_0}\frac{d^2\widetilde{\boldsymbol{q}}}{d\widetilde{t}^2} = \frac{e}{m\gamma}\left[\boldsymbol{E} + \widetilde{\boldsymbol{u}}\times\boldsymbol{B}c - \widetilde{\boldsymbol{v}}\,(\boldsymbol{E} + \widetilde{\boldsymbol{u}}\times\boldsymbol{B}c)\cdot\widetilde{\boldsymbol{v}}\right]$$

and making use of the expression $R_0 = \hbar/2mc$, we get:

$$\frac{d^2\widetilde{\boldsymbol{q}}}{d\widetilde{t}^2} = \frac{e\hbar}{2m^2c^3\gamma}\left[\boldsymbol{E} + \widetilde{\boldsymbol{u}}\times\boldsymbol{B}c - \widetilde{\boldsymbol{v}}\,(\boldsymbol{E} + \widetilde{\boldsymbol{u}}\times\boldsymbol{B}c)\cdot\widetilde{\boldsymbol{v}}\right]$$

where the external fields $\boldsymbol{E}$ y $\boldsymbol{B}$ are defined at every time in the the laboratory frame, at the center of charge position $\boldsymbol{r}$.

2.6.6 Invariant properties

If we accept the atomic principle, the intrinsic mechanical properties of the electron are not affected by any interaction. If H_m and $\boldsymbol{P}_m$ represent the mechanical energy (temporal

momentum) and mechanical linear momentum, respectively, (this is the meaning of the subindex m), they satisfy for the free particle the property

$$H_m^2 - c^2 \boldsymbol{P}_m = m^2 c^4. \tag{2.214}$$

We also know that $\boldsymbol{P}_m = H_m \boldsymbol{v}/c^2$, where $\boldsymbol{v}$ is the velocity of the center of mass. This implies that both mechanical properties can always be written as $H_m = \gamma(v)mc^2$ and $\boldsymbol{P}_m = \gamma(v)m\boldsymbol{v}$, in terms of the center of mass velocity.

For the free particle they also satisfy the equation (2.170) which defines Dirac's Hamiltonian, and which involves the spin with respect to the center of charge:

$$H_m = \boldsymbol{u} \cdot \boldsymbol{P}_m + \frac{1}{c^2} \boldsymbol{S} \cdot \left(\frac{d\boldsymbol{u}}{dt} \times \boldsymbol{u} \right) \tag{2.215}$$

However, from the interacting particle Lagrangian (2.202) the total temporal momentum and linear momentum are defined as

$$H = -T - \boldsymbol{u} \cdot \frac{d\boldsymbol{U}}{dt} = H_m + e\phi(t, \boldsymbol{r}), \quad \boldsymbol{P} = \boldsymbol{R} - \frac{d\boldsymbol{U}}{dt} = \boldsymbol{P}_m + e\boldsymbol{A}(t, \boldsymbol{r}).$$

Therefore for the interacting particle, in terms of the external potentials, the following expressions equivalent to (2.214) and (2.215) are satisfied.

$$(H - e\phi(t, \boldsymbol{r}))^2 - c^2 (\boldsymbol{P} - e\boldsymbol{A}(t, \boldsymbol{r}))^2 = m^2 c^4, \tag{2.216}$$

$$H - e\phi(t, \boldsymbol{r}) = \boldsymbol{u} \cdot (\boldsymbol{P} - e\boldsymbol{A}(t, \boldsymbol{r})) + \frac{1}{c^2} \boldsymbol{S} \cdot \left(\frac{d\boldsymbol{u}}{dt} \times \boldsymbol{u} \right) \tag{2.217}$$

where H and $\boldsymbol{P}$ are the total momenta and in terms of the external potencials ϕ and $\boldsymbol{A}$. In the quantum case (2.216) and (2.217) they will supply us with the interacting Klein-Gordon equation and interacting Dirac's equation, respectively, equations which are satisfied by the wavefunction of the electron.

If we know the dynamical equation of the momentum $\boldsymbol{P}_m$ as given in (2.203), taking the time derivative in (2.214) we get

$$2H_m \frac{dH_m}{dt} - 2c^2 \boldsymbol{P}_m \cdot \frac{d\boldsymbol{P}_m}{dt} = 0, \quad \frac{dH_m}{dt} = \boldsymbol{v} \cdot \boldsymbol{F}(t, \boldsymbol{r}),$$

i.e., $dH_m = \boldsymbol{F} \cdot d\boldsymbol{q}$, the variation of the mechanical energy is the work of the external force defined at the center of charge $\boldsymbol{r}$, along the trajectory of the center of mass $\boldsymbol{q}$.

2.7 Particles and antiparticles

The most general Lagrangian of an interacting particle is written as

$$\widetilde{L} = \widetilde{L}_0 + \widetilde{L}_I,$$

where $\widetilde{L}_0$ represents the free Lagrangian and $\widetilde{L}_I$ that part that gives rise to the interaction.

The mechanical invariant properties of the particle, which are not modified by the interaction, come from the free Lagrangian $\widetilde{L}_0$. These properties are related to the temporal momentum H_m and linear momentum $\boldsymbol{P}_m$ and the spin with respect to the center of charge $\boldsymbol{S} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W} = \boldsymbol{Z} + \boldsymbol{W}$. Because the interacting Lagrangian $\widetilde{L}_I$, cannot modify the definition of the two functions $\boldsymbol{U}$ and $\boldsymbol{W}$, the spin structure remains unmodified, according to

the atomic principle. In the relativistic context and for the particle whose center of charge is moving at the speed of light c, these observables satisfy the invariant relation,

$$H_m - \boldsymbol{P}_m \cdot \boldsymbol{u} = \frac{1}{c^2}\boldsymbol{S} \cdot \left(\frac{d\boldsymbol{u}}{dt} \times \boldsymbol{u}\right).$$

For the mechanical observables H_m and $\boldsymbol{P}_m$, coming from the part $\widetilde{L}_0$, we also have the relation

$$(H_m/c)^2 - \boldsymbol{P}_m^2 = m^2c^2,$$

where m is a positive observable which is interpreted as the mass of the particle. Between these mechanical observables there exist the relation,

$$\boldsymbol{P}_m = H_m\boldsymbol{v}/c^2$$

where $\boldsymbol{v}$ represents the velocity of the center of mass of the particle. This implies that

$$H_m = \pm\gamma(v)mc^2, \quad \boldsymbol{P}_m = \pm\gamma(v)m\boldsymbol{v}.$$

The relativistic formalism predicts the existence of two kinds of material systems of the same positive mass m, but the magnitude H_m can either be positive or negative. For the mechanical linear momentum $\boldsymbol{P}_m$, the two possibilities one in the direction of the center of mass velocity or in the opposite direction, respectively. The first object is called particle, while it is called antiparticle in the second case. The difference is that if the free Lagrangian for the first object is $\widetilde{L}_0$, the free Lagrangian for the second is $-\widetilde{L}_0$.

As far as the internal structure of the motion of the kinematical variables, implies that, once the spin direction is fixed, the center of charge motion for the particle, is antiorbital while it is orbital for the antiparticle. The unmodified mechanical properties H_m, $\boldsymbol{P}_m$ and $\boldsymbol{S}$, coming from the mechanical free Lagrangian $\widetilde{L}_0$, change their sign when derived from $-\widetilde{L}_0$, because all functions T, $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{W}$ also change their sign.

For the part $\widetilde{L}_I$ takes the general form,

$$\widetilde{L}_I = -e\phi(t,\boldsymbol{r})\dot{t} + e\boldsymbol{A}(t,\boldsymbol{r}) \cdot \dot{\boldsymbol{r}},$$

where the constant e represents the charge of the particle and its sign is undetermined. For the antiparticle

$$\widetilde{L}_I^* = -e^*\phi(t,\boldsymbol{r})\dot{t} + e^*\boldsymbol{A}(t,\boldsymbol{r}) \cdot \dot{\boldsymbol{r}},$$

where e^* is the charge of the antiparticle and ϕ and $\boldsymbol{A}$ the external potentials.

For the particle, from $\widetilde{L}_p = \widetilde{L}_0 + \widetilde{L}_I$, we get

$$\frac{d\boldsymbol{P}_m}{dt} = e\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right), \quad \boldsymbol{P}_m = \gamma(v)m\boldsymbol{v}, \quad \frac{d(\gamma(v)m\boldsymbol{v})}{dt} = e\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right),$$

while for the antiparticle, from $\widetilde{L}_a = -\widetilde{L}_0 + \widetilde{L}_I^*$, we arrive to

$$\frac{d\boldsymbol{P}_m}{dt} = e^*\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right), \quad \boldsymbol{P}_m = -\gamma(v)m\boldsymbol{v}, \quad \frac{d(\gamma(v)m\boldsymbol{v})}{dt} = -e^*\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right),$$

where $\boldsymbol{E} = -\nabla\phi - \partial\boldsymbol{A}/\partial t$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. If in front of the same external electromagnetic field, the acceleration of the center of mass of the particle is opposite to the acceleration of the antiparticle, then $-e^* = -e$, and both objects will have the same charge. This last equation for the antiparticle will be rewritten as,

$$\frac{d}{dt}\left(\gamma(v)m\boldsymbol{v}\right) = -e\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right).$$

We can define the mechanical linear momentum always in the same direction as the velocity, which implies that the observable H_m should be definite positive, and the two kinds of particles, of the same positive mass, will be different by the different sign of their charges, and they will be described by the Lagrangians

$$\widetilde{L}_p = \widetilde{L}_0 - e\phi(t,\boldsymbol{r})\dot{t} + e\boldsymbol{A}(t,\boldsymbol{r})\cdot\dot{\boldsymbol{r}},$$

$$\widetilde{L}_a = \widetilde{L}_0 + e\phi(t,\boldsymbol{r})\dot{t} - e\boldsymbol{A}(t,\boldsymbol{r})\cdot\dot{\boldsymbol{r}},$$

which corresponds simply to a change e by $-e$, and where the free common part $\widetilde{L}_0$ is that Lagrangian which leads to a positive $H_m > 0$ and $\boldsymbol{P}_m = H_m\boldsymbol{v}/c^2$. In any case the sign of the particle is undefined.

Because the dynamical equations derived from $\widetilde{L}_a$ And from $-\widetilde{L}_a$ are exactly the same, we can have two possible equivalent interpretations of the differences between particle and antiparticle. One is that both elementary objects have the same mass and charge but their mechanical properties H_m and $\boldsymbol{P}_m$ are opposite. The usual interpretation is that they have opposite charges, which brings us to adopt that the sign of the energy must necessarily be positive and that the linear momentum has the direction of the velocity of the center of mass. The requirement of the positive definitness of the energy could be related to the arrow of time. See the ananlysis performed in the section **6.10.3** about active and passive transformations of the kinematical group.

This method is valid to establish the dynamical equations of the center of mass of the particle or antiparticle. For the internal motion we have for both kinds of objects the equation

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}}{(\boldsymbol{q}-\boldsymbol{r})^2}(\boldsymbol{q}-\boldsymbol{r}),$$

which is just the definition of the center of mass position

$$\boldsymbol{q} = \boldsymbol{r} + \frac{c^2}{H_m}\left(\frac{H_m - \boldsymbol{P}_m\cdot\boldsymbol{u}}{(d\boldsymbol{u}/dt)^2}\right)\frac{d\boldsymbol{u}}{dt}.$$

The expression of the spin $\boldsymbol{S}$ in terms of these variables will be for the particle

$$\boldsymbol{S}_p = \left(\frac{H_m - \boldsymbol{P}_m\cdot\boldsymbol{u}}{(d\boldsymbol{u}/dt)^2}\right)\frac{d\boldsymbol{u}}{dt}\times\boldsymbol{u}, \quad \text{o bien} \quad \boldsymbol{S}_p = -\gamma(v)m(\boldsymbol{r}-\boldsymbol{q})\times\boldsymbol{u}$$

while for the antiparticle

$$\boldsymbol{S}_a = \left(\frac{H_m - \boldsymbol{P}_m\cdot\boldsymbol{u}}{(d\boldsymbol{u}/dt)^2}\right)\boldsymbol{u}\times\frac{d\boldsymbol{u}}{dt}, \quad \text{o bien} \quad \boldsymbol{S}_a = \gamma(v)m(\boldsymbol{r}-\boldsymbol{q})\times\boldsymbol{u},$$

with $H_m = \gamma(v)mc^2$ and $\boldsymbol{P}_m = \gamma(v)m\boldsymbol{v}$. This makes a distinction between particle and antiparticle, as far as the internal motion is concerned, because the spin $\boldsymbol{S}$, has the direction of the angular velocity for the antiparticle and in the opposite direction for the particle.

Experimentally we know that the pair electron-positron, if its total angular momentum is zero, anhilates with the emision of two photons, with opposite spins and total energy $2mc^2$. In this process it is conserved the energy, linear momentum, angular momentum and the electric charge. If the initial state is of spin 1, then the desintegration is with the emission of three or more photons. The usual interpretation that for massive particles that the antiparticle is an object of opposite charge is consistent with this experimental result.

In the case of photons, because they do not have electric charge we can think that they are their own antiparticle. This is the usual interpretation. But the same conclusion will be reached for neutrinos, because they are chargeless and they could be their own antiparticles. However,

the antineutrinos have opposite lepton number and they are different than the neutrinos. The conservation of the leptonic number requieres they should be different. However, from the mechanical point of view $\boldsymbol{S}$ and ω have the same direction for the photon and opposite direction for the antiphoton. This physical difference can be determined by the interaction with a crystal lattice. If the optics of antiphotons is the same than that for photons, they will be no difference between them. But it is possible that the interaction with the lattice, although they do not have charge, could be related to the different relative orientation af the angular momentum and angular velocity.

Since photons do not have charge, this interpretation that the linear momentum has the direction of the velocity and the charge is opposite, implies that photons and antiphotons, being chargeless, they are the same particle. However, with the first interpretation there will be no ambiguity, because the definition of the mechancial properties H and $\boldsymbol{p}$ will be opposite to each other. Today we know that in electrodynamics and chromodynamics, the interaction mechanisms between material particles (fermions of spin 1/2), is the interchange of virtual bosons of spin 1 (photons, gluons, massive bosons $W^{\pm}$, Z^0). In the electromagnetic case, if the interchange is mediated exclusively by photons, the phenomenon of atraction will not take place.

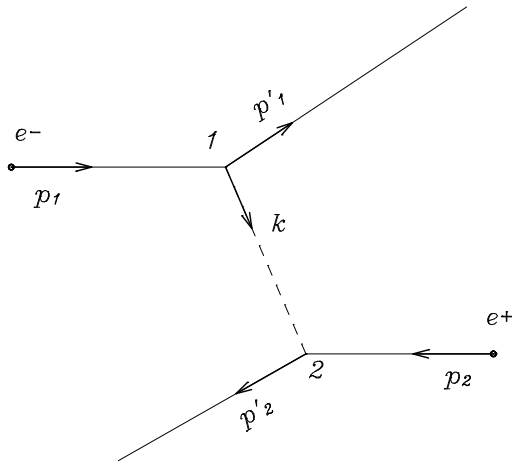


Figure 2.18: **Interaction of an electron and a positron by the interchange of a virtual photon. Both particles separate from each other.**

Let us assume, as is depicted in the figure **2.18**, that an electron and a positron, both of positive mechanical energy H_m and linear momentum in the direction of the velocity of its center of mass, interact by the interchange of a virtual photon, whihc is emited by the electron in 1 and being absorbed by the positron in 2. Due to the interchange of linear momentum and energy, th electron gets a linear momentum $\boldsymbol{p}_1' = \boldsymbol{p}_1 - \boldsymbol{k}$, while the positron ends with a linear momentum $\boldsymbol{p}_2' = \boldsymbol{p}_2 + \boldsymbol{k}$, and the two particles reppel each other. This process will be the same if the virtual photon is emitted by the positron.

Because we know experimentally that particles of opposite electric charge atract to each other, the mechanism should be that of the figure **2.19**, with the interchange of an antiphoton, emited from 1 by the electron, with linear momentum $\boldsymbol{k}$, in the opposite direction to its velocity, beind absorbed at 2 by the positron. Now we get again $\boldsymbol{p}_1' = \boldsymbol{p}_1 - \boldsymbol{k}$, and the result is that the electron approaches to the positron. The same interpretation will be obtained if the emision of the virtual antiphoton is produced by the positron.

In an atom, the existence of bound states of electrons with respect to a positively charged

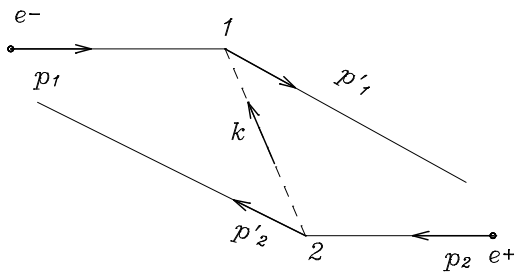


Figure 2.19: **Interaction of an electron and a positron by the interchange of a virtual antiphoton. Both particles atract to each other.**

nucleus, implies a process of electromagnetic atraction. If this process is mediated by the interchange of virtual bosons between the nucleus and the electrons, these bosons necessarilly have to be antiphotons.

2.7.1 Detection of cosmic antimatter

One of the projects to detect antimatter in the universe and to verify the existence of antimatter galaxies should consist in the detection of antimatter atoms in cosmic rays. Antiprotons and positrons are already detected, but they could be produced at the Sun or in stars of our galaxy.

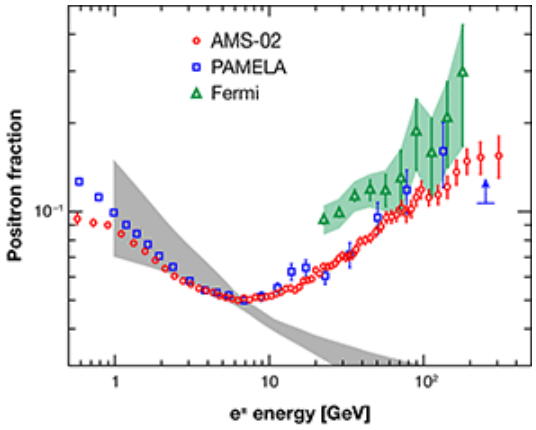


Figure 2.20: **Measurement of the positron/electron ratio performed by the detector of the AMS02. The grey band is the prediction of this ratio by astrophysicists, based on models of interaction and transport phenomena in our Galaxy. The measured ratio (red data), increases above 10 GeV.**

The simplest antimatter structure should be the nucleus of antihelium, formed by two antiprotons and two antineutrons, i.e., an antialpha particle. This idea lead to the construction of a spectrometer yo measure these objects. It is called Alpha Magnetic Spectrometer, and the AMS02 was installed at the ISS (International Space Station) on May 2011, at a mean

altitude of 350 Km. Up to 2013 had detected around 25×10^9 counts of electrons and positrons in the range from 0.5 to 350 GeV, being positrons 4×10^5, while observing an increase in the positron/electron ratio in the range from 10 to 250 GeV, with no significative difference along time and in the direction of observation. But they found *an unexplained excess of high-energy positrons in Earth-bound cosmic rays*, in Samuel Ting's words, director of the project [15]. In Decembre 2016 they inform that a small ratio of antihelium-3 in around 10^9 Helium nuclei had been detected in that year.

Recently we have analyzed the behaviour of antiphotons with mirrors and have suggested the possibility of detecting antimatter galaxies with the design of a telescope which focus antiphotons[16].

[15] S. Coutu, *Physics* **6**, 40 (2013); AMS Collaboration *Phys. Rev. Lett.* **110**, 141102 (2013)

[16] M.Rivas *Considerations about photons and antiphotons*, Indian J. Phys. **96** 583-591 (2022).

2.8 Appendix: Rotation group

We are going to describe geometrically a rotation of value α around an arbitrary axis, described by the unit vector $\boldsymbol{u}$. We interpret $\alpha > 0$ when the rotation is clockwise when looking along the direction given bu the unit vector $\boldsymbol{u}$. If $\alpha < 0$, the rotation is in the opposite sense, i.e., anticlockwise. Then, according to the figure 2.21, an arbitary point, characterized by the vector $\boldsymbol{r}$, will be rotated to the position given by the vector $\boldsymbol{r}'$,

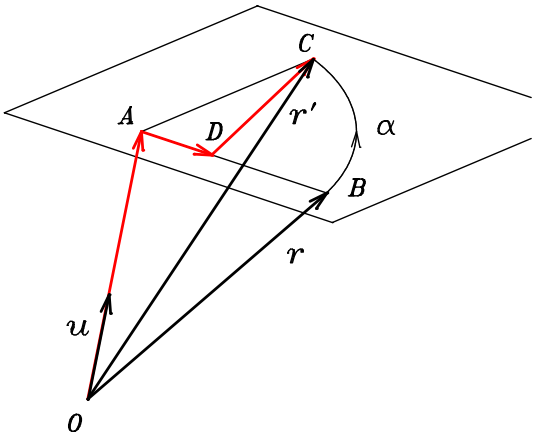


Figure 2.21: **Active rotation of value α of the vector $\boldsymbol{r}$, around the axis OA**

From the vector point of view, $\boldsymbol{r}' = \boldsymbol{OA} + \boldsymbol{AD} + \boldsymbol{DC}$, where $\boldsymbol{DC}$ is orthogonal to the vectors $\boldsymbol{OA}$ and $\boldsymbol{AB}$.

$$\boldsymbol{OA} = (\boldsymbol{r}\cdot\boldsymbol{u})\boldsymbol{u}$$

$$\boldsymbol{AD} = \boldsymbol{AB}\cos\alpha = (\boldsymbol{r} - (\boldsymbol{r}\cdot\boldsymbol{u})\boldsymbol{u})\cos\alpha$$

$$\boldsymbol{DC} = |\boldsymbol{AC}|\sin\alpha\,\boldsymbol{n}$$

where $\boldsymbol{n}$ is a unit vector orthogonal to $\boldsymbol{u}$ and $\boldsymbol{r}$, and therefore

$$\boldsymbol{n} = \frac{\boldsymbol{u}\times\boldsymbol{r}}{|\boldsymbol{u}\times\boldsymbol{r}|}$$

but $|\boldsymbol{u}\times\boldsymbol{r}| = |\boldsymbol{AC}| = |\boldsymbol{AB}|$, $|\boldsymbol{AD}| = |\boldsymbol{AC}|\cos\alpha$, $|\boldsymbol{DC}| = |\boldsymbol{AC}|\sin\alpha$, and thus

$$\boldsymbol{DC} = \boldsymbol{u}\times\boldsymbol{r}\sin\alpha$$

Finally, the vector $\boldsymbol{r}'$ is expressed as:

$$\boldsymbol{r}' = \boldsymbol{r}\cos\alpha + (\boldsymbol{r}\cdot\boldsymbol{u})\boldsymbol{u}(1-\cos\alpha) + \boldsymbol{u}\times\boldsymbol{r}\sin\alpha, \tag{2.218}$$

and its Cartesian components:

$$x'_i = x_i\cos\alpha + (x_k u_k)u_i(1-\cos\alpha) + \varepsilon_{ijk}u_j x_k\sin\alpha =$$

$$= (\delta_{ik}\cos\alpha + u_i u_k(1-\cos\alpha) + \varepsilon_{ijk}\,u_j\sin\alpha)x_k = \mathcal{R}(\alpha,\boldsymbol{u})_{ik}x_k.$$

This linear expression of x'_i in terms of x_k is expressed in terms of the matrix $\mathcal{R}(\alpha,\boldsymbol{u})_{ik}$. If we define the vector $\boldsymbol{\alpha} = \alpha\boldsymbol{u}$, then every rotation is parameterized by this three vector,

$$\mathcal{R}(\boldsymbol{\alpha})_{ik} = \delta_{ik}\cos\alpha + \frac{\alpha_i\alpha_k}{\alpha^2}(1-\cos\alpha) + \varepsilon_{ijk}\frac{\alpha_j}{\alpha}\sin\alpha \tag{2.219}$$

where the first index i represents the row and the second the column k, of the matrix which characterize this rotation. If we fix the vector $\boldsymbol{u}$, then any positive rotation of value α produces tha same rotation as another of value $2\pi - \alpha$ in the opposite direction. In order to single out a unique vector $\boldsymbol{\alpha}$, for each rotation, we have to restrict ourselves to the set of points of a sphere of radius π, (see figure 2.22) but with the constraint that opposite points on the surface of the sphere, which represent rotations of value π, represent the same rotation and have to be identified as the same point, from the topological point of view.

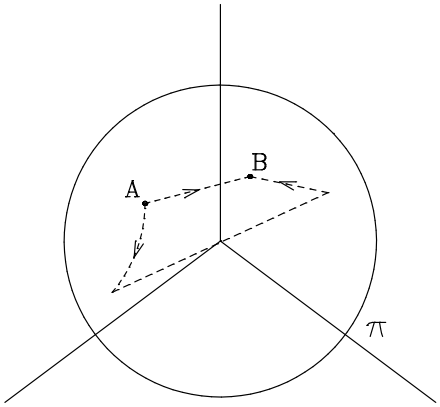


Figure 2.22: **Doubly connected and compact manifold of the group** $SO(3)$

This feature means that if we try to join two points of this manifold by a curve of points in it, there are two types of paths. These two types cannot be reduced to each other by deformation. There are paths passing through the surface and paths which do not cross the surface. This implies that the rotation group is characterized by a doubly connected, compact manifold.

Because the determinant of $\mathcal{R}(\boldsymbol{\alpha}) = 1$, then the rotation group is isomorphic to the group $SO(3)$, of 3×3 orthogonal matrices of unit determinant (Special Orthogonal group).

Other alternative parameterizations are obtained by defining a three vector $\boldsymbol{\phi} = \sin(\alpha/2)\boldsymbol{u}$ and the rotation matrix is given by:

$$\mathcal{R}(\boldsymbol{\phi})_{ik} = (1 - 2\phi^2)\,\delta_{ik} + 2\phi_i\phi_k + 2\sqrt{1-\phi^2}\,\varepsilon_{ijk}\phi_j. \tag{2.220}$$

Now the group manifold is a unit sphere with opposite points on its surface, identified.

Another interesting parameterization is given by the vector $\boldsymbol{\rho} = \tan(\alpha/2)\boldsymbol{u}$, where the matrix is

$$\mathcal{R}(\boldsymbol{\rho})_{ik} = \frac{1}{1+\rho^2}[(1-\rho^2)\,\delta_{ik} + 2\rho_i\rho_k + 2\varepsilon_{ijk}\rho_j] \tag{2.221}$$

where the manifold is the compact space $\mathbb{R}^3$, where compactification is done by adding to $\mathbb{R}^3$ the points of infinity in any direction, when the additional condition that opposite points represent the same rotation. We shall denote this manifold by $\mathbb{R}^3_c$, to enhance its compact character.

Exercise: Given the orthogonal matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

determine what kind of transformation produces.

Solution: Since the determinant is -1 it is a rotation followed by a space inversion. The trace

is zero, and the value of the rotation is $0 = 1 + 2\cos\alpha$, $\alpha = 2\pi/3$, around an axis with director cosines proporcional to $\boldsymbol{u} \sim (1, -1, 1)$.

Exercise: Calculate, by using two different parameterizations of the rotation group, the rotation matrix, in the passive sense, of value $\alpha = 30^\circ$ around an axis of director cosines proportional to $(-1, 2, 2)$.

2.8.1 Normal or Canonical parameterization of the group SO(3)

Any rotation matrix satisfies $\mathcal{R}^T\mathcal{R} = 1$. From this we have nine relations between the nine components of the matrix $\mathcal{R}$. However only six of these relations are independent. If we consider that any rotation matrix is formed, by raws or columns, as a set of three orthogonal unit vectors $\boldsymbol{e}_i$, $i = 1, 2, 3$ the above relations mean that these three vectors are orthogonal to each other and of modulus 1. The feature that the determinant is $+1$, represents that these vectors, taken in correlative order form a direct triad of unit vectors (anticlockwise). If the determinant is -1, they form a clockwise triad. Then only three values determine each rotation, and therefore the rotation group is of dimension 3. The part of the group continuously connected with the unit element, $SO(3)$, as a Lie group, has a Lie algebra of dimensiion 3. Let $\mathcal{R} = \mathbb{I} + \epsilon M$ an arbitary rotation close to the unit rotation, with ϵ infinitesimal and M a matrix to be determined. Since $\mathcal{R}^T = \mathcal{R}^{-1} = \mathbb{I} + \epsilon M^T = \mathbb{I} - \epsilon M$, implies that $M^T = -M$ and therefore M is an arbitrary 3×3 antisymmetric matrix. It is called the generator of the infinitesimal rotation.

The Lie algebre of $SO(3)$, is the real vector space of real 3×3 antisymmetric matrices. A basis of this vector space can be given by the three linearly independent antisymmetric matrices:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which clearly generate a real vector space of dimension 3.

Any Lie algebra, in addition of its structure as a real vector space, it also has another internal composition law, distributive with respect to the sum of elements, but it is not in general, neither commutative nor associative. To characterize this structure is sufficient to know this composition law for the basis vectors J_i. For matrices this law $[A, B]$ is just the commutator between them. The three J_i satisfy the following commutation rules:

$$[J_i, J_k] = \varepsilon_{ikl} J_l, \qquad i, k, l = 1, 2, 3, \tag{2.222}$$

Let $M = \sum \alpha_i J_i$ be an arbitrary linear combination of elements of the base J_i, with three arbitrary real numbers α_i. This sum we are going to write formally as $\sum \alpha_i J_i = \alpha \boldsymbol{u} \cdot \boldsymbol{J}$, where $\alpha_i = \alpha u_i$ in terms of the three components of a unit vector $\boldsymbol{u}$ and where by means of the *dot product*, $\boldsymbol{u} \cdot \boldsymbol{J} \equiv U$ what we want to express is just the sum $\sum u_i J_i$ in a compact way. If we calculate the matrix

$$\exp(M) \equiv \lim_{n\to\infty} \left(\mathbb{I} + \frac{M}{n}\right)^n \equiv \exp(\alpha U) = \exp\left\{\alpha \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}\right\} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\alpha}{1!} \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} + \frac{\alpha^2}{2!} \begin{pmatrix} -(u_2^2 + u_3^2) & u_1 u_2 & u_1 u_3 \\ u_1 u_2 & -(u_1^2 + u_3^2) & u_2 u_3 \\ u_1 u_3 & u_2 u_3 & -(u_1^2 + u_2^2) \end{pmatrix} +$$

$$+ \frac{\alpha^3}{3!} \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix} + \cdots$$

If we call

$$U = \boldsymbol{u}\cdot\boldsymbol{J} = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}, \qquad U^2 = \begin{pmatrix} -(u_2^2+u_3^2) & u_1u_2 & u_1u_3 \\ u_1u_2 & -(u_1^2+u_3^2) & u_2u_3 \\ u_1u_3 & u_2u_3 & -(u_1^2+u_2^2) \end{pmatrix},$$

then U satisfies $U^3 = -U$, $U^4 = -U^2$ and the subsequent powers, so that the above expansion can be expressed in terms of matrices U, U^2 and the unit matrix $\mathbb{I}$, in the form

$$\exp(\alpha U) \equiv \exp(\boldsymbol{\alpha}\cdot\boldsymbol{J}) = \mathbb{I} + U\left(\frac{\alpha}{1!} - \frac{\alpha^3}{3!} + \cdots\right) + U^2\left(\frac{\alpha^2}{2!} - \frac{\alpha^4}{4!} + \cdots\right),$$

i.e., the expression obtained previously in (2.219).

If we consider that two parameters α_i are zero and we analyze the one-parameter subgroup generated by the nonvanishing parameter, for instance α_1, then

$$\exp(\alpha J_1)\exp(\beta J_1) = (\mathbb{I} + \sin\alpha J_1 + (1-\cos\alpha)J_1^2)(\mathbb{I} + \sin\beta J_1 + (1-\cos\beta)J_1^2) =$$

$$\mathbb{I} + \sin(\alpha+\beta)J_1 + (1-\cos(\alpha+\beta))J_1^2 = \exp((\alpha+\beta)J_1),$$

and in this parametrization the composition of rotations of any one-parameter subgroup is just the addition of the corresponding parameters of the two elements. This parameter which defines the exponential mapping, is called the **normal or canonical parameter**.

The normal parameterization of the rotation group corresponds to that in which the group manifold is the compact sphere of radius π, and in this parameterization any rotation can also be represented by:

$$\mathcal{R}(\boldsymbol{\alpha})_{ik} = (\exp(\boldsymbol{\alpha}\cdot\boldsymbol{J}))^i{}_k = \delta_{ik}\cos\alpha + \frac{\alpha_i\alpha_k}{\alpha^2}(1-\cos\alpha) + \varepsilon_{ijk}\frac{\alpha_j}{\alpha}\sin\alpha,$$

which is the expression (2.219).

In an extended form $\mathcal{R}(\boldsymbol{\alpha})$, is:

$$\begin{pmatrix} \cos\alpha + u_1^2(1-\cos\alpha) & -u_3\sin\alpha + u_1u_2(1-\cos\alpha) & u_2\sin\alpha + u_1u_3(1-\cos\alpha) \\ u_3\sin\alpha + u_2u_1(1-\cos\alpha) & \cos\alpha + u_2^2(1-\cos\alpha) & -u_1\sin\alpha + u_2u_3(1-\cos\alpha) \\ -u_2\sin\alpha + u_3u_1(1-\cos\alpha) & u_1\sin\alpha + u_3u_2(1-\cos\alpha) & \cos\alpha + u_3^2(1-\cos\alpha) \end{pmatrix}$$

We can see that $\mathcal{R}(\boldsymbol{\alpha})^{-1} = \mathcal{R}^T(\boldsymbol{\alpha}) = \mathcal{R}(-\boldsymbol{\alpha})$ and that its trace is $1+2\cos\alpha$. The director cosines of the unit vector $\boldsymbol{u}$, which defines the direction of the rotation axis, are proportional to the terms $(\mathcal{R}_{32}-\mathcal{R}_{23}, \mathcal{R}_{13}-\mathcal{R}_{31}, \mathcal{R}_{21}-\mathcal{R}_{12})$, with the exception of a rotation of value $\alpha = \pi$, which in that case will be related to the diagonal elements because $\mathcal{R}$ is symmetric. These diagonal elements in this case are $-1+2u_1^2$, $-1+2u_2^2$ and $-1+2u_3^2$, respectively, and the two possible solutions for each u_i have to be compatible with the remaining elements of $\mathcal{R}_{ij}$.

$$1 + 2\cos\alpha = \mathcal{R}_{ii}, \quad u_i = \frac{1}{2\sin\alpha}\epsilon_{ijk}\mathcal{R}_{kj}, \quad \alpha \neq 0, \pi.$$

If $\alpha = 0$, the components $\mathcal{R}_{kj}$, $k \neq j$ vanish and the above relation in undetermined, as it corresponds to a nul rotation.

The eigenvalues of any rotation matrix are reduce to the real value 1 with eigenvector in the direction of the rotation axis, and another two eigenvalues, in general complex, of the form $e^{i\alpha}$ y $e^{-i\alpha}$, without real eigenvectors, which in the particular case $\alpha = \pi$ they are -1, and the corresponding eigenspace is the two-dimensional vector space orthogonal to the rotation axis.

Exercise: Given the following two rotation matrices determine the angle and axis of rotation.

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{10}}{5} & \frac{-\sqrt{10}}{10} \\ \frac{\sqrt{10}}{5} & \frac{1+2\sqrt{2}}{5} & \frac{\sqrt{2}-2}{5} \\ \frac{\sqrt{10}}{10} & \frac{\sqrt{2}-2}{5} & \frac{8+\sqrt{2}}{10} \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{6} & -\frac{4+\sqrt{3}}{6} & \frac{\sqrt{3}-1}{3} \\ \frac{\sqrt{3}-4}{6} & \frac{1}{6} & \frac{\sqrt{3}+1}{3} \\ -\frac{\sqrt{3}+1}{3} & \frac{1-\sqrt{3}}{3} & \frac{-1}{3} \end{pmatrix}$$

Solution: $\alpha_A = \pi/4, \quad \boldsymbol{n}_A \sim (0,-1,2). \qquad \alpha_B = 2\pi/3, \quad \boldsymbol{n}_B \sim (-2,2,1).$

The analysis of rotations we have done is called the active representation because we rotate the points in the space while leaving fixed the Cartesian reference frame. The passive interpretation consists in describing the coordinates of the same point in three-dimensional space with respect to two different frames which are rotated with respect to each other. But to rotate a coordinate system by means of the rotation $\boldsymbol{\alpha}$, the new coordinates of the point correspond to those of an active rotation in the opposite direction, of parameters $-\boldsymbol{\alpha}$. It is sufficiente to replace α by $-\alpha$ to obtain the matrix representation of a change of coordinates when we make a change of reference frame. In this case the commutation relations of the basic generators, in the passive representation, are

$$[J_i, J_k] = -\varepsilon_{ikl} J_l, \qquad i,k,l = 1,2,3$$

2.8.2 Composition law of rotations

If every rotation is represented by a vector $\boldsymbol{\alpha} \in SO(3)$, then it is possible to obtain the resultant vector of the composition of two arbitrary rotations. Let $\mathcal{R}(\boldsymbol{\gamma}) = \mathcal{R}(\boldsymbol{\alpha})\mathcal{R}(\boldsymbol{\beta})$ the composition of two rotations given by the product of the corresponding matrix representation. If the vectors are $\boldsymbol{\alpha} = \alpha \boldsymbol{u}$, $\boldsymbol{\beta} = \beta \boldsymbol{v}$ and $\boldsymbol{\gamma} = \gamma \boldsymbol{w}$, making the matrix product and after a term by term identification we get

$$\boldsymbol{w} \tan\frac{\gamma}{2} = \frac{\boldsymbol{u}\tan\alpha/2 + \boldsymbol{v}\tan\beta/2 + \tan\alpha/2\tan\beta/2(\boldsymbol{u}\times\boldsymbol{v})}{1 - \tan\alpha/2\tan\beta/2(\boldsymbol{u}\cdot\boldsymbol{v})} \tag{2.223}$$

If instead of using the normal parameterization we use the vectors

$$\boldsymbol{\rho} = \tan\frac{\alpha}{2}\boldsymbol{u}, \quad \boldsymbol{\mu} = \tan\frac{\beta}{2}\boldsymbol{v}, \quad \boldsymbol{\nu} = \tan\frac{\gamma}{2}\boldsymbol{w}$$

then $\mathcal{R}(\boldsymbol{\nu}) = \mathcal{R}(\boldsymbol{\rho})\mathcal{R}(\boldsymbol{\mu})$ implies:

$$\boldsymbol{\nu} = \frac{\boldsymbol{\rho} + \boldsymbol{\mu} + \boldsymbol{\rho}\times\boldsymbol{\mu}}{1 - \boldsymbol{\rho}\cdot\boldsymbol{\mu}} \tag{2.224}$$

We can see in the above relation that if $\alpha = \beta = \pi$, $\tan(\alpha/2) = \tan(\beta/2) = \infty$ and therefore in this limit:

$$\boldsymbol{w}\tan\frac{\gamma}{2} = \frac{\boldsymbol{v}\times\boldsymbol{u}}{\boldsymbol{u}\cdot\boldsymbol{v}}$$

so that the compound rotation is around an axis orthogonal to the previous ones in the direction of the cross product of the second times the first. If they are separated by an angle ϕ then $\tan(\gamma/2) = \sin\phi/\cos\phi = \tan\phi$, and the value of the rotation angle is $\gamma = 2\phi$, twice the angle that $\boldsymbol{u}$ and $\boldsymbol{v}$ subtend. Conversely, every rotation can always be written as the composition of two rotations of value π, around two axis orthogonal to its rotation axis and separated half the angle to be performed.

If we have a cylindrical lid and we turn around, i.e., we rotate it a value π around one of its diameters, and subsequently we make again another rotation of value π around another diameter, it is finally face up and its points have rotated an angle twice the angle subtended between the above diameters, and in the direction from the first axis to the second.

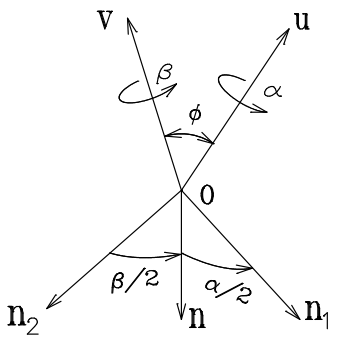


Figure 2.23: **Composition of rotations by means of rotations of value** π

This allows us to produce a geometrical picture of the composition of rotations [17] by using the decomposition of each one into two of value π. Let in the figure 2.23, $\boldsymbol{u}$ and $\boldsymbol{v}$ the two unit vectors which represent the two rotation axis of values α and β, respectively. If we construct the orthogonal planes to both vectors, passing through the point O, they intersect along a straight line characterized by the unit vector $\boldsymbol{n}$. In the perpendicular plane to vector $\boldsymbol{u}$, and in the anticlockwise direction, we locate another unit vector $\boldsymbol{n}_1$, separated from $\boldsymbol{n}$ by an angle $\alpha/2$. Similarly, in the plane orthogonal to $\boldsymbol{v}$, this time in the clockwise direction, we define the unit vector $\boldsymbol{n}_2$ separated $\beta/2$ from $\boldsymbol{n}$. Therefore:

$$\mathcal{R}(\beta,\boldsymbol{v})\mathcal{R}(\alpha,\boldsymbol{u}) = \mathcal{R}(\pi,\boldsymbol{n}_2)\mathcal{R}(\pi,\boldsymbol{n})\mathcal{R}(\pi,\boldsymbol{n})\mathcal{R}(\pi,\boldsymbol{n}_1) = \mathcal{R}(\pi,\boldsymbol{n}_2)\mathcal{R}(\pi,\boldsymbol{n}_1), \tag{2.225}$$

and thus the composite rotation is around an axis orthogonal to $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, in the sense $\boldsymbol{n}_2 \times \boldsymbol{n}_1$ of value twice the angle subtended by these two vectors.

The above analysis can also give rise to another geometrical interpretation on a unit sphere. Let us assume that, as usual each rotation is described by the rotation angle α and the unit vector $\boldsymbol{u}$, which defines the rotation axis. Let us represent both rotations on the unit sphere in the following way. Vector $\boldsymbol{u}$ defines a point, and this defines an equatorial plane orthogonal to $\boldsymbol{u}$. Along this maximal circle we depict an oriented circular segment of lentgh $\alpha/2$. Simmilarly we also depict the corresponding oriented circular segment of length y $\beta/2$ in the maximal circle orthogonal to the unit vector $\boldsymbol{v}$.

If we displace both circular segments, along the corresponding maximal circles, as in the figure 2.24, such that the segment AC is consecutive to the segment BA, then the points B and C will correspond with the end points of the unit vectors $\boldsymbol{n}_2$ and $\boldsymbol{n}_1$, respectively. Since the final rotation is orthogonal to both axis, the compound rotation axis is defined by the maximal circle passing through B y C, and the angle of rotation is twice the corresponding segment BC of value $\gamma/2$.

Because the angular separation between the two planes is $\pi - \phi$, where ϕ is the angle between the unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, by spherical trigonometry applied to the spherical triangle ABC, we find:

$$\cos\gamma/2 = \cos\alpha/2\cos\beta/2 + \sin\alpha/2\sin\beta/2\cos(\pi-\phi) =$$

[17] J.M. Aguirregabiria, A. Hernández, M. Rivas, Composition law of the rotation group, *Eur. J. Phys.*, **13**, 139-141 (1992).

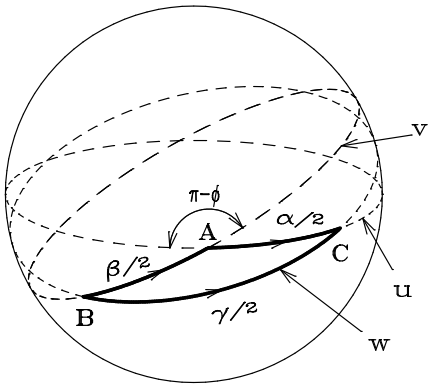


Figure 2.24: **Composition of rotations on the unit sphere**

$$= \cos\alpha/2\cos\beta/2 - \sin\alpha/2\sin\beta/2\cos\phi$$

which is a relation that can be obtained form the composition of the corresponding matrices associated to those rotations.

2.8.3 Kinematics of rotation

The description of a mechanical system with orientation, for instance a rigid body or a spinning elementary particle, is by means of three unit vectors $\boldsymbol{e}_i, i = 1,2,3$, of the three orthogonal axis associated to a moving point. In the case of the rigid body, these axes can be the principal axes of inertia around the center of mass of the body. In the case of an elementary particle, an arbitrary Cartesian frame located at the center of charge.

If these three unit vectors are written as column vectors, consecutively, they form an orthogonal 3×3 matrix of unit determinant, i.e., a rotation matrix. Then, only three essential parameters $\alpha_i, i = 1,2,3$, characterize the independent degrees of freedom associated to the change of orientation.

$$\mathcal{R}(\boldsymbol{\alpha}) = ((\boldsymbol{e}_1),\ (\boldsymbol{e}_2),\ (\boldsymbol{e}_3))$$

If at instant $t = 0$, we select the laboratory axis in coincidence with the body axis, then at instant t, the matrix $\mathcal{R}(\boldsymbol{\alpha}(t))$ represents the active rotation I have to produce to the laboratory axis to transform them into the body axis.

If we consider now another inertial observer O' related to O by means a Galilei transformation, then the relative spacetime measurement of some spacetime event is given by

$$t' = t + b, \quad \boldsymbol{r}' = \mathcal{R}(\boldsymbol{\mu})\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}.$$

This means that the three unit vectors linked to the body transform among inertial observers in the form

$$\boldsymbol{e}'_i = \mathcal{R}(\boldsymbol{\mu})\boldsymbol{e}_i,$$

and if we collect them in the form of a matrix in both members, at any instant t:

$$\big((\boldsymbol{e}'_1),\ (\boldsymbol{e}'_2),\ (\boldsymbol{e}'_3)\big) \equiv \mathcal{R}(\boldsymbol{\alpha}'(t')) = (\mathcal{R}(\boldsymbol{\mu})(\boldsymbol{e}_1),\ \mathcal{R}(\boldsymbol{\mu})(\boldsymbol{e}_2),\ \mathcal{R}(\boldsymbol{\mu})(\boldsymbol{e}_3)) = \mathcal{R}(\boldsymbol{\mu})\mathcal{R}(\boldsymbol{\alpha}(t)) \quad (2.226)$$

For the observer O, $\mathcal{R}(\boldsymbol{\alpha}(t))$ is the orientation of the body at the instant t and $\mathcal{R}(\boldsymbol{\alpha}(t+dt))$ the orientation at the instant $t+dt$. This can be written as a rotation $\mathcal{R}(\boldsymbol{\alpha}(t))$ followed by the infinitesimal rotation $\mathbb{I}+\Omega dt$, i.e.,

$$\mathcal{R}(\boldsymbol{\alpha}(t+dt)) = (\mathbb{I}+\Omega dt)\mathcal{R}(\boldsymbol{\alpha}(t)) = \mathcal{R}(\boldsymbol{\alpha}(t)) + \Omega\mathcal{R}(\boldsymbol{\alpha}(t))dt = \mathcal{R}(\boldsymbol{\alpha}(t)) + \dot{\mathcal{R}}(\boldsymbol{\alpha}(t))dt,$$

and the matrix Ω, is

$$\Omega(t)\mathcal{R}(\boldsymbol{\alpha}(t)) = \dot{\mathcal{R}}(\boldsymbol{\alpha}(t)), \quad \rightarrow \quad \Omega(t) = \dot{\mathcal{R}}(\boldsymbol{\alpha}(t))\mathcal{R}^{-1}(\boldsymbol{\alpha}(t)) = \dot{\mathcal{R}}(\boldsymbol{\alpha}(t))\mathcal{R}^{T}(\boldsymbol{\alpha}(t))$$

and Ω is an antisymmetric matrix with three essential components wich define the components of the instantaneous angular velocity $\boldsymbol{\omega}(t)$, $\omega_i = \frac{1}{2}\epsilon_{ijk}\Omega_{jk}$.

> In fact, for any rotation matrix $\mathcal{R}\mathcal{R}^T = \mathbb{I}$, and also at any instant t, $\mathcal{R}(t)\mathcal{R}^T(t) = \mathbb{I}$, and thus taking the time derivative
> $$\dot{\mathcal{R}}\mathcal{R}^T + \mathcal{R}\dot{\mathcal{R}}^T = 0, \quad \Omega + \Omega^T = 0.$$

The relation $\dot{\mathcal{R}}(\boldsymbol{\alpha}(t)) = \Omega(t)\mathcal{R}(\boldsymbol{\alpha}(t))$, if we analyze by columns is equivalent to

$$\frac{d\boldsymbol{e}_i}{dt} = \Omega\boldsymbol{e}_i \equiv \boldsymbol{\omega}\times\boldsymbol{e}_i.$$

The kinematics corresponds to an instantaneous rotation around an axis in the direction of $\boldsymbol{\omega}$.

If we express the rotations in terms of the vector $\boldsymbol{\alpha} = \alpha\boldsymbol{u}$, the angular velocity is given by

$$\boldsymbol{\omega} = \boldsymbol{u}\frac{d\alpha}{dt} + \sin\alpha\frac{d\boldsymbol{u}}{dt} + (1-\cos\alpha)\boldsymbol{u}\times\frac{d\boldsymbol{u}}{dt}. \tag{2.227}$$

> **Exercise.** Show that if we use the parameterization of the orientation by the three-vector $\boldsymbol{\rho} = \tan(\alpha/2)\boldsymbol{n}$, where $\boldsymbol{n}$ is the unit vector along the rotation axis and α the rotated angle, the angular velocity can be written as
> $$\boldsymbol{\omega} = \frac{2}{1+\rho^2}(\dot{\boldsymbol{\rho}} + \boldsymbol{\rho}\times\dot{\boldsymbol{\rho}}), \quad \boldsymbol{w} = R^T(\boldsymbol{\rho})\boldsymbol{\omega} = \frac{2}{1+\rho^2}(\dot{\boldsymbol{\rho}} - \boldsymbol{\rho}\times\dot{\boldsymbol{\rho}}).$$
> where $\boldsymbol{w}$ is the angular velocity vector with respect to the body frame.

If in (2.226) we take the derivative of both sides with respect to t', taking into account that $\partial t/\partial t' = 1$, gives

$$\dot{\mathcal{R}}(\boldsymbol{\alpha}'(t')) = \mathcal{R}(\boldsymbol{\mu})\dot{\mathcal{R}}(\boldsymbol{\alpha}(t))$$

and taking the transpose of (2.226)

$$\mathcal{R}^T(\boldsymbol{\alpha}'(t')) = \mathcal{R}^T(\boldsymbol{\alpha}(t))\mathcal{R}^T(\boldsymbol{\mu})$$

and thus the matrices Ω transform between inertial observers

$$\Omega'(t') = \dot{\mathcal{R}}(\boldsymbol{\alpha}'(t'))\mathcal{R}^T(\boldsymbol{\alpha}'(t')) = \mathcal{R}(\boldsymbol{\mu})\dot{\mathcal{R}}(\boldsymbol{\alpha}(t))\mathcal{R}^T(\boldsymbol{\alpha}(t))\mathcal{R}^T(\boldsymbol{\mu}) = \mathcal{R}(\boldsymbol{\mu})\,\Omega(t)\,\mathcal{R}^T(\boldsymbol{\mu})$$

which corresponds to the transformation equations of a second rank antisymmetric tensor, such that for its essential components, gives

$$\boldsymbol{\omega}'(t') = \mathcal{R}(\boldsymbol{\mu})\boldsymbol{\omega}(t).$$

From expression (2.221) we get that the unit vectors associated to the body axis $\boldsymbol{e}_k$, in the $\boldsymbol{\rho}$ representation of rotations, admit the following representation

$$(\boldsymbol{e}_k)_i = \frac{1}{1+\rho^2}[(1-\rho^2)\,\delta_{ik} + 2\rho_i\rho_k + 2\varepsilon_{ijk}\rho_j]. \tag{2.228}$$

2.8.4 Dynamics of rotation

If we want to make the Lagrangian description of a body with orientation $\boldsymbol{\alpha}$, because the rotation group has no central extensions, and the dynamical equations must be rotation invariant, then the Lagrangian has to be an invariant function $L(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})$, of the variables we use to describe the orientation $\boldsymbol{\alpha}$, and its time derivatives $\dot{\boldsymbol{\alpha}}$. It must be a function of them through its dependence of the angular velocity ω_i, $L(\omega_i)$. In this way, Euler-Lagrange dynamical equations are

$$\frac{\partial L}{\partial \alpha_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}_i}\right) = \frac{\partial L}{\partial \omega_j}\frac{\partial \omega_j}{\partial \alpha_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \omega_j}\frac{\partial \omega_j}{\partial \dot{\alpha}_i}\right) = 0,$$

If we call $W_j = \partial L/\partial \omega_j$, we propose to the reader (is relatively simpler in the $\boldsymbol{\rho}$ parameterization) to show that the above equations lead to

$$\frac{d\boldsymbol{W}}{dt} = \boldsymbol{\omega} \times \boldsymbol{W}, \quad W_i = \frac{\partial L}{\partial \omega_i}.$$

The angular momentum components with respect to the body axis, are constants of the motion. Let us call $T_i = \boldsymbol{W} \cdot \boldsymbol{e}_i$. Its time derivative gives

$$\frac{dT_i}{dt} = \frac{d\boldsymbol{W}}{dt} \cdot \boldsymbol{e}_i + \boldsymbol{W} \cdot \frac{d\boldsymbol{e}_i}{dt} = (\boldsymbol{\omega} \times \boldsymbol{W}) \cdot \boldsymbol{e}_i + \boldsymbol{W} \cdot (\boldsymbol{\omega} \times \boldsymbol{e}_i) = 0.$$

In the case of a nonrelativistic elementary particle, if it is a rigid body and its spin is a constant of the motion, then $\boldsymbol{W} \sim \boldsymbol{\omega}$ and the Lagrangian has to be an arbitrary function of $\boldsymbol{\omega}^2$. A simple case corresponds to

$$L = \frac{1}{2} I \boldsymbol{\omega}^2$$

an object with spherical symmetry, i.e., with the three principal moments of inertia of the same value, and the angular momentum $\boldsymbol{S} = \boldsymbol{W} = I\boldsymbol{\omega}$. We have to remark that an object with the three principal inertia momenta of the same value does not mean that its shape is that of a sphere. The same thing happens to a cube.

If the three principal momenta are different

$$L = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2),$$

and Euler-Lagrange equations are, with $S_i = I_i\omega_i$, (no addition on indes i)

$$I_1\frac{d\omega_1}{dt} = (I_3 - I_2)\omega_2\omega_3, \quad I_2\frac{d\omega_2}{dt} = (I_1 - I_3)\omega_3\omega_1, \quad I_3\frac{d\omega_3}{dt} = (I_2 - I_1)\omega_1\omega_2.$$

The ω_1 component will be a constant of the motion if $I_3 = I_2$, and the same criteria for the others.

2.9 Appendix: Galilei group

The Galilei group is a group of space-time transformations characterized by ten parameters $g \equiv (b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$. The action of g on a space-time point $x \equiv (t, \boldsymbol{r})$ is given by $x' = gx$, and is considered in the form

$$x' = \exp(bH)\exp(\boldsymbol{a}\cdot\boldsymbol{P})\exp(\boldsymbol{v}\cdot\boldsymbol{K})\exp(\boldsymbol{\alpha}\cdot\boldsymbol{J})\,x$$

as the action of a rotation of value $\boldsymbol{\alpha}$, followed by a pure Galilei transformation of velocity $\boldsymbol{v}$ and finally a space and time translation of values $\boldsymbol{a}$ and b, respectively. In this way all parameters that define each one-parameter subgroup are normal, because the exponential mapping works. Explicitly

$$\begin{aligned} t' &= t + b, && (2.229)\\ \boldsymbol{r}' &= R(\boldsymbol{\alpha})\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}, && (2.230) \end{aligned}$$

and the composition law of the group $g'' = g'g$ is:

$$\begin{aligned} b'' &= b' + b, && (2.231)\\ \boldsymbol{a}'' &= R(\boldsymbol{\alpha}')\boldsymbol{a} + \boldsymbol{v}'b + \boldsymbol{a}', && (2.232)\\ \boldsymbol{v}'' &= R(\boldsymbol{\alpha}')\boldsymbol{v} + \boldsymbol{v}', && (2.233)\\ R(\boldsymbol{\alpha}'') &= R(\boldsymbol{\alpha}')R(\boldsymbol{\alpha}). && (2.234) \end{aligned}$$

For rotations we shall alternatively use two different parameterizations. One is the normal or canonical parameterization in terms of a three vector $\boldsymbol{\alpha} = \alpha\boldsymbol{n}$, where $\boldsymbol{n}$ is a unit vector along the rotation axis, and $\alpha \in [0, \pi]$ is the clockwise rotation angle in radians, when looking along $\boldsymbol{n}$. Another, in terms of a three vector $\boldsymbol{\mu} = \boldsymbol{n}\tan(\alpha/2)$, which is more suitable to represent algebraically the composition of rotations.

The rotation matrix $R(\boldsymbol{\alpha}) = \exp(\boldsymbol{\alpha}\cdot\boldsymbol{J})$ is expressed in terms of the normal parameters α_i and in terms of the antisymmetric matrix generators J_i which have the usual matrix representation

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and satisfy the commutation relations $[J_i, J_k] = \epsilon_{ikl}J_l$, such that if we write the normal parameters $\boldsymbol{\alpha} = \alpha\boldsymbol{n}$ in terms of the rotation angle α and the unit vector $\boldsymbol{n}$ along the rotation axis, it is written as

$$R(\boldsymbol{\alpha})_{ij} = \delta_{ij}\cos\alpha + n_i n_j(1-\cos\alpha) - \epsilon_{ijk}n_k\sin\alpha, \quad i,j,k = 1,2,3. \tag{2.235}$$

In the parametrization $\boldsymbol{\mu} = \boldsymbol{n}\tan(\alpha/2)$, the rotation matrix is

$$R(\boldsymbol{\mu})_{ij} = \frac{1}{1+\mu^2}\left((1-\mu^2)\delta_{ij} + 2\mu_i\mu_j - 2\epsilon_{ijk}\mu_k\right), \quad i,j,k = 1,2,3. \tag{2.236}$$

In terms of these variables, $R(\boldsymbol{\mu}'') = R(\boldsymbol{\mu}')R(\boldsymbol{\mu})$ is equivalent to

$$\boldsymbol{\mu}'' = \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} + \boldsymbol{\mu}'\times\boldsymbol{\mu}}{1 - \boldsymbol{\mu}'\cdot\boldsymbol{\mu}}. \qquad \text{(active)} \tag{2.237}$$

This can be seen in a simple manner by using the homomorphism between the rotation group and the group $SU(2)$, of 2×2 unitary matrices of unit determinant. The matrix generators

of $SU(2)$ are $\boldsymbol{J} = -i\boldsymbol{\sigma}/2$ in terms of $\boldsymbol{\sigma}$ Pauli matrices. In the normal parameterization the rotation matrix $\exp(\boldsymbol{\alpha}\cdot\boldsymbol{J}) = \exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}/2)$ is written in the form

$$R(\boldsymbol{\alpha}) = \cos(\alpha/2)\mathbb{I} - i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sin(\alpha/2).$$

By defining $\boldsymbol{\mu} = \boldsymbol{n}\tan(\alpha/2)$, this rotation matrix is expressed as

$$R(\boldsymbol{\mu}) = \frac{1}{\sqrt{1+\mu^2}}\left(\mathbb{I} - i\boldsymbol{\mu}\cdot\boldsymbol{\sigma}\right), \tag{2.238}$$

where $\mathbb{I}$ is the 2×2 unit matrix and in this form we can get the composition law (2.237). [18]

If the rotation is of value π, then eqs. (2.235) or (2.236) lead to

$$R(\boldsymbol{n},\pi)_{ij} = -\delta_{ij} + 2n_i n_j.$$

Even if the two rotations $R(\boldsymbol{\mu})$ and $R(\boldsymbol{\mu}')$ involved in (2.237) are of value π, although $\tan(\pi/2) = \infty$, this expression is defined and gives:

$$\boldsymbol{n}''\tan(\alpha''/2) = \frac{\boldsymbol{n}\times\boldsymbol{n}'}{\boldsymbol{n}\cdot\boldsymbol{n}'}.$$

The absolute value of this relation leads to $\tan(\alpha''/2) = \tan\theta$, *i.e.*, $\alpha'' = 2\theta$, where θ is the angle between the two unit vectors $\boldsymbol{n}$ and $\boldsymbol{n}'$. We obtain the known result that every rotation of value α around an axis $\boldsymbol{n}$ can be obtained as the composition of two rotations of value π around two axes orthogonal to $\boldsymbol{n}$ and separated by an angle $\alpha/2$.

Because every transformation of the Galilei group corresponds to a change of reference frame, it is necessary to consider the rotations from the passive point of view. This amounts, when compared with the active point of view a simple change of sign in the group parameter. In this way, the composition of rotations in the passive representation is:

$$\boldsymbol{\mu}'' = \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} - \boldsymbol{\mu}'\times\boldsymbol{\mu}}{1 - \boldsymbol{\mu}'\cdot\boldsymbol{\mu}}. \qquad \text{(passive)} \tag{2.239}$$

For the orientation variables we shall use throughout the book the early Greek variables $\boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots$ whenever we consider the normal parametrization, while for the $\tan(\alpha/2)$ parameterization we will express rotations in terms of the intermediate Greek variables $\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\rho}, \ldots$. In this last notation, transformation equations (2.231-2.234) should be replaced by

$$b'' = b' + b, \tag{2.240}$$
$$\boldsymbol{a}'' = R(\boldsymbol{\mu}')\boldsymbol{a} + \boldsymbol{v}'b + \boldsymbol{a}', \tag{2.241}$$
$$\boldsymbol{v}'' = R(\boldsymbol{\mu}')\boldsymbol{v} + \boldsymbol{v}', \tag{2.242}$$
$$\boldsymbol{\mu}'' = \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} - \boldsymbol{\mu}'\times\boldsymbol{\mu}}{1 - \boldsymbol{\mu}'\cdot\boldsymbol{\mu}}. \tag{2.243}$$

The neutral element of the Galilei group is $(0, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$ and the inverse of every element is

$$(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})^{-1} = (-b, -R(-\boldsymbol{\alpha})(\boldsymbol{a} - b\boldsymbol{v}), -R(-\boldsymbol{\alpha})\boldsymbol{v}, -\boldsymbol{\alpha}).$$

The generators of the group in the realization (2.229, 2.230) are the differential operators

$$H = \partial/\partial t, \quad P_i = \partial/\partial r_i, \quad K_i = t\partial/\partial r_i, \quad J_k = \varepsilon_{kli} r_l \partial/\partial r_i \tag{2.244}$$

and the commutation rules of the Galilei Lie algebra are

$$[\boldsymbol{J}, \boldsymbol{J}] = -\boldsymbol{J}, \quad [\boldsymbol{J}, \boldsymbol{P}] = -\boldsymbol{P}, \quad [\boldsymbol{J}, \boldsymbol{K}] = -\boldsymbol{K}, \quad [\boldsymbol{J}, H] = 0, \tag{2.245}$$

[18] D. Hestenes, *Space-time algebra*, Gordon and Breach, NY (1966).

$$[H, \boldsymbol{P}] = 0,\ [H, \boldsymbol{K}] = \boldsymbol{P},\ [\boldsymbol{P}, \boldsymbol{P}] = 0,\ [\boldsymbol{K}, \boldsymbol{K}] = 0,\ [\boldsymbol{K}, \boldsymbol{P}] = 0. \tag{2.246}$$

All throughout this book, except when explicitly stated, we shall use the following shorthand notation for commutators of scalar and 3-vector operators, that as usual, are represented by bold face characters:

$$\begin{aligned}
[\boldsymbol{A}, \boldsymbol{B}] &= \boldsymbol{C}, \quad \Longrightarrow \quad [A_i, B_j] = \epsilon_{ijk} C_k, \\
[\boldsymbol{A}, \boldsymbol{B}] &= C, \quad \Longrightarrow \quad [A_i, B_j] = \delta_{ij} C, \\
[\boldsymbol{A}, B] &= \boldsymbol{C}, \quad \Longrightarrow \quad [A_i, B] = C_i, \\
[B, \boldsymbol{A}] &= \boldsymbol{C}, \quad \Longrightarrow \quad [B, A_i] = C_i,
\end{aligned}$$

where $\delta_{ij} = \delta_{ji}$ is Kronecker's delta and ϵ_{ijk} is the completely antisymmetric symbol, so that Latin indexes match on both sides of commutators.

The group action (2.229)-(2.230) represents the relationship between the coordinates $(t, \boldsymbol{r})$ of a space-time event as measured by the inertial observer O and the corresponding coordinates $(t', \boldsymbol{r}')$ of the same space-time event as measured by another inertial observer O'. The ten group parameters have the following meaning. If we consider the event $(0, \boldsymbol{0})$ measured by O, for instance the flashing of a light beam from its origin at time $t = 0$, it takes the values $(b, \boldsymbol{a})$ in O', where b is the time parameter that represents the time translation and $\boldsymbol{a}$ is the space translation. The parameter $\boldsymbol{v}$ of dimensions of velocity represents the velocity of the origin of the Cartesian frame of O as measured by O', and finally the parameters $\boldsymbol{\alpha}$, or $R(\boldsymbol{\alpha})$, represent the orientation of the Cartesian frame of O as measured by O'. In a certain sense the ten parameters $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ with dimensions respectively of time, position, velocity and orientation describe the relative motion of the Cartesian frame of O by O'.

The Galilei group has non-trivial exponents given by [19]

$$\xi(g, g') = m \left(\frac{1}{2} \boldsymbol{v}^2 b' + \boldsymbol{v} \cdot R(\boldsymbol{\alpha}) \boldsymbol{a}' \right). \tag{2.247}$$

They are characterized by the non-vanishing parameter m.

The central extension of the Galilei group [20] is an 11-parameter group with an additional generator I which commutes with the other ten,

$$[I, H] = [I, \boldsymbol{P}] = [I, \boldsymbol{K}] = [I, \boldsymbol{J}] = 0, \tag{2.248}$$

and the remaining commutation relations are the same as above (2.245, 2.246), except the last one which appears as

$$[K_i, P_j] = -m \delta_{ij} I, \quad \text{or} \quad [\boldsymbol{K}, \boldsymbol{P}] = -mI, \tag{2.249}$$

using our shorthand notation, in terms of a non-vanishing parameter m. If we define the following polynomial operators on the group algebra

$$\boldsymbol{W} = I\boldsymbol{J} - \frac{1}{m} \boldsymbol{K} \times \boldsymbol{P}, \quad U = IH - \frac{1}{2m} \boldsymbol{P}^2, \tag{2.250}$$

U commutes with all generators of the extended Galilei group and $\boldsymbol{W}$ satisfies the commutation relations:

$$[\boldsymbol{W}, \boldsymbol{W}] = -I\boldsymbol{W}, \quad [\boldsymbol{J}, \boldsymbol{W}] = -\boldsymbol{W}, \quad [\boldsymbol{W}, \boldsymbol{P}] = [\boldsymbol{W}, \boldsymbol{K}] = [\boldsymbol{W}, H] = 0,$$

so that $\boldsymbol{W}^2$ also commutes with all generators. It turns out that the extended Galilei group has three functionally independent Casimir operators which, in those representations in which the

[19] V. Bargmann, On unitary ray representations of continuous groups, *Ann. Math.* **5**, 1 (1954).

[20] J.M. Levy-Leblond, *Galilei Group and Galilean Invariance*, in E.M. Loebl, *Group Theory and its applications*, Acad. Press, NY (1971), vol. 2, p. 221.

operator I becomes the unit operator, for instance in irreducible representations, are interpreted as the mass, $M = mI$, the internal energy $H_0 = H - P^2/2m$, and the absolute value of the spin with respect to the center of mass

$$S^2 = \left(\boldsymbol{J} - \frac{1}{m} \boldsymbol{K} \times \boldsymbol{P} \right)^2 . \tag{2.251}$$

The spin operator $\boldsymbol{S}$ in those representations in which $I = \mathbb{I}$, satisfy the commutation relations:

$$[\boldsymbol{S}, \boldsymbol{S}] = -\boldsymbol{S}, \quad [\boldsymbol{J}, \boldsymbol{S}] = -\boldsymbol{S}, \quad [\boldsymbol{S}, \boldsymbol{P}] = [\boldsymbol{S}, H] = [\boldsymbol{S}, \boldsymbol{K}] = 0,$$

i.e., it is an angular momentum operator, transforms like a vector under rotations and is invariant under space and time translations and under Galilei boosts, respectively.

2.10 Appendix: Poincaré group

The Poincaré group is the group of transformations of Minkowski's space-time that leave invariant the separation between any two close space-time events $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - d\boldsymbol{r}^2$. We shall consider the contravariant components $x^\mu \equiv (ct, \boldsymbol{r})$, and $x' = gx$ is expressed as $x'^\mu = \Lambda^\mu{}_\nu\, x^\nu + a^\mu$, in terms of a constant 4×4 matrix Λ and a constant translation four-vector $a^\mu \equiv (cb, \boldsymbol{a})$. We take for the covariant components of Minkowski's metric tensor $\eta_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$. Then $dx'^\mu = \Lambda^\mu{}_\nu dx^\nu$ and $ds^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\sigma\rho} dx^\sigma dx^\rho$ implies for the matrix Λ

$$\eta_{\mu\nu} \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho = \eta_{\sigma\rho}. \tag{2.252}$$

Relations (2.252) represent ten conditions among the 16 components of the matrix Λ, so that each matrix depends on six essential parameters, which can be chosen in many ways. Throughout this book we shall take three of them as the components of the relative velocity $\boldsymbol{v}$ between inertial observers and the remaining three as the orientation $\boldsymbol{\alpha}$ of their Cartesian frames, expressed in a suitable parametrization of the rotation group.

Therefore, every element of the Poincaré group $\mathcal{P}$ will be represented, as in the previous case of the Galilei group, by the ten parameters $g \equiv (b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ and the group action on a space-time point $x \equiv (t, \boldsymbol{r})$ will be interpreted in the same way, *i.e.*, $x' = gx$:

$$x' = \exp(bH) \exp(\boldsymbol{a} \cdot \boldsymbol{P}) \exp(\boldsymbol{\beta} \cdot \boldsymbol{K}) \exp(\boldsymbol{\alpha} \cdot \boldsymbol{J}) x, \tag{2.253}$$

as the action of a rotation of value $\boldsymbol{\alpha}$, followed by a boost or pure Lorentz transformation of normal parameter $\boldsymbol{\beta}$ and finally a space and time translation of values $\boldsymbol{a}$ and b, respectively. It is explicitly given on the space-time variables by

$$t' = \gamma t + \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r})/c^2 + b, \tag{2.254}$$

$$\boldsymbol{r}' = R(\boldsymbol{\mu})\boldsymbol{r} + \gamma \boldsymbol{v} t + \gamma^2 (\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r})\boldsymbol{v}/(1+\gamma)c^2 + \boldsymbol{a}. \tag{2.255}$$

Parameter $\boldsymbol{\beta}$ in (2.253) is the normal parameter for the pure Lorentz transformations, that in terms of the relative velocity among observers $\boldsymbol{v}$ is expressed as $\boldsymbol{\beta}/\beta \tanh\beta = \boldsymbol{v}/c$ as we shall see below. The dimensions and domains of the parameters b, $\boldsymbol{a}$ and $\boldsymbol{\mu}$ are the same as those of the Galilei group, and the parameter $\boldsymbol{v} \in \mathbb{R}^3$, with the upper bound $v < c$, has also dimensions of velocity. The physical meaning of these ten parameters, that relate any two inertial observers, is the same as in the Galilei case. The parameter $\boldsymbol{v}$ is the velocity of the origin of the observer O, as measured by O', and $R(\boldsymbol{\mu})$ represents the orientation of the Cartesian frame O relative to O', once O' is boosted with velocity $\boldsymbol{v}$. The factor $\gamma(v) = (1 - v^2/c^2)^{-1/2}$.

The composition law of the group is obtained from $x'' = \Lambda' x' + a' = \Lambda'(\Lambda x + a) + a'$ that by identification with $x'' = \Lambda'' x + a''$ reduces to $\Lambda'' = \Lambda'\Lambda$ and $a'' = \Lambda' a + a'$, *i.e.*, the composition law of the Lorentz transformations, that we will find in the next Section 2.10.1, and a Poincaré transformation (Λ', a') of the four-vector a^μ. In this parameterization $g'' = g'g$, is: [21]

$$b'' = \gamma' b + \gamma'(\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{a})/c^2 + b', \tag{2.256}$$

$$\boldsymbol{a}'' = R(\boldsymbol{\mu}')\boldsymbol{a} + \gamma' \boldsymbol{v}' b + \frac{\gamma'^2}{(1+\gamma')c^2} (\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{a})\boldsymbol{v}' + \boldsymbol{a}', \tag{2.257}$$

$$\boldsymbol{v}'' = \frac{R(\boldsymbol{\mu}')\boldsymbol{v} + \gamma'\boldsymbol{v}' + \dfrac{\gamma'^2}{(1+\gamma')c^2}(\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{v})\boldsymbol{v}'}{\gamma'(1 + \boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\,\boldsymbol{v}/c^2)}, \tag{2.258}$$

$$\boldsymbol{\mu}'' = \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} - \boldsymbol{\mu}' \times \boldsymbol{\mu} + \boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})}{1 - \boldsymbol{\mu}' \cdot \boldsymbol{\mu} + G(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})}, \tag{2.259}$$

[21] M.Rivas, M.Valle and J.M.Aguirregabiria, Composition law and contractions of the Poincaré group, *Eur. J. Phys.* **6**, 128 (1986).

where $\boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})$ and $G(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})$ are the real functions:

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu}) &= \frac{\gamma\gamma'}{(1+\gamma)(1+\gamma')c^2}\big[\boldsymbol{v}\times\boldsymbol{v}' + \boldsymbol{v}(\boldsymbol{v}'\cdot\boldsymbol{\mu}') + \boldsymbol{v}'(\boldsymbol{v}\cdot\boldsymbol{\mu}) \\ &+ \boldsymbol{v}\times(\boldsymbol{v}'\times\boldsymbol{\mu}') + (\boldsymbol{v}\times\boldsymbol{\mu})\times\boldsymbol{v}' + (\boldsymbol{v}\cdot\boldsymbol{\mu})(\boldsymbol{v}'\times\boldsymbol{\mu}') \\ &+ (\boldsymbol{v}\times\boldsymbol{\mu})(\boldsymbol{v}'\cdot\boldsymbol{\mu}') + (\boldsymbol{v}\times\boldsymbol{\mu})\times(\boldsymbol{v}'\times\boldsymbol{\mu}')\big], \end{aligned} \tag{2.260}$$

$$\begin{aligned}G(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu}) &= \frac{\gamma\gamma'}{(1+\gamma)(1+\gamma')c^2}\big[\boldsymbol{v}\cdot\boldsymbol{v}' + \boldsymbol{v}\cdot(\boldsymbol{v}'\times\boldsymbol{\mu}') + \boldsymbol{v}'\cdot(\boldsymbol{v}\times\boldsymbol{\mu}) \\ &- (\boldsymbol{v}\cdot\boldsymbol{\mu})(\boldsymbol{v}'\cdot\boldsymbol{\mu}') + (\boldsymbol{v}\times\boldsymbol{\mu})\cdot(\boldsymbol{v}'\times\boldsymbol{\mu}')\big]. \end{aligned} \tag{2.261}$$

The unit element of the group is $(0, \mathbf{0}, \mathbf{0}, \mathbf{0})$ and the inverse of any arbitrary element $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})$ is

$$(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})^{-1} = (-\gamma b + \gamma\boldsymbol{v}\cdot\boldsymbol{a}/c^2, -R(-\boldsymbol{\mu})(\boldsymbol{a} - \gamma\boldsymbol{v}b + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v}), -R(-\boldsymbol{\mu})\boldsymbol{v}, -\boldsymbol{\mu}).$$

The group generators in the realization (2.254, 2.255), and in terms of the normal parameters $(b, \boldsymbol{a}, \boldsymbol{\beta}, \boldsymbol{\alpha})$, are

$$H = \partial/\partial t,\ P_i = \partial/\partial r_i,\ K_i = ct\partial/\partial r_i + (r_i/c)\partial/\partial t,\ J_k = \varepsilon_{kli} r_l \partial/\partial r_i.$$

Thus, $\boldsymbol{K}$ and $\boldsymbol{J}$ are dimensionless and the commutation relations become

$$[\boldsymbol{J}, \boldsymbol{J}] = -\boldsymbol{J},\ [\boldsymbol{J}, \boldsymbol{P}] = -\boldsymbol{P},\ [\boldsymbol{J}, \boldsymbol{K}] = -\boldsymbol{K},\ [\boldsymbol{J}, H] = 0,\ [H, \boldsymbol{P}] = 0, \tag{2.262}$$

$$[H, \boldsymbol{K}] = c\boldsymbol{P},\ [\boldsymbol{P}, \boldsymbol{P}] = 0,\ [\boldsymbol{K}, \boldsymbol{K}] = \boldsymbol{J},\ [\boldsymbol{K}, \boldsymbol{P}] = -H/c. \tag{2.263}$$

If, as usual, we call $x^0 = ct$, $p^0 = H/c$, $p^i = P_i$ and $K_i = J_{0i} = -J_{i0}$ and $J_k = \frac{1}{2}\epsilon_{klr}J_{lr}$, $x_\mu = \eta_{\mu\nu}x^\nu$, $\mu = 0, 1, 2, 3$ and $\partial_\nu \equiv \partial/\partial x^\nu$, then,

$$p_\mu = \partial_\mu, \quad J_{\mu\nu} = -J_{\nu\mu} = x_\mu\partial_\nu - x_\nu\partial_\mu.$$

In covariant notation the commutation relations appear:

$$\begin{aligned}[p_\mu, p_\nu] &= 0, \\ [J_{\mu\nu}, p_\sigma] &= -\eta_{\mu\sigma}p_\nu + \eta_{\nu\sigma}p_\mu, \\ [J_{\mu\nu}, J_{\rho\sigma}] &= -\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho} + \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho}.\end{aligned}$$

The Poincaré group has two functionally independent Casimir invariants. One is interpreted as the squared mass of the system,

$$p^\mu p_\mu = (H/c)^2 - \boldsymbol{P}^2 = m^2c^2, \tag{2.264}$$

and the other is the square of the Pauli-Lubanski four-vector w^μ. The Pauli-Lubanski four-vector is defined as

$$w^\mu = \frac{1}{2}\varepsilon^{\mu\nu\sigma\lambda}p_\nu J_{\sigma\lambda} \equiv (\boldsymbol{P}\cdot\boldsymbol{J}, H\boldsymbol{J}/c - \boldsymbol{K}\times\boldsymbol{P}) \equiv (\boldsymbol{P}\cdot\boldsymbol{S}, H\boldsymbol{S}/c), \tag{2.265}$$

which is by construction orthogonal to p_μ, *i.e.*, $w^\mu p_\mu = 0$.

It is related to the spin with respect to the center of mass $\boldsymbol{S}_{CM}$, defined through the relation

$$\boldsymbol{S}_{CM} = \boldsymbol{J} - \boldsymbol{q}\times\boldsymbol{P}, \quad H\boldsymbol{S}_{CM}/c = H\boldsymbol{J}/c - \boldsymbol{K}\times\boldsymbol{P}, \tag{2.266}$$

after writing $\boldsymbol{K} = H\boldsymbol{q}/c^2 - \boldsymbol{P}t$, so that its time component $w^0 = \boldsymbol{P}\cdot\boldsymbol{S} = \boldsymbol{P}\cdot\boldsymbol{J} = \boldsymbol{P}\cdot\boldsymbol{S}_{CM}$ is the helicity of the particle, and the spatial part is the vector (6.45).

The other Casimir operator is thus

$$w^\mu w_\mu = (\boldsymbol{P}\cdot\boldsymbol{J})^2 - (H\boldsymbol{J}/c - \boldsymbol{K}\times\boldsymbol{P})^2 = -m^2c^2S^2, \tag{2.267}$$

where it depends on S^2, the absolute value squared of the spin with respect to the CM. We see in the relativistic case that the two parameters m and S characterize the two Casimir invariants and therefore they are the intrinsic properties of the elementary particle the formalism provides. In the quantum case, since the representation must be irreducible $S^2 = s(s+1)\hbar^2$, for any $s = 0, 1/2, 1, \ldots$, depending on the value of the quantized spin of the particle, but in the classical case S^2 can take any continuous value.

These w^μ operators satisfy the commutation relations:

$$[w^\mu, w^\nu] = \epsilon^{\mu\nu\sigma\rho} w_\sigma p_\rho, \tag{2.268}$$

where we take $\epsilon^{0123} = +1$, and

$$[p^\mu, w^\nu] = 0, \qquad [J_{\mu\nu}, w_\sigma] = -\eta_{\mu\sigma}w_\nu + \eta_{\nu\sigma}w_\mu. \tag{2.269}$$

The Poincaré group has no non-trivial exponents, so that gauge functions when restricted to homogeneous spaces of $\mathcal{P}$ vanish.

2.10.1 Lorentz group

The Lorentz group $\mathcal{L}$ is the subgroup of the Poincaré group $\mathcal{P}$ of transformations of the form $(0, \boldsymbol{0}, \boldsymbol{v}, \boldsymbol{\mu})$, and every Lorentz transformation $\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$ will be interpreted as $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu})$, as mentioned before where $L(\boldsymbol{v})$ is a boost or pure Lorentz transformation and $R(\boldsymbol{\mu})$ a spatial rotation. Expressions (2.258, 2.259) come from $\Lambda(\boldsymbol{v}'', \boldsymbol{\mu}'') = \Lambda(\boldsymbol{v}', \boldsymbol{\mu}')\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$. Expression (2.258) is the relativistic composition of velocities since

$$\begin{aligned} L(\boldsymbol{v}'')R(\boldsymbol{\mu}'') &= L(\boldsymbol{v}')R(\boldsymbol{\mu}')L(\boldsymbol{v})R(\boldsymbol{\mu}) \\ &= L(\boldsymbol{v}')R(\boldsymbol{\mu}')L(\boldsymbol{v})R(-\boldsymbol{\mu}')R(\boldsymbol{\mu}')R(\boldsymbol{\mu}), \end{aligned}$$

but the conjugate of the boost $R(\boldsymbol{\mu}')L(\boldsymbol{v})R(-\boldsymbol{\mu}') = L(R(\boldsymbol{\mu}')\boldsymbol{v})$ is another boost and thus

$$L(\boldsymbol{v}'')R(\boldsymbol{\mu}'') = L(\boldsymbol{v}')L(R(\boldsymbol{\mu}')\boldsymbol{v})R(\boldsymbol{\mu}')R(\boldsymbol{\mu}).$$

The product $L(\boldsymbol{v}')L(R(\boldsymbol{\mu}')\boldsymbol{v}) = L(\boldsymbol{v}'')R(\boldsymbol{w})$ where $\boldsymbol{v}''$ is the relativistic composition of the velocities $\boldsymbol{v}'$ and $R(\boldsymbol{\mu}')\boldsymbol{v}$, and $R(\boldsymbol{w})$ is the Thomas-Wigner rotation associated to the boosts $L(\boldsymbol{v}')$ and $L(R(\boldsymbol{\mu}')\boldsymbol{v})$.

Therefore, expression (2.258) is equivalent to

$$L(\boldsymbol{v}'') = L(\boldsymbol{v}')L(R(\boldsymbol{\mu}')\boldsymbol{v})R(-\boldsymbol{w}), \tag{2.270}$$

and (2.259) is

$$R(\boldsymbol{\mu}'') = R(\boldsymbol{w})R(\boldsymbol{\mu}')R(\boldsymbol{\mu}) \equiv R(\boldsymbol{\phi})R(\boldsymbol{\mu}). \tag{2.271}$$

The Thomas-Wigner rotation matrix $R(\boldsymbol{w})$ is:

$$R(\boldsymbol{w})_{ij} = \delta_{ij} + \frac{1}{1+\gamma''}\left(\frac{\gamma'^2}{c^2}\left(\frac{1-\gamma}{1+\gamma'}\right)v'_i v'_j + \frac{\gamma^2}{c^2}\left(\frac{1-\gamma'}{1+\gamma}\right)R'_{ik}v_k R'_{jl}v_l\right.$$

$$\left. + \frac{\gamma'\gamma}{c^2}(v'_i R'_{jk}v_k - v'_j R'_{ik}v_k) + \frac{2\gamma'^2\gamma^2(v'_k R'_{kl}v_l)}{(1+\gamma')(1+\gamma)c^2}v'_i R'_{jk}v_k\right),$$

and the factor

$$\gamma'' = \gamma'\gamma\left(1 + \frac{\boldsymbol{v}' \cdot R(\boldsymbol{\mu})\boldsymbol{v}}{c^2}\right).$$

Matrix $R(\boldsymbol{w})$ is written in terms of the vector parameter $\boldsymbol{w}$, which is a function of $\boldsymbol{v}'$, $\boldsymbol{\mu}'$ and $\boldsymbol{v}$, given by

$$\boldsymbol{w} = \frac{\boldsymbol{F}(\boldsymbol{v}', \boldsymbol{0}, R(\boldsymbol{\mu}')\boldsymbol{v}, \boldsymbol{0})}{1 + G(\boldsymbol{v}', \boldsymbol{0}, R(\boldsymbol{\mu}')\boldsymbol{v}, \boldsymbol{0})}, \tag{2.272}$$

and the parameter $\boldsymbol{\phi}$, such that $R(\boldsymbol{\phi}) = R(\boldsymbol{w})R(\boldsymbol{\mu}')$ is

$$\boldsymbol{\phi} = \frac{\boldsymbol{\mu}' + \boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{0})}{1 + G(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{0})}. \tag{2.273}$$

If any one of the two velocities $\boldsymbol{v}$ or $\boldsymbol{v}'$ vanishes, $R(\boldsymbol{w})_{ij} = \delta_{ij}$.

The composition law is obtained by the homomorphism between the Lorentz group $\mathcal{L}$ and the group $SL(2, \mathbb{C})$ of 2×2 complex matrices of determinant $+1$. The Lie algebra of this group has as generators $\boldsymbol{J} = -i\boldsymbol{\sigma}/2$ and $\boldsymbol{K} = \boldsymbol{\sigma}/2$, where σ_i are Pauli spin matrices. A rotation of angle α around a rotation axis given by the unit vector $\boldsymbol{n}$ is given by the 2×2 unitary matrix $\exp(\boldsymbol{\alpha} \cdot \boldsymbol{J})$,

$$R(\boldsymbol{\alpha}) = \cos(\alpha/2)\sigma_0 - i\boldsymbol{n} \cdot \boldsymbol{\sigma} \sin(\alpha/2). \tag{2.274}$$

In terms of the vector $\boldsymbol{\mu} = \tan(\alpha/2)\boldsymbol{n}$,

$$R(\boldsymbol{\mu}) = \frac{1}{\sqrt{1+\mu^2}}(\sigma_0 - i\boldsymbol{\mu} \cdot \boldsymbol{\sigma}), \tag{2.275}$$

where σ_0 is the 2×2 unit matrix. A pure Lorentz transformation of normal parameters β_i is represented by the hermitian matrix $\exp(\boldsymbol{\beta} \cdot \boldsymbol{K})$. This matrix is:

$$L(\boldsymbol{\beta}) = \cosh(\beta/2)\sigma_0 + \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\beta}}{\beta} \sinh(\beta/2). \tag{2.276}$$

In terms of the relative velocity parameters, taking into account the functions $\cosh\beta = \gamma(v)$, $\sinh\beta = \gamma v/c$ and the trigonometric relations $\cosh(\beta/2) = \sqrt{(\cosh\beta + 1)/2}$ and $\tanh(\beta/2) = \sinh\beta/(1 + \cosh\beta)$, the matrix can be written as

$$L(\boldsymbol{v}) = \sqrt{\frac{1+\gamma}{2}}\left(\sigma_0 + \frac{\gamma}{1+\gamma}\frac{\boldsymbol{\sigma} \cdot \boldsymbol{v}}{c}\right). \tag{2.277}$$

Then, every element of $SL(2, \mathbb{C})$ is parametrized by the six real numbers $(\boldsymbol{v}, \boldsymbol{\mu})$, and interpreted as

$$A(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu}). \tag{2.278}$$

We thus see that every 2×2 matrix $A \in SL(2, \mathbb{C})$ can be written in terms of a complex four-vector a^μ and the four Pauli matrices σ_μ. As $A = a^\mu\sigma_\mu$, and $\det A = 1$ leads to $a^\mu a_\mu = 1$ or $(a^0)^2 - \boldsymbol{a}^2 = 1$. The general form of (2.278) is

$$A(\boldsymbol{v}, \boldsymbol{\mu}) = \sqrt{\frac{1+\gamma}{2(1+\mu^2)}}\left[\sigma_0\left(1 - i\frac{\boldsymbol{\mu} \cdot \boldsymbol{u}}{1+\gamma}\right) + \boldsymbol{\sigma} \cdot \left(\frac{\boldsymbol{u} + \boldsymbol{u} \times \boldsymbol{\mu}}{1+\gamma} - i\boldsymbol{\mu}\right)\right], \tag{2.279}$$

here the dimensionless vector $\boldsymbol{u} = \gamma(v)\boldsymbol{v}/c$.

Conversely, since $\mathrm{Tr}\,(\sigma_\mu\sigma_\nu) = 2\delta_{\mu\nu}$, we obtain $a^\mu = (1/2)\mathrm{Tr}\,(A\sigma_\mu)$. If we express (2.279) in the form $A(\boldsymbol{v}, \boldsymbol{\mu}) = a^\mu\sigma_\mu$ we can determine $\boldsymbol{\mu}$ and $\boldsymbol{v}$, and thus $\boldsymbol{u}$, from the components of the complex four-vector a^μ as:

$$\boldsymbol{\mu} = -\frac{\mathrm{Im}\,(\boldsymbol{a})}{\mathrm{Re}\,(a^0)}, \tag{2.280}$$

$$\boldsymbol{u} = 2\left[\mathrm{Re}\,(a^0)\mathrm{Re}\,(\boldsymbol{a}) + \mathrm{Im}\,(a^0)\mathrm{Im}\,(\boldsymbol{a}) + \mathrm{Re}\,(\boldsymbol{a}) \times \mathrm{Im}\,(\boldsymbol{a})\right], \tag{2.281}$$

where $\mathrm{Re}\,(a^\mu)$ and $\mathrm{Im}\,(a^\mu)$ are the real and imaginary parts of the corresponding components of the four-vector a^μ. When $\mathrm{Re}\,(a^0)=0$ expression (2.280) is defined and represents a rotation of value π along the axis in the direction of vector $\mathrm{Im}\,(\boldsymbol{a})$.

If we represent every Lorentz transformation in terms of a rotation and a boost, *i.e.*, in the reverse order, $\Lambda(\boldsymbol{v},\boldsymbol{\mu})=R(\boldsymbol{\mu})L(\boldsymbol{v})$, then the general expression of A is the same as (2.279) with a change of sign in the cross product term $\boldsymbol{u}\times\boldsymbol{\mu}$. Therefore, the decomposition is also unique, the rotation $R(\boldsymbol{\mu})$ is the same as before but the Lorentz boost is given in terms of the variables a^μ by

$$\boldsymbol{u}=2\left[\mathrm{Re}\,(a^0)\mathrm{Re}\,(\boldsymbol{a})+\mathrm{Im}\,(a^0)\mathrm{Im}\,(\boldsymbol{a})+\mathrm{Im}\,(\boldsymbol{a})\times\mathrm{Re}\,(\boldsymbol{a})\right].$$

Note the difference in the third term which is reversed when compared with (2.281).

In the four-dimensional representation of the Lorentz group on Minkowski space-time, a boost is expressed as $L(\boldsymbol{\beta})=\exp(\boldsymbol{\beta}\cdot\boldsymbol{K})$ in terms of the dimensionless normal parameters β_i and the 4×4 boost generators K_i given by

$$K_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ K_2=\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\ K_3=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}.$$

If we call $B=\boldsymbol{\beta}\cdot\boldsymbol{K}\equiv\sum_i\beta_iK_i$, we have

$$B=\begin{pmatrix}0&\beta_1&\beta_2&\beta_3\\\beta_1&0&0&0\\\beta_2&0&0&0\\\beta_3&0&0&0\end{pmatrix},\quad B^2=\begin{pmatrix}\beta^2&0&0&0\\0&\beta_1\beta_1&\beta_1\beta_2&\beta_1\beta_3\\0&\beta_2\beta_1&\beta_2\beta_2&\beta_2\beta_3\\0&\beta_3\beta_1&\beta_3\beta_2&\beta_3\beta_3\end{pmatrix},$$

with $\beta^2=\beta_1^2+\beta_2^2+\beta_3^2$ and $B^3=\beta^2B$, and so on for the remaining powers of B, so that the final expression for $L(\boldsymbol{\beta})=\exp(\boldsymbol{\beta}\cdot\boldsymbol{K})$ is

$$\exp(\boldsymbol{\beta}\cdot\boldsymbol{K})=\exp(B)=\mathbb{I}+\frac{1}{1!}B+\frac{1}{2!}B^2+\frac{1}{3!}B^3+\cdots=\mathbb{I}+\frac{1}{1!}B+\frac{1}{2!}B^2+\frac{1}{3!}\beta^2B+\frac{1}{4!}\beta^2B^2+\cdots$$

and the addition term by term converges to

$$\begin{pmatrix}C&(\beta_1/\beta)S&(\beta_2/\beta)S&(\beta_3/\beta)S\\(\beta_1/\beta)S&1+\dfrac{\beta_1\beta_1}{\beta^2}(C-1)&\dfrac{\beta_1\beta_2}{\beta^2}(C-1)&\dfrac{\beta_1\beta_3}{\beta^2}(C-1)\\(\beta_2/\beta)S&\dfrac{\beta_2\beta_1}{\beta^2}(C-1)&1+\dfrac{\beta_2\beta_2}{\beta^2}(C-1)&\dfrac{\beta_2\beta_3}{\beta^2}(C-1)\\(\beta_3/\beta)S&\dfrac{\beta_3\beta_1}{\beta^2}(C-1)&\dfrac{\beta_3\beta_2}{\beta^2}(C-1)&1+\dfrac{\beta_3\beta_3}{\beta^2}(C-1)\end{pmatrix}$$

where $S=\sinh\beta$ and $C=\cosh\beta$.

What is the physical interpretation of the normal parameters β_i? Let us assume that observers O and O' relate their space-time measurements x and x' by $x'^\mu=L(\boldsymbol{\beta})^\mu{}_\nu x^\nu$. Observer O sends at time t and at a later time $t+dt$ two light signals from a source placed at the origin of its Cartesian frame. These two signals when measured by O' take place at points $\boldsymbol{r}'$ and $\boldsymbol{r}'+d\boldsymbol{r}'$ and at instants t' and $t'+dt'$, respectively. They are related by

$$cdt'=L^0{}_0cdt,\qquad dx'^i=L^i{}_0cdt$$

because $dx^i=0$. The quotient dx'^i/dt' is just the velocity of the light source v^i, *i.e.*, of the origin of the O frame as measured by observer O', and then this velocity $v^i=cL^i{}_0/L^0{}_0=c(\beta_i/\beta)S/C$,

such that the relation between the normal parameters and the relative velocity between observers is

$$\frac{\boldsymbol{v}}{c} = \frac{\boldsymbol{\beta}}{\beta}\tanh\beta$$

and therefore $\tanh\beta = v/c$. Function $\cosh\beta \equiv \gamma(v) = (1 - v^2/c^2)^{-1/2}$ and when the transformation is expressed in terms of the relative velocity it takes the form of the symmetric matrix:

$$L(\boldsymbol{v}) = \begin{pmatrix} \gamma & \gamma v_x/c & \gamma v_y/c & \gamma v_z/c \\ \gamma v_x/c & 1 + \frac{v_x^2}{c^2}\frac{\gamma^2}{\gamma+1} & \frac{v_x v_y}{c^2}\frac{\gamma^2}{\gamma+1} & \frac{v_x v_z}{c^2}\frac{\gamma^2}{\gamma+1} \\ \gamma v_y/c & \frac{v_y v_x}{c^2}\frac{\gamma^2}{\gamma+1} & 1 + \frac{v_y^2}{c^2}\frac{\gamma^2}{\gamma+1} & \frac{v_y v_z}{c^2}\frac{\gamma^2}{\gamma+1} \\ \gamma v_z/c & \frac{v_z v_x}{c^2}\frac{\gamma^2}{\gamma+1} & \frac{v_z v_y}{c^2}\frac{\gamma^2}{\gamma+1} & 1 + \frac{v_z^2}{c^2}\frac{\gamma^2}{\gamma+1} \end{pmatrix}. \tag{2.282}$$

The inverse transformation $L^{-1}(\boldsymbol{v}) = L(-\boldsymbol{v})$. The orthogonal 4×4 rotation matrix takes the block form

$$R(\boldsymbol{\mu}) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{R}(\boldsymbol{\mu}) \end{pmatrix}, \tag{2.283}$$

where $\widetilde{R}(\boldsymbol{\mu})$ is the 3×3 orthogonal matrix (2.235). We can also give a matrix representation to the Lorentz boosts (2.282) in the form

$$L(\boldsymbol{v}) = \begin{pmatrix} \gamma & \gamma \boldsymbol{v}^T/c \\ \gamma \boldsymbol{v}/c & \mathbb{I}_3 + \frac{\gamma^2}{(1+\gamma)c^2}\boldsymbol{v}\boldsymbol{v}^T \end{pmatrix}, \tag{2.284}$$

where $\boldsymbol{v}^T$ is the row vector transposed of the three-dimensional column vector $\boldsymbol{v}$, and $\mathbb{I}_3$ the 3×3 unit matrix. In this way the component of the 3×3 matrix, $(\boldsymbol{v}\boldsymbol{v}^T)_{ij} = v_i v_j$. Is easy to see that in this representation the conjugate transformation

$$R(\boldsymbol{\mu})L(\boldsymbol{v})R(\boldsymbol{\mu})^{-1} = L(R(\boldsymbol{\mu})\boldsymbol{v}).$$

In fact

$$R(\boldsymbol{\mu})L(\boldsymbol{v}) = \begin{pmatrix} \gamma & \gamma \boldsymbol{v}^T/c \\ \gamma R(\boldsymbol{\mu})\boldsymbol{v}/c & R(\boldsymbol{\mu}) + \frac{\gamma^2}{(1+\gamma)c^2}(R(\boldsymbol{\mu})\boldsymbol{v})\boldsymbol{v}^T \end{pmatrix},$$

and when acting on the right with $R(\boldsymbol{\mu})^{-1} = R(\boldsymbol{\mu})^T$, we arrive to

$$\begin{pmatrix} \gamma & (R(\boldsymbol{\mu})\boldsymbol{v})^T/c \\ \gamma R(\boldsymbol{\mu})\boldsymbol{v}/c & \mathbb{I}_3 + \frac{\gamma^2}{(1+\gamma)c^2}(R(\boldsymbol{\mu})\boldsymbol{v})(R(\boldsymbol{\mu})\boldsymbol{v})^T \end{pmatrix} = L(R(\boldsymbol{\mu})\boldsymbol{v}).$$

When a Lorentz transformation is expressed in the form $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu})$, then by construction the first column of $\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$ is just the first column of (2.282) where the velocity parameters $\boldsymbol{v}$ are defined. Therefore, given the general Lorentz transformation $\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$, from its first column we determine the parameters $\boldsymbol{v}$ and thus the complete $L(\boldsymbol{v})$ can be worked out. The rotation involved can be easily calculated as $L(-\boldsymbol{v})\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = R(\boldsymbol{\mu})$. If expressed in the reverse order $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = R(\boldsymbol{\mu})L(\boldsymbol{v})$, then it is the first row of Λ that coincides with the first row of (2.282). It turns out that, given any general Lorentz transformation $\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$, then $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu}) = R(\boldsymbol{\mu})L(\boldsymbol{v}')$ with the same rotation in both sides as derived in (2.280) and $L(\boldsymbol{v}') = R(-\boldsymbol{\mu})L(\boldsymbol{v})R(\boldsymbol{\mu}) = L(R(-\boldsymbol{\mu})\boldsymbol{v})$, *i.e*, the velocity $\boldsymbol{v}' = R(-\boldsymbol{\mu})\boldsymbol{v}$. In any case, the decomposition of a general Lorentz transformation as a product of a rotation and a boost is a unique one, in terms of the same rotation $R(\boldsymbol{\mu})$ and a boost to be determined, depending on the order in which we take these two operations.

Matrix Λ can be considered as a tetrad (*i.e.*, a set of four orthonormal four-vectors, one time-like and the other three space-like) attached by observer O' to the origin of observer O.

In fact, if the matrix is considered in the form $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu})$, then the first column of Λ is the four-velocity of the origin of the O Cartesian frame and the other three columns are just the three unit vectors of the O reference frame, rotated with rotation $R(\boldsymbol{\mu})$ and afterwards boosted with $L(\boldsymbol{v})$.

Chapter 3

Quantization of the models

Quantization of generalized Lagrangian systems will suggest that wave functions for elementary particles must be squared integrable functions defined on the kinematical space.

We shall use Feynman's quantization method to show the structure of the wave function and the way it transforms under the kinematical or symmetry group of the theory. Once the Hilbert space structure of the state space is determined, this leads to a specific representation of the generators of the group as self-adjoint operators and the remaining analysis is done within the usual quantum mechanical context, *i.e.*, by choosing the complete commuting set of operators to properly determine a set of orthogonal basis vectors of the Hilbert space. Special emphasis is devoted to the analysis of the different angular momentum operators the formalism supplies. They have a similar structure to the classical ones, and this will help us to properly obtain the identification of the spin observables.

The structure of the spin operator depends on the kind of translation invariant kinematical variables we use to describe the particle, and the way these variables transform under the rotation group. Since in the Galilei and Poincaré case, as we have seen previously, these variables are the velocity $\boldsymbol{u}$ and orientation $\boldsymbol{\alpha}$ and they transform in the same way under rotations in both approaches, then the mathematical structure of the spin as a differential operator is exactly the same in both relativistic and nonrelativistic formalisms.

In fact the spin operators are related to the compact part of the velocity variable $\boldsymbol{u}$, i.e., its direction given by the two angles, the polar angle θ and the azimuthal angle ϕ, and to the three variables which characterize the orientation of the cartesian frame linked to the particle, and therefore they will be differential operators with respect to these five compact, angular variables.

Half integer spins depend on the kind of the differential operators and on the manifold they act. If the angular momentum operators act on a two-dimensional manifold, like the surface of the unit sphere, we do not obtain all representations of the rotation group but only those related to integer spin. It is necessary that the operators act on the three dimensional manifold of the whole rotation group, to obtain both integer and half integer representations. This implies that the classical spin has to depend on the angular variables which describe the classical orientation of the particle.

As we have seen in the classical description the position of the charge of the particle and its center of mass are different points, and the spin is related to the rotation and internal motion (zitterbewegung) of the charge around the center of mass of the particle. The magnetic properties of the particle are connected only with the motion of the charge and therefore to the zitterbewegung part of spin. It is this double spin structure that gives rise to the concept of gyromagnetic ratio when expressing the magnetic moment in terms of the total spin. If the Lagrangian shows no dependence on the acceleration, the spin is only of rotational nature, and the position and center of mass position define the same point. Spin 1/2 particles arise if the

corresponding classical model rotates but no half integer spins are obtained for systems with spin of orbital nature related only to the zitterbewegung. On the manifold spanned by non-compact variables $\boldsymbol{u}$ no half-integer spins can be found, because the spin operator has the form of an orbital angular momentum and eigenvectors are but spherical harmonics.

Dirac's equation will be obtained when quantizing the classical relativistic spinning particles whose center of charge is circling around its center of mass at the speed c. In that case, the internal orientation of the electron completely characterizes its Dirac algebra.

3.1 Feynman's quantization of Lagrangian systems

Let us consider a generalized Lagrangian system as described in previous chapters and whose evolution is considered on the kinematical space between points x_1 and x_2.

The variational formulation requieres to know the boundary states, and the particular solution of the Euler-Lagrange equations passing through them, singles out the evolution of the particle. However, from the experimental point of view it is impossible to get a precise determination of these boundary states, because any measurement means to interact with the particle, and when we measure some property other properties become distorted, and their uncertainty increases. This means that we do not know accurately the values of the point x_1, but some average values around x_1, with a certain probability. The same happens with respect to x_2, so that finding the path described by the particle is equivalent to determine among all paths coming from a region $\mathcal{R}_1$ around x_1 to the region $\mathcal{R}_2$ around x_2. What we have is a kind of thick tube of paths, linking both regions, so that to determine a unique trajectory like in the classical description, is mathematically impossible. We have to replace the variational formulation by a theory which predicts the probability that a mechanical system starting from a region $\mathcal{R}_1$ in kinematical space, reaches the region $\mathcal{R}_2$.

For quantizing these generalized Lagrangian systems we shall follow Feynman's path integral method [1]. The Quantization Principle is introduced in Feynman's approach by the condition that if no measurement is performed to determine the trajectory followed by the system from x_1 to x_2, then all paths $x(\tau)$ are allowed with the same probability. Therefore a probability definition $P[x(\tau)]$, must be given for every path. The variational formalism does not longer works and it is substituted by a quantization principle which considers that all paths have the same probability.

The probability associated to each possible path $P[x(\tau)]$, is calculated in terms of a complex number $\phi[x(\tau)]$, associated to every path, and called the **probability amplitude**, such that

$$P[x(\tau)] = |\phi[x(\tau)]|^2, \quad \forall x(\tau), \quad 0 \leq P[x(\tau)] \leq 1.$$

Since all paths have the same probability all probability amplitudes are complex numbers of the same absolute value and they only have a different phase. Thus, to every possible trajectory followed by the system, $x(\tau)$ in X space, Feynman associates a complex number $\phi[x(\tau)]$ called the probability amplitude of this alternative, given by

$$\phi[x(\tau)] = N \exp\left\{\frac{i}{\hbar}\int_{\tau_1}^{\tau_2} L(x(\tau), \dot{x}(\tau))d\tau\right\} = N \exp\left\{\frac{i}{\hbar} A_{[x]}(x_1, x_2)\right\}, \tag{3.1}$$

where N is a normalization factor, the same for all paths, and where the phase of this complex number in units of $\hbar$ is the classical action of the system $A_{[x]}(x_1, x_2)$ along the corresponding path $x(\tau)$. Once we perform the integration along the path, this probability amplitude becomes clearly a complex function of the initial and final points in X space, x_1 and x_2, respectively.

[1] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, MacGraw Hill, NY (1965), p. 36.

In this Feynman statistical procedure, the probability amplitude of the occurrence of any alternative of a set of independent alternatives is the sum of the corresponding probability amplitudes of the different independent events. The probability of the whole process is the square of the absolute value of the total probability amplitude. This produces the effect that the probability of the whole process can be less than the probability of any single alternative of the set. This is what Feynman calls **interfering statistics**.

> The idea that underlies in Feynman's quantization method is the two slit interfering process. We have a monochromatic light beam impinging on a two slit pannel. It difracts and arrives to a screen. At every point of the screen arrive photons coming from both slits. However there are points on the screen where no photons arrive. In those points some interference has been produced.
> As an electromagnetic wave we can think that along the light ray there is a transversal orthogonal plane which contains the electric and magnetic field of the wave. Let us assume that the light corresponds to circularly polarized light. The electric field is rotating with constant angular velocity ω. When the wave travels the phase of this field is changing, but not its intensity. We can also imagine that this transversal plane represents a complex plane and that the electric field corresponds to the complex number $\phi[x(\tau)]$, of constant absolute value and whose phase is changing along the trajectory. If at a point on the screen two photons arrive, each one coming from a different slit, the electric field at that point will be the sum of the corresponing electric fields, but the energy willl be the squared of the intensity of this field. The probability of arrival of the light at that point is the squared of the sum of the amplitudes $\boldsymbol{E}_1 + \boldsymbol{E}_2$, i.e., $|\phi[x_1(\tau)] + \phi[x_2(\tau)]|^2$. There are points on the screen where photons arrive coming from one of the slits, but when consider the effect of both slits the resultant electric field is zero. We do not add the probablities of arrival but rather the probability amplitudes.

If we extend this idea to an arbitrary system we can imagine that at every point of the evolution of the mechanical system on the kinematical space we associate a complex plane where we depict the amplitude $\phi[x(\tau)]$, like in the figure **3.1**,

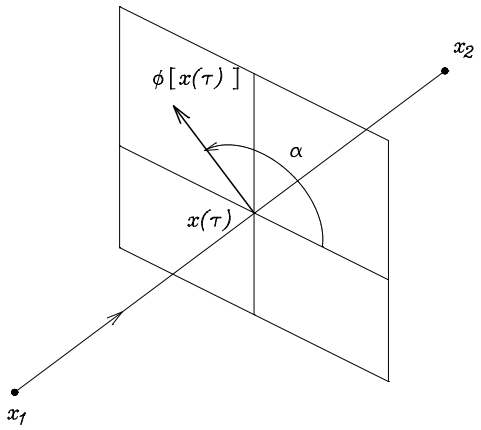


Figure 3.1: **Trajectory of the mechanical system on its kinematical space where we have depicted a transversal complex plane at the point $x(\tau)$, where we represent the probability amplitude $\phi[x(\tau)]$. The phase α of this complex number is the action of the mechanical system, in units of $\hbar$, along that trajectory up to this point.**

Then, the total probability amplitude that the system arrives at point x_2 coming from x_1, *i.e.*, Feynman's kernel $K(x_1, x_2)$, is obtained as the sum or integration over all paths, of terms of the form of Eq. (3.1). Feynman writes this probability amplitude as

$$K(x_1, x_2) = \int_{x_1}^{x_2} \phi[x(\tau)]\mathcal{D}(x(\tau)),$$

where $\mathcal{D}(x(\tau))$ represents a measure over the kind of paths $x(\tau)$ going from x_1 to x_2.

Feynman's kernel $K(x_1, x_2)$, will be in general a function, or more precisely a distribution, on the $X \times X$ manifold. If information concerning the initial point is lost, and the final point is left arbitrary, say x, the kernel reduces to the probability amplitude for finding the system at point x, *i.e.*, the usual interpretation of the quantum mechanical wave function $\Phi(x)$. By the above discussion we see that the wave functions must be complex functions of the kinematical variables but not of other kind of variables. The Hilbert space of pure states is the vector space $\mathbb{L}^2(X)$ of squared integrable complex functions on the kinematical space.

We thus see that Feynman's quantization method enhances the role of the kinematical variables to describe the quantum state of an arbitrary system, in spite of the independent degrees of freedom. We consider that this is one of the reasons why the kinematical variables have to play a leading role also in the classical approach.

We are used to consider in quantum mechanics, instead of a single function $\Phi(x)$, multicomponent wave functions, *i.e*, a set of linearly independent functions $\psi_i(t, \boldsymbol{r})$ defined on space-time and labeled with a discrete subindex that runs over a finite range, such that it can be considered as a vector valued function in a finite dimensional complex vector space. In general this finite space carries some irreducible representation of the rotation group and each component ψ_i represents a definite spin state of the system. Nevertheless, our wave function $\Phi(x)$ depends on more variables than space-time variables. Once we define later the complete commuting set of observables to obtain, in terms of their simultaneous eigenvectors, an orthonormal basis for the Hilbert space of states, we shall find that $\Phi(x)$ can be separated in two parts. One part $\phi(t, \boldsymbol{r})$ depending on space-time variables and another part χ that depends on the remaining compact translation invariant kinematical variables, that in our case will reduce to the velocity $\boldsymbol{u}$ and orientation $\boldsymbol{\alpha}$. It is this possible separation of our wave function that will produce the emergence of the different components of the usual formalism.

3.1.1 Transformation of the wave function

To see how the wave function transforms between inertial observers, and therefore to obtain its transformation under the kinematical groups, let us consider that O and O' are two inertial observers related by means of a transformation $g \in G$, such that the kinematical variables transform as:

$$x'^i = f^i(x, g). \tag{3.2}$$

If observer O considers that the system follows the path $\bar{x}(\tau)$, then it follows for O' the path $\bar{x}'(\tau) = f(\bar{x}(\tau), g)$ and because the action along classical paths transforms according to Eq. (1.13), the probability amplitude for observer O' is just

$$\phi'[\bar{x}'(\tau)] = N \exp\left\{\frac{i}{\hbar}\int_{\tau_1}^{\tau_2} L(\bar{x}'(\tau), \dot{\bar{x}}'(\tau))d\tau\right\}$$

$$= N \exp\left\{\frac{i}{\hbar}\int_{\tau_1}^{\tau_2} L(\bar{x}(\tau), \dot{\bar{x}}(\tau))d\tau\right\} \exp\left\{\frac{i}{\hbar}\int_{\tau_1}^{\tau_2} \frac{d\alpha(g; \bar{x}(\tau))}{d\tau} d\tau\right\},$$

i.e.,

$$\phi'[\bar{x}'(\tau)] = \phi[\bar{x}(\tau)] \exp\left\{\frac{i}{\hbar}\left(\alpha(g; x_2) - \alpha(g; x_1)\right)\right\},$$

where the last phase factor is independent of the integration path. If we add all probability amplitudes of this form, it turns out that Feynman's kernel transforms as:

$$K'(x'_1, x'_2) = K(x_1, x_2) \exp\left\{\frac{i}{\hbar}\left(\alpha(g; x_2) - \alpha(g; x_1)\right)\right\}. \tag{3.3}$$

If information concerning the initial point x_1 is lost, the wave function transforms as the part related to the variables x_2, up to an arbitrary function on G,

$$\Phi'(x'(x)) = \Phi'(gx) = \Phi(x) \exp\left\{\frac{i}{\hbar}\left(\alpha(g;x) + \theta(g)\right)\right\}, \tag{3.4}$$

or in terms of unprimed x variables

$$\Phi'(x) = \Phi(g^{-1}x) \exp\left\{\frac{i}{\hbar}\left(\alpha(g;g^{-1}x) + \theta(g)\right)\right\}, \tag{3.5}$$

where $\theta(g)$ is some function defined on G but independent of x.

Since our system is somewhere in X space, the probability of finding the system anywhere is 1. Then we have to define the way of adding probabilities at different points $x \in X$. If we define a measure on X, $\mu(x)$, such that $d\mu(x)$ is the volume element in X space and $|\Phi(x)|^2 d\mu(x)$ is interpreted as the probability of finding the system inside the volume element $d\mu(x)$ around point x, the probability of finding it anywhere in X must be unity, so that

$$\int_X |\Phi(x)|^2 d\mu(x) = 1. \tag{3.6}$$

Since from (3.5)

$$|\Phi'(x')|^2 = |\Phi(x)|^2, \tag{3.7}$$

it is sufficient for the conservation of probability to assume that the measure to be defined $\mu(x)$ is group invariant. In that case, equation (3.7) implies also that inertial observers measure locally the same probability. This will have strong consequences about the possibility of invariance of the formalism under arbitrary changes of phase of the wave function. But the phase can be changed in a different manner at different points x. We can use this fact to further impose the local gauge invariance of the theory. It must be remarked that this arbitrary change of phase $\beta(x)$ is not only a phase on space-time, but rather on the whole kinematical space of the system and this enlarges the possibilities of analyzing different transformation groups that can be more general than the original kinematical groups, because they act on a larger manifold.

3.1.2 Hilbert space structure of the probability amplitudes

The complex function $\Phi(x)$, if interpreted as the probability amplitude for finding the system around the point $x \in X$, coming from anywhere, satisfies (3.6). It means that $\Phi(x)$ is a complex, squared integrable function defined on the kinematical space. Because probability amplitudes add to form new probability amplitudes when properly normalized, the set of possible functions $\Phi(x)$ forms a complex vector space, because we can add and multiply them by arbitray complex numbers to produce new complex functions which will describe new probability amplitudes.

Consequently, the Hilbert space $\mathcal{H}$ whose unit rays represent the pure states of the system is the space of squared-integrable functions $\mathbb{L}^2(X,\mu)$ defined on the kinematical space X, $\mu(x)$ being an invariant measure such that the scalar product on $\mathcal{H}$ is defined as

$$< \Phi|\Psi >= \int_X \Phi^*(x)\Psi(x)d\mu(x), \tag{3.8}$$

$\Phi^*(x)$ being the complex conjugate function of $\Phi(x)$. There is an arbitrariness in the election of the invariant measure $\mu(x)$ but this will be guided by physical arguments. Nevertheless, the invariance condition will restrict the possible measures to be used.

The absolute value of the above (3.8) $| < \Phi|\Psi > |$ represents the probability that preparing the system in the state given by $\Psi(x)$ we find the system in the state $\Phi(x)$, and conversely, because $| < \Phi|\Psi > | = | < \Psi|\Phi > |$.

3.1.3 Representation of Observables

Wigner's theorem [2,3], implies that to every symmetry $g \in G$ of a continuous group, there exists a one to one mapping of unit rays into unit rays that is induced on $\mathcal{H}$ by a unitary operator $U(g)$ defined up to a phase that maps a wave function defined on x into an arbitrary wave function of the image unit ray in x'. The Relativity Principle is a strong symmetry of physical systems that defines the equivalence between the set of inertial observers whose space-time measurements are related by means of a transformation of a kinematical group G. Now, if we interpret $\Phi(x)$ as the wave function that describes the state of the system for the observer O and $\Phi'(x)$ for O', then we have

$$U(g)\Phi(x) = \Phi'(x) = \Phi(g^{-1}x)\exp\left\{\frac{i}{\hbar}\alpha(g;g^{-1}x) + \theta(g)\right\}. \tag{3.9}$$

Since the $\theta(g)$ function gives rise to a constant phase we can neglect it and take as the definition of the unitary representation of the group G on the Hilbert space $\mathcal{H}$, the following

$$\Phi'(x) = U(g)\Phi(x) = \Phi(g^{-1}x)\exp\left\{\frac{i}{\hbar}\alpha(g;g^{-1}x)\right\}. \tag{3.10}$$

Gauge functions satisfy (1.15), and therefore the phase term can be replaced by

$$\alpha(g;g^{-1}x) = -\alpha(g^{-1};x) + \alpha(0;x) + \xi(g,g^{-1}) = -\alpha(g^{-1};x) + \zeta(g), \tag{3.11}$$

because gauge functions can always be chosen such that $\alpha(0;x) = 0$ and the group function $\zeta(g) = \xi(g,g^{-1})$ giving rise also to a constant phase, can be suppressed. We thus define the transformation of the wave function by

$$\Phi'(x) = U(g)\Phi(x) = \Phi(g^{-1}x)\exp\left\{-\frac{i}{\hbar}\alpha(g^{-1};x)\right\}. \tag{3.12}$$

If the unitary operator is represented in terms of the corresponding self-adjoint generators of the Lie algebra in the form

$$U(g) = \exp\left\{-\frac{i}{\hbar}\, g^\sigma X_\sigma\right\}, \tag{3.13}$$

then, for an infinitesimal transformation of parameters δg^σ its inverse transformation has infinitesimal parameters $-\delta g^\sigma$, we obtain at first order in δg^σ

$$U(\delta g)\Phi(x) = \left(\mathbb{I} - \frac{i}{\hbar}\delta g^\sigma X_\sigma\right)\Phi(x) = \Phi(x) - \frac{i}{\hbar}\delta g^\sigma X_\sigma\,\Phi(x),$$

while

$$\Phi(\delta g^{-1}x) \equiv \Phi(f(x,\delta g^{-1})) = \Phi(x) - \delta g^\sigma u^i_\sigma(x)\frac{\partial\Phi(x)}{\partial x^i},$$

and

$$\exp\left\{-\frac{i}{\hbar}\alpha(\delta g^{-1};x)\right\} = 1 - \frac{i}{\hbar}\alpha(\delta g^{-1};x).$$

But because $\alpha(0;x) = 0$,

$$\alpha(\delta g^{-1};x) = \left.\frac{\partial\alpha(g;x)}{\partial g^\sigma}\right|_{g=0}(-\delta g^\sigma),$$

[2] E.P. Wigner, *Group theory and its application to the quantum mechanics of atomic spectra*, Acad. Press, NY (1959).

[3] V. Bargmann, *J. Math. Phys.* **5**, 862 (1964).

and the substitution of the above terms in (3.12) and further identification of the first order terms in δg^σ imply that the self-adjoint operators X_σ when acting on the wave functions have the differential representation

$$X_\sigma = \frac{\hbar}{i}\, u^j_\sigma(x)\, \frac{\partial}{\partial x^j} - \lambda_\sigma(x), \tag{3.14}$$

where

$$u^j_\sigma(x) = \left.\frac{\partial f^j(x,g)}{\partial g^\sigma}\right|_{g=0}, \qquad \lambda_\sigma(x) = \left.\frac{\partial \alpha(g;x)}{\partial g^\sigma}\right|_{g=0}. \tag{3.15}$$

If we restrict ourselves to transformations of the enlarged configuration space (t, q_i) that can be extended to the whole kinematical space $x \equiv (t, q_i, \ldots, q_i^{(k-1)})$, then, using the same notation as in (1.18)-(1.21), if the infinitesimal transformation is of the form

$$t' = t + M_{0\sigma}\delta g^\sigma,\ q'_i = q_i + M_{i\sigma}\delta g^\sigma,\ \ldots, q'^{(k-1)}_i = q_i^{(k-1)} + M^{(k-1)}_{i\sigma}\delta g^\sigma,$$

these generators take the form

$$X_\sigma = \frac{\hbar}{i}\left(M_{0\sigma}\frac{\partial}{\partial t} + M_{i\sigma}\frac{\partial}{\partial q_i} + \ldots + M^{(k-1)}_{i\sigma}\frac{\partial}{\partial q_i^{(k-1)}}\right) - \lambda_\sigma(x). \tag{3.16}$$

When compared with the Noether constants of the motion (1.36) written in the form

$$-N_\sigma = -H\, M_{0\sigma} + p^i_{(s+1)} M^{(s)}_{i\sigma} - \lambda_\sigma(x), \tag{3.17}$$

we see a certain kind of **'correspondence recipe'**. When restricted to kinematical groups, the functions $\lambda_\sigma(x)$ of (1.36), are obtained from the Lagrangian gauge functions $\alpha(g;x)$, by (1.14), which is exactly the same derivation as the functions $\lambda_\sigma(x)$ above in (3.15). Now, by identifying the different classical observables and generalized momenta that appear in (3.17) with the corresponding differential operators of (3.16) that multiply the corresponding $M^{(s)}_{i\sigma}$ function, we get: the generalized Hamiltonian $H = p^i_{(s)} q_i^{(s)} - L$, is multiplied in (3.17) by the function $M_{0\sigma}$, is identified with the operator $i\hbar\partial/\partial t$ which is also in front of the function M_σ in (3.16), and similarly, the generalized momentum $p^i_{(s+1)}$, the factor that multiplies the function $M^{(s)}_{i\sigma}$, with the differential operator $-i\hbar\partial/\partial q_i^{(s)}$, for $s = 0, \ldots, k-1$.

Recipe: Remember that $p^i_{(s+1)}$ and $q_i^{(s)}$ are canonical conjugate variables. Then, each generalized momentum $p^i_{(s+1)}$ is replaced by $(\hbar/i)$ times the differential operator that differentiates with respect to its conjugate generalized coordinate $q_i^{(s)}$ and the generalized Hamiltonian H by the differential operator $i\hbar\partial/\partial t$.

$$p^i_{(s+1)} \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial q_i^{(s)}}, \qquad H \longrightarrow i\hbar\frac{\partial}{\partial t}.$$

In the case of elementary particles, the kinematical variables are $t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho}$, the generalized variables we have $\boldsymbol{r}, \boldsymbol{u}$ and $\boldsymbol{\rho}$ and the corresponding conjugate momenta are $\boldsymbol{p}_r = \boldsymbol{P}$, $\boldsymbol{p}_u = \boldsymbol{U}$ and $\boldsymbol{p}_\rho = \boldsymbol{V}$, and H the Hamiltonian, these will be given by the differential operators

$$\boldsymbol{P} = \frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{r}}, \quad \boldsymbol{U} = \frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{u}}, \quad \boldsymbol{V} = \frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{\rho}}, \quad H = i\hbar\frac{\partial}{\partial t}. \tag{3.18}$$

Instead of the momentum $\boldsymbol{V} = \partial\widetilde{L}/\partial\dot{\boldsymbol{\rho}}$, we have used the function $\boldsymbol{W} = \partial\widetilde{L}/\partial\boldsymbol{\omega}$, which produces the part of the spin related to the rotation of the particle, which will be described as a differential

operator with respect to the orientation variables $\boldsymbol{\rho}$, in the form which is described in the appendix about general spinors 3.5, at the end of this chapter. This takes the form

$$\boldsymbol{W} = \frac{\hbar}{2i}\left\{\nabla_\rho + \boldsymbol{\rho}\times\nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)\right\}, \tag{3.19}$$

where $\nabla_\rho \equiv \partial/\partial\boldsymbol{\rho}$. This representation of the momenta (3.18) is valid even when the particle is under any interaction, because its mathematical structure depends only on the kinematical variables.

We know that $V_i = \partial\widetilde{L}/\partial\dot{\rho}_i$ and $W_j = \partial\widetilde{L}/\partial\omega_j = \partial\widetilde{L}/\partial\dot{\rho}_i\,\partial\dot{\rho}_i/\partial\omega_j = V_i\partial\dot{\rho}_i/\partial\omega_j$. Since in the passive representation of rotations

$$\omega_i = \frac{2}{1+\rho^2}(\dot{\rho}_i + \epsilon_{ijk}\rho_j\dot{\rho}_k), \quad \dot{\rho}_i = \frac{1}{2}\left(\omega_i - \epsilon_{ikj}\rho_k\omega_j + \rho_i(\boldsymbol{\rho}\cdot\boldsymbol{\omega})\right),$$

$$\frac{\partial\dot{\rho}_i}{\partial\omega_j} = \frac{1}{2}\left(\delta_{ij} - \epsilon_{ikj}\rho_k + \rho_i\rho_j\right), \quad W_j = V_i\frac{\partial\dot{\rho}_i}{\partial\omega_j} = \frac{\hbar}{2i}\left(\frac{\partial}{\partial\rho_j} + \epsilon_{jik}\rho_i\frac{\partial}{\partial\rho_k} + \rho_j\rho_i\frac{\partial}{\partial\rho_i}\right),$$

i.e.,(3.19).

The Heisenberg representation is that representation in which the time dependence has been withdrawn from the wave function by means of a time dependent unitary transformation. Then the wave function in this representation depends on the kinematical variables with the time excluded, *i.e.*, it depends only on the generalized coordinates $q_i^{(r)}$. Therefore, when acting on the wave function in the Heisenberg representation $\psi(q_i, q_i^{(1)}, \ldots, q_i^{(k-1)})$, the observables $q_i^{(r)}$ and $p^j_{(s)}$ satisfy the canonical commutation relations

$$[q_i^{(r)}, p^j_{(s+1)}] = i\hbar\delta_i^j\delta_s^r.$$

If the functions $\lambda_\sigma(x)$ in (3.14) vanish, the X_σ generators satisfy the commutation relations of the group G. But if some $\lambda_\sigma(x) \neq 0$ the X_σ generators do not satisfy in general the commutation relations of the initial group G, but rather the commutation relations of a central extension of G. The group representation on the Hilbert space is not a true representation but a projective representation of G as shown by Bargmann [4].

In fact, from (3.10) we get

$$U(g_1)\Phi(x) = \Phi(g_1^{-1}x)\exp\{\frac{i}{\hbar}\alpha(g_1; g_1^{-1}x)\},$$

acting now on the left with $U(g_2)$,

$$U(g_2)U(g_1)\Phi(x) = U(g_2)\Phi(g_1^{-1}x)\exp\{\frac{i}{\hbar}\alpha(g_1; g_1^{-1}x)\}$$

$$= \Phi((g_2g_1)^{-1}x)\exp\{\frac{i}{\hbar}\alpha(g_2; g_2^{-1}x)\}\exp\{\frac{i}{\hbar}\alpha(g_1; (g_2g_1)^{-1}x)\}, \tag{3.20}$$

while acting on $\Phi(x)$ with $U(g_2g_1)$,

$$U(g_2g_1)\Phi(x) = \Phi((g_2g_1)^{-1}x)\exp\{\frac{i}{\hbar}\alpha(g_2g_1; (g_2g_1)^{-1}x)\}. \tag{3.21}$$

If we define $(g_2g_1)^{-1}x = g_1^{-1}g_2^{-1}x = z$, then $g_1z = g_2^{-1}x$ and because gauge functions satisfy (1.15), we write

$$\alpha(g_2; g_1z) + \alpha(g_1; z) = \alpha(g_2g_1; z) + \xi(g_2, g_1), \tag{3.22}$$

[4] V. Bargmann, *Ann. Math.* **59**, 1 (1954).

and by comparing (3.20) with (3.21), taking into account (3.22), we obtain

$$U(g_2)U(g_1)\Phi(x) = U(g_2g_1)\Phi(x)\exp\{\frac{i}{\hbar}\xi(g_2;g_1)\}. \tag{3.23}$$

Since $\Phi(x)$ is arbitrary, we have a projective unitary representation of the group G characterized by the non-trivial exponent $\xi(g,g')$.

For both Galilei and Poincaré particles the kinematical space is the ten-dimensional manifold spanned by the variables $(t,\boldsymbol{r},\boldsymbol{u},\boldsymbol{\alpha})$, t being the time, $\boldsymbol{r}$ the charge position, $\boldsymbol{u}$ the velocity and $\boldsymbol{\alpha}$ the orientation of the particle. Thus in the quantum formalism the wave function of the most general elementary particle is a squared-integrable function $\Phi(t,\boldsymbol{r},\boldsymbol{u},\boldsymbol{\alpha})$ of these kinematical variables. For point particles, the kinematical space is just the four-dimensional space-time, so that wave functions are only functions of time and position, but spinning particles will have to depend on the additional variables like velocity and orientation. The spin structure will thus be related to these additional compact variables.

3.2 Nonrelativistic spinning particles

3.2.1 Nonrelativistic spinning particles. Bosons

Now let us apply the formalism to the most interesting case of spinning particles. Let us consider next Galilei particles with (anti)orbital spin. This corresponds for example to particles whose kinematical variables are time, position and velocity. A particular classical example is given in Chapter 2, Section 2.2 by the free Lagrangian

$$L = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 - \frac{m}{2\omega^2}\left(\frac{d\boldsymbol{u}}{dt}\right)^2, \tag{3.24}$$

with $\boldsymbol{u} = d\boldsymbol{r}/dt$. For the free particle, the center of mass $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k}$ has a straight motion while the relative position vector $\boldsymbol{k}$ follows an elliptic trajectory with frequency ω around its center of mass. The spin with respect to the center of mass is expressed as $\boldsymbol{S}_{CM} = -m\boldsymbol{k}\times d\boldsymbol{k}/dt$.

The kinematical variables transform under $\mathcal{G}$ in the form

$$t'(\tau) = t(\tau) + b, \tag{3.25}$$
$$\boldsymbol{r}'(\tau) = R(\boldsymbol{\alpha})\boldsymbol{r}(\tau) + \boldsymbol{v}t(\tau) + \boldsymbol{a}, \tag{3.26}$$
$$\boldsymbol{u}'(\tau) = R(\boldsymbol{\alpha})\boldsymbol{u}(\tau) + \boldsymbol{v}. \tag{3.27}$$

The wave functions are complex functions on X and thus functions of the variables $(t,\boldsymbol{r},\boldsymbol{u})$. On this kinematical space the gauge function is the same as in (2.41), where m defines the mass of the particle. Taking into account the correspondence recipe for the Hamiltonian $H \to i\hbar\partial/\partial t$, the first generalized momentum $\boldsymbol{p}_r \equiv \boldsymbol{P} \to -i\hbar\nabla$ and the other generalized momentum $\boldsymbol{p}_u \equiv \boldsymbol{U} \to -i\hbar\nabla_u$, the generators of the projective representation are given by

$$H = i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{P} = \frac{\hbar}{i}\nabla, \quad \boldsymbol{K} = m\boldsymbol{r} - t\frac{\hbar}{i}\nabla - \frac{\hbar}{i}\nabla_u, \tag{3.28}$$

$$\boldsymbol{J} = \boldsymbol{r}\times\frac{\hbar}{i}\nabla + \boldsymbol{u}\times\frac{\hbar}{i}\nabla_u = \boldsymbol{L} + \boldsymbol{Z}, \tag{3.29}$$

where ∇ is the gradient operator with respect to the $\boldsymbol{r}$ variables and ∇_u the gradient operator with respect to the $\boldsymbol{u}$ variables. It is important to stress that this representation of the generators is independent of the particular Lagrangian that describes the particle. It depends only on the kinematical variables $(t,\boldsymbol{r},\boldsymbol{u})$ and the usual Galilei gauge function. We write the symbol

$\boldsymbol{Z} = \boldsymbol{u} \times \boldsymbol{U}$ for the angular momentum with respect to the center of charge, related to the **Zitterbewegung** motion of the particle and we shall see that quantizes with integer values.

If we define $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k} = (\boldsymbol{K} + \boldsymbol{P}t)/m$, it satisfies the commutation relations with $\boldsymbol{P}$,

$$[q_i, P_j] = i\hbar\,\delta_{ij},$$

which are the canonical commutation relations between the linear momentum and position for a point particle and therefore these canonical commutation relations between the total linear momentum and the center of mass position for a spinning particle are already contained in the commutation relations of the extended Lie algebra of the kinematical group. Therefore the quantum mechanical operator

$$\boldsymbol{q} = \boldsymbol{r} - \frac{\hbar}{im}\nabla_u, \tag{3.30}$$

can be interpreted as the center of mass position operator. Discussion of other possibilities for the center of mass position operator can be found in the book by the author.

In this representation, one Casimir operator is the internal energy $H - \boldsymbol{P}^2/2m$. We see that the spin operator with respect to the center of mass is defined as usual

$$\boldsymbol{S}_{CM} = \boldsymbol{J} - \frac{1}{m}\boldsymbol{K} \times \boldsymbol{P} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{k} \times \boldsymbol{P} = \boldsymbol{u} \times \frac{\hbar}{i}\nabla_u + \frac{\hbar}{im}\nabla_u \times \frac{\hbar}{i}\nabla,$$

which is written in terms of two non-commuting terms. It satisfies

$$[\boldsymbol{S}_{CM}, \boldsymbol{S}_{CM}] = i\hbar\boldsymbol{S}_{CM}, \quad [\boldsymbol{J}, \boldsymbol{S}_{CM}] = i\hbar\boldsymbol{S}_{CM}, \quad [\boldsymbol{S}_{CM}, \boldsymbol{P}] = [\boldsymbol{S}_{CM}, H] = [\boldsymbol{S}_{CM}, \boldsymbol{K}] = 0,$$

i.e., it is an angular momentum operator, transforms like a vector under rotations and is invariant under space and time translations and under Galilei boosts, respectively. The second part of the spin operator is of order $\hbar^2$ so that it produces a very small correction to the first $\boldsymbol{Z}$ part.

The angular momentum operators $\boldsymbol{Z}$, or spin with respect to the center of charge, satisfy the commutation relations

$$[\boldsymbol{Z}, \boldsymbol{Z}] = i\hbar\boldsymbol{Z}, \quad [\boldsymbol{J}, \boldsymbol{Z}] = i\hbar\boldsymbol{Z}, \quad [\boldsymbol{Z}, \boldsymbol{P}] = [\boldsymbol{Z}, H] = 0,$$

$$[\boldsymbol{Z}, \boldsymbol{K}] = -i\hbar\boldsymbol{U} = -\hbar^2\nabla_u,$$

i.e., $\boldsymbol{Z}$ is an angular momentum operator, transforms like a vector under rotations and is invariant under space and time translations but not under Galilei boosts. It is usually considered as the quantum mechanical spin operator, because commutes with H and $\boldsymbol{P}$.

We see however, that the angular momentum operator $\boldsymbol{J}$ is split into two commuting terms $\boldsymbol{r} \times \boldsymbol{P}$ and $\boldsymbol{Z}$. They both commute with H, but the first one is not invariant under space translations. The $\boldsymbol{Z}$ operators are angular momentum operators that only differentiate the wave function with respect to the velocity variables, and consequently commute with H and $\boldsymbol{P}$, and although it is not the true Galilei invariant spin operator, we can find simultaneous eigenstates of the three commuting operators $H - \boldsymbol{P}^2/2m$, Z^2 and Z_3. Because the $\boldsymbol{Z}$ operators only affect the wave function in its dependence on $\boldsymbol{u}$ variables, we can choose functions with the variables separated in the form $\Phi(t, \boldsymbol{r}, \boldsymbol{u}) = \sum_i \psi_i(t, \boldsymbol{r})\chi_i(\boldsymbol{u})$ so that

$$(H - \boldsymbol{P}^2/2m)\psi_i(t, \boldsymbol{r}) = E\psi_i(t, \boldsymbol{r}), \tag{3.31}$$

$$Z^2\chi_i(\boldsymbol{u}) = z(z+1)\hbar^2\chi_i(\boldsymbol{u}), \tag{3.32}$$

$$Z_3\chi_i(\boldsymbol{u}) = m_z\hbar\chi_i(\boldsymbol{u}). \tag{3.33}$$

The space-time dependent wave function $\psi_i(t, \boldsymbol{r})$, satisfies Schroedinger's equation and is uncoupled with the spin part $\chi(\boldsymbol{u})$.

Due to the structure of Z^2 in terms of the $\boldsymbol{u}$ variables, which is that of an orbital angular momentum, the spin part of the wave function is of the form

$$\chi(\boldsymbol{u}) = f(u)Y_z^{m_z}(\theta,\phi), \tag{3.34}$$

$f(u)$ being an arbitrary function of the modulus of $\boldsymbol{u}$ and $Y_z^{m_z}(\theta,\phi)$ the spherical harmonics on the direction of $\boldsymbol{u}$.

For the center of mass observer, $\boldsymbol{S} = \boldsymbol{Z}$ and both angular momentum operators are the same. But for an arbitrary observer, $\boldsymbol{Z}$ operators do not commute with the boosts generators so that its absolute value is not Galilei invariant, while $\boldsymbol{S}$ is. But the splitting of the wave function into a multiple-component function that reflects its spin structure is an intrinsic property that can be done in any frame.

It turns out that if for a free particle $\boldsymbol{Z}$ is not conserved, $\boldsymbol{r}\times\boldsymbol{P}$ is not the conserved orbital angular momentum, because $\boldsymbol{r}$ does not represent the position of the center of mass of the particle.

When there is an interaction with an external electromagnetic field, equation (3.31) is satisfied for the mechanical parts $H_m = H - e\phi$ and $\boldsymbol{P}_m = \boldsymbol{P} - e\boldsymbol{A}$ and we thus obtain the usual equation

$$\left(H - e\phi - \frac{(\boldsymbol{P}-e\boldsymbol{A})^2}{2m}\right)\psi_i(t,\boldsymbol{r}) = E\psi_i(t,\boldsymbol{r}). \tag{3.35}$$

This formalism, when the classical spin is of orbital nature, does not lead to half integer spin values, and therefore, from the quantum mechanical point of view these particles can be used only as models for representing bosons.

3.2.2 Nonrelativistic spinning particles. Fermions

Other examples of nonrelativistic spinning particles are those which have orientation and thus angular velocity. For instance, if $X = \mathcal{G}/\mathbb{R}^3_v$, $\mathbb{R}^3_v$ being the subgroup $\{\mathbb{R}^3,+\}$ of pure Galilei transformations, then the kinematical space is spanned by the variables $(t,\boldsymbol{r},\boldsymbol{\alpha})$. This corresponds for instance to the Lagrangian system described by

$$L = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 + \frac{I}{2}\boldsymbol{\omega}^2. \tag{3.36}$$

The particle travels freely at constant velocity while it rotates with constant angular velocity $\boldsymbol{\omega}$. The classical spin is just $\boldsymbol{S} = I\boldsymbol{\omega}$, and the center of charge and center of mass represent the same point.

To describe orientation we can think of the three orthogonal unit vectors $\boldsymbol{e}_i$, $i = 1,2,3$ linked to the body, similarly as in a rigid rotator. If initially they are taken parallel to the spatial Cartesian axis of the laboratory inertial frame, then their nine components considered by columns define an orthogonal rotation matrix $R_{ij}(\boldsymbol{\alpha})$ that describes the triad evolution with the initial condition $R_{ij}(t=0) = \delta_{ij}$.

Now, kinematical variables t, $\boldsymbol{r}$ and $\boldsymbol{\rho}$ transform under $\mathcal{G}$ in the form

$$t'(\tau) = t(\tau) + b, \tag{3.37}$$

$$\boldsymbol{r}'(\tau) = R(\boldsymbol{\alpha})\boldsymbol{r}(\tau) + \boldsymbol{v}t(\tau) + \boldsymbol{a}, \tag{3.38}$$

$$\boldsymbol{\rho}'(\tau) = \frac{\boldsymbol{\mu} + \boldsymbol{\rho}(\tau) + \boldsymbol{\mu}\times\boldsymbol{\rho}(\tau)}{1 - \boldsymbol{\mu}\cdot\boldsymbol{\rho}(\tau)}. \tag{3.39}$$

On the corresponding Hilbert space, the Galilei generators are given by:

$$H = i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{P} = \frac{\hbar}{i}\nabla, \quad \boldsymbol{K} = m\boldsymbol{r} - t\,\frac{\hbar}{i}\,\nabla, \tag{3.40}$$

$$\boldsymbol{J} = \frac{\hbar}{i}\,\boldsymbol{r}\times\nabla + \frac{\hbar}{2i}\left\{\nabla_\rho + \boldsymbol{\rho}\times\nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)\right\} = \boldsymbol{L} + \boldsymbol{W}, \tag{3.41}$$

∇_ρ being the gradient operator with respect to the $\boldsymbol{\rho}$ variables and in the $\boldsymbol{\rho}$ parameterization of the rotation group.

The $\boldsymbol{W}$ part comes from the general group analysis. The group generators in this parametrization X_i will be obtained from (3.39) and according to (1.48) and (1.50). They are obtained as

$$X_i = \left(\frac{\partial\rho'^k}{\partial\mu^i}\right)\Bigg|_{\mu=0}\frac{\partial}{\partial\rho^k},$$

that can be written in vector notation as

$$\boldsymbol{X} = \nabla_\rho + \boldsymbol{\rho}\times\nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)$$

They satisfy the commutation relations

$$[X_i, X_k] = -2\epsilon_{ikl}X_l$$

and therefore the operators $W_k = \frac{\hbar}{2i}X_k$, or in vector notation

$$\boldsymbol{W} = \frac{\hbar}{2i}\left\{\nabla_\rho + \boldsymbol{\rho}\times\nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)\right\}, \tag{3.42}$$

will satisfy the angular momentum commutation relations

$$[\boldsymbol{W}, \boldsymbol{W}] = i\hbar\boldsymbol{W}. \tag{3.43}$$

In this way since $\boldsymbol{L}$ and $\boldsymbol{W}$ commute among each other, we also get $[\boldsymbol{J}, \boldsymbol{J}] = i\hbar\boldsymbol{J}$.

In this example the center of mass and center of charge are the same point, $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{P}$ is the orbital angular momentum associated to the center of mass motion and $\boldsymbol{W} \equiv \boldsymbol{S}$ is the spin operator with respect to the CM. The spin operator commutes with H, $\boldsymbol{P}$ and $\boldsymbol{K}$ and the wave function can be separated as $\Phi(t, \boldsymbol{r}, \boldsymbol{\rho}) = \sum_i \psi_i(t, \boldsymbol{r})\chi_i(\boldsymbol{\rho})$ leading to the equations

$$(H - \boldsymbol{P}^2/2m)\psi_i(t, \boldsymbol{r}) = E\psi_i(t, \boldsymbol{r}), \tag{3.44}$$

$$S^2\chi_i(\boldsymbol{\rho}) = s(s+1)\hbar^2\chi_i(\boldsymbol{\rho}), \tag{3.45}$$

$$S_3\chi_i(\boldsymbol{\rho}) = m_s\hbar\chi_i(\boldsymbol{\rho}). \tag{3.46}$$

Bopp and Haag [5] succeeded in finding $s = 1/2$ solutions for the system of equations (3.45) and (3.46). They are called Wigner's functions [6]. Solutions of (3.45) for arbitrary spin s are but a linear combination of the matrix elements of a $(2s+1)\times(2s+1)$ irreducible matrix representation of the rotation group as can be derived from the Peter-Weyl theorem on finite representations of compact groups [7,8,9]. We shall deal with the $s = 1/2$ functions in the Appendix Section 3.5, where explicit expressions and a short introduction to the Peter-Weyl theorem, will be given.

To describe fermions, the classical particles must necessarily have compact orientation variables as kinematical variables, otherwise no spin 1/2 values can be obtained when the classical spin is related only to the zitterbewegung.

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[6] L.C. Biedenharn and J.D. Louck, *Angular Momentum in Quantum Physics. Theory and Application*, Cambridge U. P., Cambridge, England (1989).

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3.3 Photon quantization

If we take axis OZ as the trajectory of the photon, the kinematical variables reduce to (t, z, α), where α is the orientation of a Cartesian frame with one of the axis along OZ and α represents the rotated angle of this local frame while moving along OZ. The Lagrangian which describes a classical photon of spin S and helicity $\epsilon = \pm 1$, is described in section **2.5.1**,

$$\widetilde{L} = \epsilon S \frac{\dot{z}\dot{\alpha}}{c\dot{t}}.$$

The Noether cosntants of the motions are

$$H = -\frac{\partial \widetilde{L}}{\partial \dot{t}} = \epsilon S\omega, \quad P_z = \frac{\partial \widetilde{L}}{\partial \dot{z}} = \epsilon S\omega/c = \epsilon S k, \quad S_z = \frac{\partial \widetilde{L}}{\partial \dot{\alpha}} = \epsilon S,$$

with $\omega = d\alpha/dt$ is the angular velocity of the local frame, and $k = \omega/c$ the wave number. For the photon $\epsilon\omega > 0$ and $\epsilon\omega < 0$ for the antiphoton. In this way $H > 0$ and $P_z > 0$ for the photon and negative for the antiphoton. According to Planck this particle represents a quantum of electromagnetic energy of value $h\nu$, where ν is the frequency of the radiation. If we identify with the angular frequency of the particle, this implies that $H = S2\pi\nu = h\nu$, and therefore the value $S = \hbar$. From the quantum point of view the spin S_z can only take the integer values $\pm\hbar$, and the particle represents a boson. If the spin were different than that value, the energy of the photon will not be $\hbar\omega = h\nu$. To single out a unique solution between the extremal values of the kinematical variables is not sufficient to fix t_1, z_1, α_1 with $0 \leq \alpha_1 \leq 2\pi$ and the same kind of values at 2. We also need to give the complete number of turns n rotated by the particle, such that the final phase will be expressed as $2\pi n + \alpha_2$, con $0 \leq \alpha_2 \leq 2\pi$. In this way

$$\omega = \frac{2\pi n + \alpha_2 - \alpha_1}{t_2 - t_1}, \quad z_2 - z_1 = c(t_2 - t_1).$$

In the quantum case, the selfadjoint generators of translations and rotations are

$$H = i\hbar\frac{\partial}{\partial t}, \quad P_z = -i\hbar\frac{\partial}{\partial z}, \quad S_z = -i\hbar\frac{\partial}{\partial \alpha}.$$

The wave function which describes the states of the photon will be a squared integrable function $\psi(t, z, \alpha)$, which is an eigenvector of these three commuting generators. We can take the solution in separate variables

$$\psi(t, z, \alpha) = e^{-i\epsilon\omega t} e^{i\epsilon k z} e^{i\epsilon\alpha} = \exp(-i\epsilon(\omega t - kz - \alpha)),$$

which is an eigenvector with the corresponding eigenvalues. This corresponds with the description given in (3.48) once the two spin sates have been replaced by the two component column vector. The identification of these two polarization states corresponds to

$$e^{i\alpha} \quad \longrightarrow \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e^{-i\alpha} \quad \longrightarrow \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In this basis the representation of the spin operator is reduced to

$$S_z = \hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These two basic states correspond to photons of circularly polarized light, left and right, respectively. If we include the eigenstates of the antiphoton then the vector state will be a four column vector.

In this way any polarized photon will be a linear combination

$$\psi(t,z) = e^{i(kz-\omega t)} \begin{pmatrix} a \\ b \end{pmatrix}, \quad |a|^2 + |b|^2 = 1$$

and it looks like a plane wave, travelling along the positive direction of the axis OZ. The action funtion between the initial and final states, takes the form

$$A(x_1,x_2) = \epsilon\hbar \frac{(z_2-z_1)(2\pi n + \alpha_2 - \alpha_1)}{c(t_2-t_1)} = \epsilon\hbar(2\pi n + \alpha_2 - \alpha_1) = \epsilon\hbar\omega(t_2-t_1),$$

if we consider that $(z_2-z_1)/(t_2-t_1) = c$. In units of $\hbar$ this function is the phase of the wave function, or state of the photon, according to Feynman.

Since the values of the observables H, P_z and S_z are defined accurately because they are eigenfunctions of all of them, the corresponding conjugate variables have a great uncertainty. In fact, since the commutators are $[t,H] = -i\hbar$, $[z,P_z] = i\hbar$, $[\alpha,S_z] = i\hbar$, this implies that

$$\Delta t \Delta H \geq |[t,H]| = \hbar, \quad \Delta z \Delta P_z \geq |[z,P_z]| = \hbar, \quad \Delta\alpha\Delta S_z \geq |[\alpha,S_z]| = \hbar,$$

which means that we cannot locate exactly the photon and its phase is unobservable.

The measure to define the scalar product at constant t can be

$$< \psi|\phi >= \frac{1}{(z_2-z_1)(2\pi n + \alpha_2 - \alpha_1)} \int_{z_1}^{z_2} \int_{\alpha_1}^{2\pi n+\alpha_2} \psi^*\phi \; dzd\alpha,$$

extended to the whole range of the kinematical variables during this evolution. Therefore, the expected value of the position operator in any one of these states is

$$< z >=< \psi|z|\psi >= \frac{1}{(z_2-z_1)(2\pi n + \alpha_2 - \alpha_1)} \int_{z_1}^{z_2} \int_{\alpha_1}^{2\pi n+\alpha_2} z \; dzd\alpha = \frac{1}{2}(z_1+z_2),$$

i.e., the middle point $\bar{z}$ of the trajectory, and its uncertainty

$$(\Delta z)^2 =< (z- < z >)^2 >= \frac{1}{3}(z_1^2 + z_1z_2 + z_2^2) - \frac{1}{4}(z_1+z_2)^2 = \frac{1}{12}(z_2-z_1)^2.$$

The uncertainty $\Delta z = (z_2-z_1)/3.46$, and therefore the probability of finding the photon in the range $\bar{z} \pm \Delta z$, is of 68%.

3.4 Appendix: Light polarization states

The internal part of the wave function which describes the spin states of the photon is a two-dimensional complex Hilbert space $\mathbb{C}^2$. Therefore any pure state is described by a vector $|\Phi>\in \mathbb{C}^2$ of unit norm, $<\Phi|\Phi>=1$. Non pure states, or statistical mixtures, will be characterized by density operators ρ, i.e., selfadjoint operators of unit trace. They will be represented as 2×2 hermitian matrices of unit trace.

The set of hermitian matrices 2×2 is a real four-dimensional vector space, and a basis of this vector space can be given by the four linearly independent hermitian matrices:

$$\sigma_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{3.47}$$

and every hermitian matrix can be expressed in the form $\rho=a^\mu\sigma_\mu$ with $a^\mu\in\mathbb{R}$, four real numbers. The condition that ρ represents some state is that its trace is 1, which leads to $a^0=1/2$. If ρ is a pure state, also called a vector state, then $\rho=|\Phi><\Phi|$ in terms of the vector $|\Phi>$, and it results a projection operator. Pure states are characterized by the condition $\rho^2=\rho$, and thus $\mathrm{Tr}\rho^2=\mathrm{Tr}\rho=1$, while for any arbitrary state $\mathrm{Tr}\rho^2\leq\mathrm{Tr}\rho=1$.

This condition, taking into account that $\mathrm{Tr}\sigma_\mu\sigma_\nu=2\delta_{\mu\nu}$, leads to $2(a^0)^2+2\boldsymbol{a}^2\leq 1$, and since $a^0=1/2$, $\boldsymbol{a}^2\leq 1/4$, and we can characterize any state by a real three-dimensional vector $\boldsymbol{a}$, of absolute value $|\boldsymbol{a}|\leq 1/2$, being $1/2$ in the case of a pure state. The set of states of light will be given by the points of a sphere in $\mathbb{R}^3$ of radius $1/2$, sometimes called the **Poincaré sphere**, where the points on its surface represent the pure or vector states, while the points of the interior represent the mixture states. The set of all states is a convex set such that any state can be described as a convex linear combination of pure states.

This kind of description is completely general for quantum systems with two basic internal states. In this case it describes polarized photons whose Hilbert space of states can be realized as $\mathbb{L}^2(\mathbb{R}^3)\otimes\mathbb{C}^2$, where the two possible states of polarization are described by the part $\mathbb{C}^2$ of this Hilbert space. Let us assume that at time t we have the vector $|\Psi>$:

$$|\Psi>=\begin{pmatrix}\alpha\\\beta\end{pmatrix}\exp(i(kz-\omega t)) \tag{3.48}$$

where α and β are two complex numbers such that $|\alpha|^2+|\beta|^2=1$. It is an eigenvector of the operator $H=i\hbar\partial/\partial t$ with eigenvalue $\hbar\omega$, and of the operator $\boldsymbol{P}=-i\hbar\nabla$ with eigenvalue $(0,0,\hbar k)$. If we write $\alpha=a\exp(i\delta_a)$ and $\beta=b\exp(i\delta_b)$ and represent the state with the phase of the first component real, then

$$|\Psi>=\begin{pmatrix}\alpha\\\beta\end{pmatrix}\exp(i(kz-\omega t))=\begin{pmatrix}a\\be^{i\delta}\end{pmatrix}\exp(i(kz-\omega t+\delta_a)) \tag{3.49}$$

where $\delta=\delta_b-\delta_a$.

The projection operator on this state is $|\Psi><\Psi|$ given by:

$$|\Psi><\Psi|=\begin{pmatrix}a^2&abe^{-i\delta}\\abe^{i\delta}&b^2\end{pmatrix}=\begin{pmatrix}a^0+a^3&a^1-ia^2\\a^1+ia^2&a^0-a^3\end{pmatrix} \tag{3.50}$$

and the four-vector related to it is:

$$[(a^2+b^2)/2, ab\cos\delta, ab\sin\delta, (a^2-b^2)/2],\quad 2a^0\equiv a^2+b^2=1. \tag{3.51}$$

In terms of the three-dimensional vector $\boldsymbol{a}$, the different pure states are the points of the surface of the sphere of radius $1/2$ of the figure, while mixed states are points of the interior.

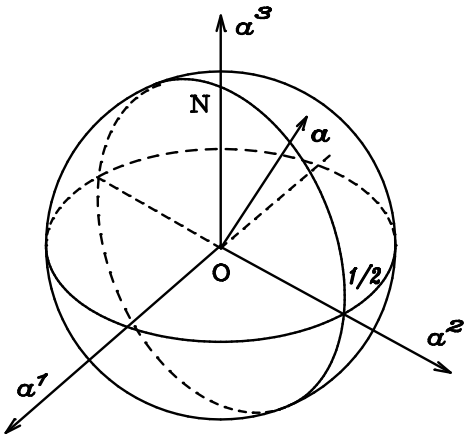


Figure 3.2: **The Poincaré sphere represents the different states of polarization of light. The surface of the sphere represents the pure states or vector states, while the inner points represent mixture states characterized by density operators ρ. The sphere is a convex set.**

In particular the orthogonal states $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ give rise to the projection operators $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $\begin{pmatrix}0&0\\0&1\end{pmatrix}$ described respectively by the vectors $\boldsymbol{a}$ And $-\boldsymbol{a}$. The vector $\boldsymbol{a} \equiv (0,0,1/2)$, is the North pole and the vector $-\boldsymbol{a} \equiv (0,0,-1/2)$ is the South pole of the sphere.

If E_a and E_b are two projection operators characterized by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively and they represent orthogonal states, then $E_a E_b = 0$. In terms of Pauli matrices $E_a E_b = a^\mu b^\nu \sigma_\mu \sigma_\nu = 0$, and taking the trace of this, taking into account that $a^0 = b^0 = 1/2$, we get $(1/4 + \boldsymbol{a}\cdot\boldsymbol{b}) = 0$, and since $\boldsymbol{a}$ and $\boldsymbol{b}$ are of modulus $1/2$, the solution is $\boldsymbol{a} = -\boldsymbol{b}$. Orthogonal pure states are represented by opposite points on the surface of the sphere. Taking by pairs they characterize every possible ortonormal basis of $\mathbb{C}^2$. This means that every pure state is expressed as a coherent superposition of any of these two orthogonal states and any other state as a linear convex combination of pure states.

3.4.1 Stokes' parameters

The above description of the polarization states in terms of the four-vector a^μ is equivalent to the description made by Stokes (1852). If we have an elliptic polarized wave where the electric fields are

$$E_x = a\cos(kz - \omega t), \quad E_y = b\cos(kz - \omega t + \delta),$$

the field describes on the plane XOY an ellipse contained in the rectangle of sides $2a$ and $2b$. [10] The points of contact of the ellipse with the rectangle are $A, A' \equiv (\pm a, \pm b\cos\delta)$ and $B, B' \equiv (\pm a\cos\delta, \pm b)$. The Stokes' parameters are

$$s_0 = a^2 + b^2, \quad s_1 = a^2 - b^2, \quad s_2 = 2ab\cos\delta, \quad s_3 = 2ab\sin\delta,$$

which are not independent since $s_0^2 = s_1^2 + s_2^2 + s_3^2$ and where s_0 represents the intensity of the electromagnetic wave. If we compare them with the previous description $s_0 = 2a^0$, $s_1 = 2a^3$, $s_2 = 2a^1$ and $s_3 = 2a^2$. The radius of the Poincaré sphere in this case is not $1/2$ but it is s_0. The poles N and S represent circularly polarized light right and left, respectively. We can interpret

[10] M. Born and E. Wolf, Principles of Optics, Cambridge Univ. Press, (1993), ch.1

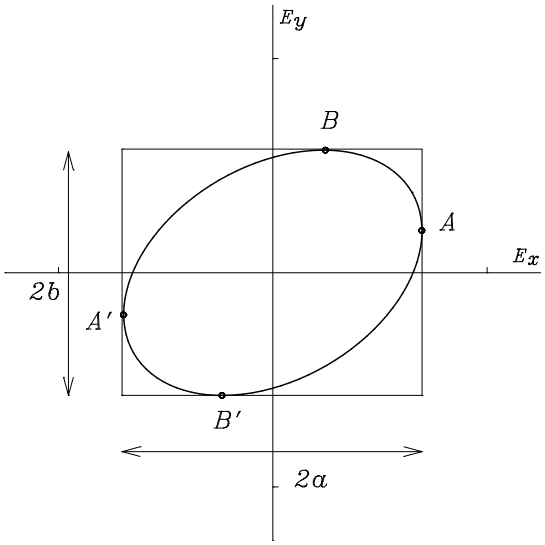


Figure 3.3: **Electric field of an elliptic polarized wave.**

the quantum decription of the photon in such a way that the real part of a non-normalized wave function would represent the transversal part of the electric field associated to the photon. The normalization of the wave function would be interpreted as the density of energy of the photon. The different elliptic polarized states are in correspondence with the pure states of the quantum description.

3.4.2 Coherent Superposition

If we have two pure states $|\Phi>$ and $|\Psi>$, the vector $\alpha|\Phi> +\beta|\Psi>$ once normalized will represent another pure state and we say that it is the coherent superposition of both states. In this case the relative phase of both vectors is important.

Lat us assume that we take the two states $\begin{pmatrix}1\\0\end{pmatrix}\exp(i(kz-\omega t))$ and $\begin{pmatrix}0\\1\end{pmatrix}\exp(i(kz-\omega t))$, and that α ans β are taken real with the condition $\alpha^2+\beta^2=1$. Then the vector state of this superposition is $\begin{pmatrix}\alpha\\\beta\end{pmatrix}\exp(i(kz-\omega t))$ and because the projector is $\begin{pmatrix}\alpha^2 & \alpha\beta\\ \alpha\beta & \beta^2\end{pmatrix}$ it is represented on the surface of the sphere by $[\alpha\beta, 0, (\alpha^2-\beta^2)/2]$ and therefore on the meridian of the plane XOZ at the point P represented in the figure **3.4** where $NP=\beta$ and $PS=\alpha$.

In fact, $a^3=(\alpha^2-\beta^2)/2=1/2-\beta^2=\alpha^2-1/2$, and the distances of the point P to the tangent planes through the poles are β^2 and α^2, respectively. If $\alpha=\beta$, the representative point will be on the Equator. These values α^2 and β^2 are the probabilities that our state will be either N or S, respectively, or the proportion of the states N and S which are used in the construction of the new state.

If α and β are in general complex, once the phase of α has been neglected, the state is reduced to (3.50) and (3.51), and the coherent superposition with the same a And b and different δ, will produce all points of the same parallel. The angle δ is the azimuth in the clockwise sense around an axis from the second state to the first.

The situation is completely symmetric if we make it from any two arbitrary orthogonal states. We have on the sphere two opposite points. We determine first the superposition with $\alpha=\beta=1/\sqrt{2}$, which gives rise the position of the corresponding meridian. From now on we rotate clockwise an angle δ, and the parallel will correspond of the relative values a and b.

The coherent superposition is performed on the surface of the sphere, such that to every

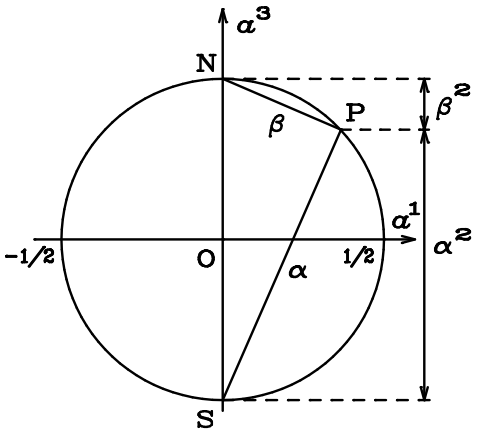


Figure 3.4: **The point P represents the coherent superposition of the two states N and S with real coefficients α and β.**

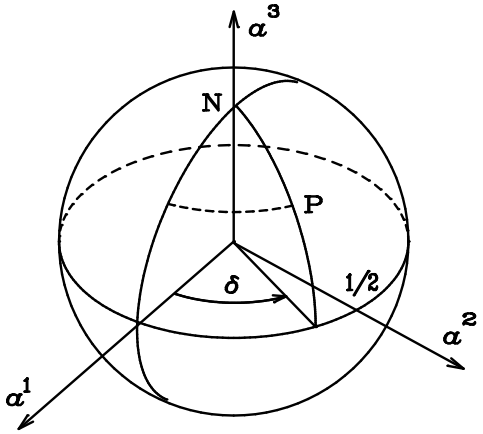


Figure 3.5: **The point P is the coherent superposition of N And S with coefficients a and $be^{i\delta}$.**

pair of opposite points given the numbers a and $be^{i\delta}$, there corresponds another point of the surface obtained by the above method.

3.4.3 Incoherent superposition

Let $|\Phi_1>$ and $|\Phi_2>$ two orthogonal pure states characterized by the projection operators E_1 and E_2, respectively and for the real vectors $\boldsymbol{a}$ and $-\boldsymbol{a}$. If we define the state $\rho = \alpha E_1 + (1-\alpha)E_2$, with $0 < \alpha < 1$, it represents an incoherent superposition of both states proportionally to α and $1-\alpha$. In this representation, the resultant vector is $\alpha\boldsymbol{a} - (1-\alpha)\boldsymbol{a} = (2\alpha - 1)\boldsymbol{a}$. The characteristic point in the figure **3.6**, is in the line joining the two pure states, in such a way that $PA = (1-\alpha)$ and $PB = \alpha$. It is the center of mass of two masses α and $(1-\alpha)$ located at A and B, respectively.

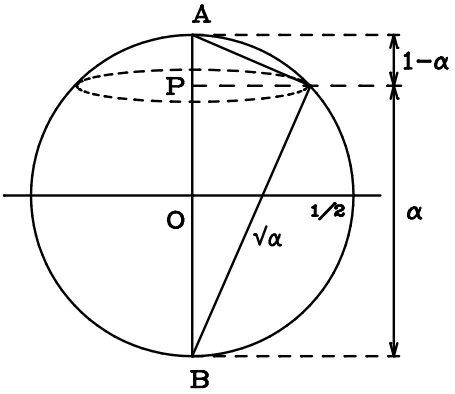


Figure 3.6: **The point P is the incoherent superposition of the pure states A and B proportionally to α and $1-\alpha$, respectively.**

We can interpret the incoherent superposition as the center of mass of all possible coherent superpositions with the same proportions A and B, where the relative phase is left free. In this case we shall obtain all possible states of the parallel separated of the Poles by $(1-\alpha)$ and α, respectively.

If $\rho_1, \rho_2, \ldots, \rho_k$ are k pure states, and therefore represented by points on the surface of the sphere and $\alpha_1, \alpha_2, \ldots, \alpha_k$, $0 \leq \alpha_i \leq 1$ such that $\sum \alpha_i = 1$, then $\rho = \sum \alpha_i \rho_i$ is said the incoherent mixture, with weights α_i of the k pure states ρ_i. The representative point will be given by the center of mass of the corresponding points ρ_i with weights of value α_i. The pure states are the only states which can never be obtained as a convex linear combination of other pure states.

The incoherent superposition becomes the convex linear combination of vectors of $\mathbb{R}^3$, on the surface of the sphere of radius $1/2$.

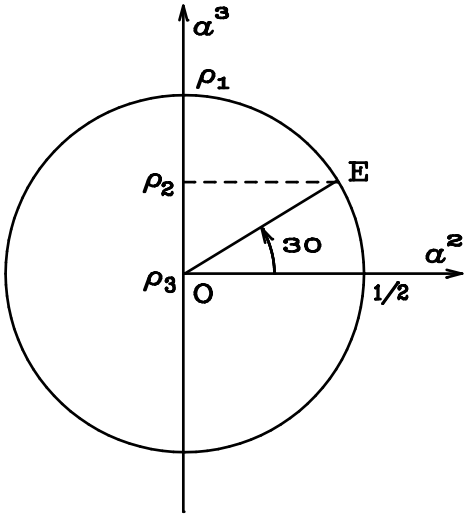
3.4.4 Filters

Filters are observables such that acting on any state of the quantum system it is projected into a possible pure state. They are represented by the projection operator on the state onto they project. If the initial state of the photon is ρ, then the measurement of the filter E in the state ρ is given by $< E >= \mathrm{Tr}\rho E$, while the photon goes after measurement into a

new state $\rho' = E\rho E/\mathrm{Tr}(\rho E)$. For exemple, let us assume that $\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and E is the projection operator onto the state given by the vector $\boldsymbol{a}$ of components $[0, \sqrt{3}/4, 1/4]$, i.e., $E = \begin{pmatrix} 3/4 & -\sqrt{3}i/4 \\ \sqrt{3}i/4 & 1/4 \end{pmatrix}$. The expected value of E in the state ρ_1 is $3/4$, which precisely the proportion in which the state E is produce by the superposition of ρ_1 and its orthogonal state. After the measurement, the state of the photon is:

$$\rho' = E\rho_1 E/Tr(\rho_1 E) = (4/3)E\rho_1 E = \begin{pmatrix} 3/4 & -\sqrt{3}i/4 \\ \sqrt{3}i/4 & 1/4 \end{pmatrix} = E$$

i.e, the state E.



From the vector point of view, because the state is a vector state $|\rho_1 > \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and therefore the expected value $< E; \rho_1 > = < \rho_1 | E\rho_1 > = 3/4$ and the final state will be $|\rho_1' > = E|\rho_1 > /||E|\rho_1 > || = \begin{pmatrix} \sqrt{3}/2 \\ i/2 \end{pmatrix}$ which is precisely the state E, because $|\rho_1' >< \rho_1'| = E$.

If the initial state is not a pure state, for example $\rho_2 = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$, which corresponds to an incoherent superposition of $3/4$ of the previous state ρ_1 and $1/4$ of its orthogonal state, then $< E; \rho_2 > = Tr\rho_2 E = 5/8$, i.e., that part of ρ_2 which is projected into E, and the final state ρ_2' is anew the state E.

If the initial state is the completely incoherent state, like the one given by the density operator $\rho_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, represented by the point O, Then $< E; \rho_3 > = 1/2$, because that incoherent state can also be made from E and the 50% of its orthogonal state.

3.5 Appendix: Spinors

In this section of mathematical content we shall review the main properties of spinors, in particular those connected with the possible representation of the wave function to describe spin 1/2 particles. We shall describe the representations in terms of eigenfunctions of the different commuting spin operators. But it must be remarked that in addition to the spin operators in the laboratory frame we also have spin operators projected on the body frame, because our general spinning particle has orientation, and therefore, a local Cartesian frame linked to its motion. This produces the result that for a spin 1/2 particle the wave function necessarily is a four-component object.

All calculations in this Appendix can be obtained in the Mathematica[11] notebook file by the author `http://tp.lc.ehu.es/documents/SpinorsNotesBilbao.nb`.

The general wave function is a function of the ten kinematical variables, $\Phi(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho})$, and the spin with respect to the center of charge is related to the kinematical variables $\boldsymbol{u}$ and $\boldsymbol{\rho}$, as

$$\boldsymbol{S} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W} = \boldsymbol{Z} + \boldsymbol{W}, \tag{3.52}$$

where $\boldsymbol{Z}$ and $\boldsymbol{W}$ are given by

$$\boldsymbol{Z} = \boldsymbol{u} \times \frac{\hbar}{i} \nabla_u, \quad \boldsymbol{W} = \frac{\hbar}{2i} \left\{ \nabla_\rho + \boldsymbol{\rho} \times \nabla_\rho + \boldsymbol{\rho} (\boldsymbol{\rho} \cdot \nabla_\rho) \right\}, \tag{3.53}$$

in the $\tan(\alpha/2)$ representation of the rotation group, as has been deduced in previous sections. ∇_u and ∇_ρ are respectively the gradient operators with respect to $\boldsymbol{u}$ and $\boldsymbol{\rho}$ variables. These operators always commute with the $H = i\hbar\partial/\partial t$ and $\boldsymbol{P} = -i\hbar\nabla$ operators, and therefore they are translation invariant. This feature allows the separation of the general wave function in terms of space-time variables and velocity-orientation variables to describe the translation invariant properties of the system.

The above spin operators satisfy the commutation relations

$$[\boldsymbol{Z}, \boldsymbol{Z}] = i\hbar \boldsymbol{Z}, \quad [\boldsymbol{W}, \boldsymbol{W}] = i\hbar \boldsymbol{W}, \quad [\boldsymbol{Z}, \boldsymbol{W}] = 0, \tag{3.54}$$

and thus

$$[\boldsymbol{S}, \boldsymbol{S}] = i\hbar \boldsymbol{S}.$$

3.5.1 Unit vectors

Because we are describing the orientation of the particle by attaching to it a system of three unit vectors $\boldsymbol{e}_i$, whose orientation in space is described by variables $\boldsymbol{\rho}$ or $\boldsymbol{\alpha}$, then, if at initial instant $\tau = 0$ we choose the body axes coincident with the laboratory axes, the components of the unit vectors $\boldsymbol{e}_i$ at any time are

$$(\boldsymbol{e}_i)_j = R_{ji}(\boldsymbol{\alpha}) = \delta_{ji} \cos\alpha + n_j n_i (1 - \cos\alpha) - \epsilon_{jik} n_k \sin\alpha, \tag{3.55}$$

in the normal parameterization and also in the $\boldsymbol{\rho}$ parameterization by

$$(\boldsymbol{e}_i)_j = R_{ji}(\boldsymbol{\rho}) = \frac{1}{1+\rho^2} ((1-\rho^2)\delta_{ji} + 2\rho_j \rho_i - 2\epsilon_{jik}\rho_k), \tag{3.56}$$

where the Cartesian components of the rotation axis unit vector $\boldsymbol{n}$ are:

$$n_1 = \sin\theta\cos\phi, \qquad n_2 = \sin\theta\sin\phi, \qquad n_3 = \cos\theta, \tag{3.57}$$

[11] Mathematica is the registered computer program edited by Wolfram

where θ is the polar angle and ϕ the usual azimuth angle. Explicitly:

$$\begin{aligned}
e_{11} &= \cos\alpha + \sin^2\theta\cos^2\phi(1-\cos\alpha),\\
e_{12} &= \cos\theta\sin\alpha + \sin^2\theta\sin\phi\cos\phi(1-\cos\alpha),\\
e_{13} &= -\sin\theta\sin\phi\sin\alpha + \sin\theta\cos\theta\cos\phi(1-\cos\alpha),\\
\\
e_{21} &= -\cos\theta\sin\alpha + \sin^2\theta\sin\phi\cos\phi(1-\cos\alpha),\\
e_{22} &= \cos\alpha + \sin^2\theta\sin^2\phi(1-\cos\alpha),\\
e_{23} &= \sin\theta\cos\phi\sin\alpha + \sin\theta\cos\theta\sin\phi(1-\cos\alpha),\\
\\
e_{31} &= \sin\theta\sin\phi\sin\alpha + \sin\theta\cos\theta\cos\phi(1-\cos\alpha),\\
e_{32} &= -\sin\theta\cos\phi\sin\alpha + \sin\theta\cos\theta\sin\phi(1-\cos\alpha),\\
e_{33} &= \cos\alpha + \cos^2\theta(1-\cos\alpha),
\end{aligned}$$

in the $\boldsymbol{\alpha} = \alpha\boldsymbol{n}$, or normal parametrization of the rotation group. In the $\boldsymbol{\rho} = \tan(\alpha/2)\boldsymbol{n}$ parametrization the body frame is

$$\begin{aligned}
e_{11} &= (1+\rho_1^2-\rho_2^2-\rho_3^2)/(1+\rho^2),\\
e_{12} &= (2\rho_1\rho_2+2\rho_3)/(1+\rho^2),\\
e_{13} &= (2\rho_1\rho_3-2\rho_2)/(1+\rho^2),\\
\\
e_{21} &= (2\rho_2\rho_1-2\rho_3)/(1+\rho^2),\\
e_{22} &= (1-\rho_1^2+\rho_2^2-\rho_3^2)/(1+\rho^2),\\
e_{23} &= (2\rho_2\rho_3+2\rho_1)/(1+\rho^2),\\
\\
e_{31} &= (2\rho_1\rho_3+2\rho_2)/(1+\rho^2),\\
e_{32} &= (2\rho_3\rho_2-2\rho_1)/(1+\rho^2),\\
e_{33} &= (1-\rho_1^2-\rho_2^2+\rho_3^2)/(1+\rho^2),
\end{aligned}$$

where $\rho^2 \equiv \rho_1^2+\rho_2^2+\rho_3^2 = \tan^2(\alpha/2)$.

3.5.2 Spin projection on the unit vectors

In addition to the different components of the spin operators S_i, Z_i and W_i in the laboratory frame, we also have another set of spin operators. They are the spin projections on the body axes $\boldsymbol{e}_i$, *i.e.*, the operators $R_i = \boldsymbol{e}_i\cdot\boldsymbol{S}$, $M_i = \boldsymbol{e}_i\cdot\boldsymbol{Z}$ and $T_i = \boldsymbol{e}_i\cdot\boldsymbol{W}$, respectively. In particular, spin operators T_i, collecting terms from (3.56) and (3.53), take the expression

$$\begin{aligned}
T_i = \sum_{k=1}^{k=3}(e_i)_k W_k &= \frac{\hbar}{2i(1+\rho^2)}\sum_{k=1}^{k=3}\left((1-\rho^2)\delta_{ik}+2\rho_i\rho_k-2\epsilon_{kij}\rho_j\right)\\
&\quad\times\left(\frac{\partial}{\partial\rho_k}+\epsilon_{klr}\rho_l\frac{\partial}{\partial\rho_r}+\rho_k(\boldsymbol{\rho}\cdot\nabla_\rho)\right),
\end{aligned}$$

and after some tedious manipulations we reach the final result, written in vector notation as

$$\boldsymbol{T} = \frac{\hbar}{2i}\left\{\nabla_\rho - \boldsymbol{\rho}\times\nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)\right\}. \tag{3.58}$$

We see, by inspection, that this result can also be obtained from the expression of $\boldsymbol{W}$ in (3.53), just by replacing $\boldsymbol{\rho}$ by $-\boldsymbol{\rho}$, followed by a global change of sign. This is because we describe the orientation of the particle by vector $\boldsymbol{\rho}$ in the laboratory frame from the active viewpoint, *i.e.*, with the laboratory reference frame fixed. However, its orientation with respect to the body frame is described by the motion of the laboratory frame, whose orientation for the body is $-\boldsymbol{\rho}$, and the global change of sign comes from the change from the active point of view to the passive one. This is the difference in the spin description in one frame or another.

It satisfies the following commutation relations

$$[\boldsymbol{T},\boldsymbol{T}] = -i\hbar\boldsymbol{T}, \quad [\boldsymbol{T},\boldsymbol{W}] = 0.$$

and in general all spin projections on the body frame R_i, M_i and T_i, commute with all the spin projections on the laboratory frame S_i, Z_i and W_i. This is in agreement with the quantum mechanical uncertainty principle, because spin components with respect to different frames are compatible observables.

3.5.3 Spinor wave functions

To find eigenstates of the spin operator we have to solve equations of the form:

$$S^2\chi(\boldsymbol{u},\boldsymbol{\rho}) = s(s+1)\hbar^2\chi(\boldsymbol{u},\boldsymbol{\rho}), \quad S_3\chi(\boldsymbol{u},\boldsymbol{\rho}) = m\hbar\chi(\boldsymbol{u},\boldsymbol{\rho}).$$

But we also have the orientation of the particle, and therefore the spin projections on the body axes. These projections commute with S^2 and S_3, and it is possible to choose another commuting spin operator, like the T_3 operator, and therefore our wave function can be taken also as an eigenvector of T_3,

$$T_3\chi(\boldsymbol{u},\boldsymbol{\rho}) = n\hbar\chi(\boldsymbol{u},\boldsymbol{\rho}),$$

so that the complete commuting set of operators that describe the spin structure must also include spin projections on the body axes.

The spin squared operator is

$$S^2 = \boldsymbol{Z}^2 + \boldsymbol{W}^2 + 2\boldsymbol{Z}\cdot\boldsymbol{W}, \tag{3.59}$$

and we see from (3.54) is expressed as the sum of three commuting terms and its eigenvectors can be obtained as the simultaneous eigenvectors of the three commuting operators on the right-hand side of (3.59). Operators $\boldsymbol{Z}$ and $\boldsymbol{W}$ produce derivatives of the wave function with respect to $\boldsymbol{u}$ and $\boldsymbol{\rho}$ variables, separately. Thus, each $\chi(\boldsymbol{u},\boldsymbol{\rho})$ can again be separated as

$$\chi(\boldsymbol{u},\boldsymbol{\rho}) = \sum_j U_j(\boldsymbol{u})\,V_j(\boldsymbol{\rho}), \tag{3.60}$$

where the sum runs over a finite range, and where $U_j(\boldsymbol{u})$ will be eigenfunctions of $\boldsymbol{Z}^2$ and $V_j(\boldsymbol{\rho})$ of $\boldsymbol{W}^2$, respectively.

Functions $U_j(\boldsymbol{u})$ are multiples of spherical harmonics defined on the orientation of the velocity vector $\boldsymbol{u}$, because the $\boldsymbol{Z}$ operator has the structure of an orbital angular momentum in terms of the $\boldsymbol{u}$ variables, and thus its eigenvalues are integer numbers. The global factor left out is an arbitrary function depending on the absolute value of the velocity u.

In fact, if the velocity is expressed in polar spherical coordinates, $\boldsymbol{u} \equiv (u,\beta,\lambda)$, where β is the polar angle and λ the azimuthal angle

$$u_x = u\sin\beta\cos\lambda, \quad u_y = u\sin\beta\sin\lambda, \quad u_z = u\cos\beta,$$

the components of the angular momentum Z_i are:

$$Z_1 = i\hbar\left(\sin\lambda\frac{\partial}{\partial\beta} + \frac{\cos\beta}{\sin\beta}\cos\lambda\frac{\partial}{\partial\lambda}\right), \quad Z_2 = -i\hbar\left(\cos\lambda\frac{\partial}{\partial\beta} - \frac{\cos\beta}{\sin\beta}\sin\lambda\frac{\partial}{\partial\lambda}\right), \quad Z_3 = -i\hbar\frac{\partial}{\partial\lambda},$$

$$Z_\pm = Z_1 \pm iZ_2 = \hbar e^{\pm i\lambda}\left\{\pm\frac{\partial}{\partial\beta} + i\,\frac{\cos\beta}{\sin\beta}\,\frac{\partial}{\partial\lambda}\right\}. \tag{3.61}$$

We see that they are independent of the variable u, because the rotation group is not acting on the whole $\mathbb{R}^3$ space but only on the surface of the unit sphere, parameterized by β and λ.

The operator Z^2 commutes with th three Z_i, and takes the form

$$Z^2 = -\hbar^2\left[\frac{\partial^2}{\partial\beta^2} + \frac{\cos\beta}{\sin\beta}\,\frac{\partial}{\partial\beta} + \frac{1}{\sin^2\beta}\,\frac{\partial^2}{\partial\lambda^2}\right]. \tag{3.62}$$

We have to search for eigenfucntions of Z^2 and Z_3 in separate variables in the form $f(u)G(\beta,\lambda)$, with $f(u)$ arbitrary and as far as the angular part is concerned

$$Z^2\,Y_l^m(\beta,\lambda) = l(l+1)\hbar^2 Y_l^m(\beta,\lambda), \quad Z_3\,Y_l^m(\beta,\lambda) = m\hbar Y_l^m(\beta,\lambda).$$

Only solutions for integer eigenvalues of l and $m = -l, -l+1, \ldots, l$, can be found for this system of differential equations.

The functions $|l,m>\equiv Y_l^m(\beta,\lambda)$, defined on the unit sphere, are called **spherical harmonics**. The normalized rotational invariant measure on the unit sphere is

$$\int_0^\pi d\beta \int_0^{2\pi} \frac{1}{4\pi}\sin\beta d\lambda = 1 \tag{3.63}$$

The spherical harmonics are orthogonal with respect to the hermitian scalar product defined by

$$< l,m|s,n >= \frac{1}{4\pi}\int_0^{2\pi} d\lambda \int_0^\pi \sin\beta d\beta\; Y_l^{m*}(\beta,\lambda)Y_s^n(\beta,\lambda) = \delta_{mn}\delta_{ls},$$

i.e., with respect to the normalized invariant measure on the unit sphere $(1/4\pi)\sin\beta d\beta d\lambda$. The solution of this system is to find functions $Y_l^l(\beta,\lambda)$ of the separate variables $Y_l^l(\beta,\lambda) = A_l(\beta)B_l(\lambda)$, which satisfy

$$Z_+A_l(\beta)B_l(\lambda) = 0, \quad Z_3A_l(\beta)B_l(\lambda) = l\hbar A_l(\beta)B_l(\lambda),$$

i.e.,

$$A_l' - l(\cos\beta/\sin\beta)A_l = 0, \quad -iB_l' = lB_l.$$

They have to be proportional to the functions $A_l(\beta) \simeq \sin^l\beta$ and $B_l(\lambda) \simeq \exp(il\lambda)$. Because on the unit sphere the point (β,λ) is the same than the point $(\beta,\lambda+2\pi)$, it implies that $\exp(il\lambda) = \exp(il(\lambda+2\pi))$, and therefore necessarily l must be an integer number.

These functions, normalized with respect to the measure (3.63) can be written as

$$Y_l^l(\beta,\lambda) = (-1)^l\sqrt{\frac{(2l+1)(2l)!}{2^{2l}(l!)^2}}\,\sin^l\beta\, e^{il\lambda}, \tag{3.64}$$

and the remaining eigenvectors are obtained by the action on them of the operator Z_-. There are no half integer eigenvectors, because the surface of the unit sphere is not the most general homogeneous space of the rotation group. We can see that $Y_l^{m*} = (-1)^m Y_l^{-m}$, and the first normalized spherical harmonics are:

$$|0,0>= 1,$$

$$|1,1>=-\sqrt{\frac{3}{2}}\sin\beta\, e^{i\lambda},\quad |1,0>=\sqrt{3}\cos\beta,\quad |1,-1>=\sqrt{\frac{3}{2}}\sin\beta\, e^{-i\lambda},\tag{3.65}$$

$$|2,2>=\sqrt{\frac{15}{8}}\sin^2\beta\, e^{2i\lambda},\quad |2,1>=-\sqrt{\frac{15}{2}}\sin\beta\cos\beta\, e^{i\lambda},\quad |2,0>=\sqrt{\frac{5}{4}}(3\cos^2\beta-1).$$

It turns out that to find the most general spinor is necessary to seek also solutions of the $V_j(\boldsymbol{\rho})$ part, depending on the orientation variables. This goal will be achieved in the next section, where we consider the action of the rotation group on itself as a transformation group.

3.5.4 Spinor representation on SU(2)

We shall describe now in detail the orientation part of the general wave function, $V_i(\boldsymbol{\rho})$. If there is no contribution to spin from the zitterbewegung part $\boldsymbol{Z}$, the spin operator (3.52) reduces to the $\boldsymbol{W}$ operator given in (3.53). To solve the corresponding eigenvalue equations we shall first represent the spin operators in spherical coordinates.

If we represent vector $\boldsymbol{\rho}=\tan(\alpha/2)\boldsymbol{n}=r\boldsymbol{n}$ in spherical coordinates (r,θ,ϕ), with $r=|\boldsymbol{\rho}|=\tan(\alpha/2)$ and θ and ϕ the usual polar and azimuth angles, respectively, then unit vector $\boldsymbol{n}$ has the Cartesian components given in (3.57). If from now on we take $\hbar=1$, the spin operators (3.53) are represented by the differential operators

$$W_1=\frac{1}{2i}\left[(1+r^2)\sin\theta\,\cos\phi\,\frac{\partial}{\partial r}+\left(\frac{1}{r}\cos\theta\,\cos\phi-\sin\phi\right)\frac{\partial}{\partial\theta}-\left(\frac{\sin\phi}{r\sin\theta}+\frac{\cos\theta\,\cos\phi}{\sin\theta}\right)\frac{\partial}{\partial\phi}\right],$$

$$W_2=\frac{1}{2i}\left[(1+r^2)\sin\theta\,\sin\phi\,\frac{\partial}{\partial r}+\left(\frac{1}{r}\cos\theta\,\sin\phi+\cos\phi\right)\frac{\partial}{\partial\theta}-\left(\frac{\cos\theta\,\sin\phi}{\sin\theta}-\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right],$$

$$W_3=\frac{1}{2i}\left[(1+r^2)\cos\theta\,\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\phi}\right].$$

The Casimir operator of the rotation group W^2 is:

$$W^2=-\frac{1+r^2}{4}\left[(1+r^2)\frac{\partial^2}{\partial r^2}+\frac{2(1+r^2)}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left\{\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\right].$$

The up and down spin operators defined as usual by $W_\pm=W_1\pm iW_2$ are

$$W_+=\frac{e^{i\phi}}{2i}\left[(1+r^2)\sin\theta\,\frac{\partial}{\partial r}+\left(\frac{\cos\theta+ir}{r}\right)\frac{\partial}{\partial\theta}-\left(\frac{r\cos\theta-i}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right],$$

$$W_-=\frac{e^{-i\phi}}{2i}\left[(1+r^2)\sin\theta\,\frac{\partial}{\partial r}+\left(\frac{\cos\theta-ir}{r}\right)\frac{\partial}{\partial\theta}-\left(\frac{r\cos\theta+i}{r\sin\theta}\right)\frac{\partial}{\partial\phi}\right].$$

They satisfy the commutation relations

$$[W_3,W_+]=W_+,\quad [W_3,W_-]=-W_-,\quad [W_+,W_-]=2W_3.$$

We can check that $(W_i)^*=-W_i$ and $W_+=-(W_-)^*$, where * means to take the complex conjugate of the corresponding operator.

If $F_s^m(r,\theta,\phi)$ is an eigenfunction of W^2 and W_3, it satisfies the differential equations:

$$W^2F_s^m(r,\theta,\phi)=s(s+1)F_s^m(r,\theta,\phi),\quad W_3F_s^m(r,\theta,\phi)=mF_s^m(r,\theta,\phi).$$

To find solutions of the above system we know that we can proceed in the following way. Let us compute first the eigenfunctions of the form F_s^s. Then operator W_+ annihilates this state $W_+F_s^s=0$ and by acting on this function with operator W_- we can obtain the remaining

eigenstates F_s^m of the same irreducible representation characterized by parameter s and for $-s \leq m \leq s$. Then our task will be to obtain first the F_s^s functions.

Now, let us consider eigenfunctions F_s^s that can be written in separate variables as $F_s^s(r,\theta,\phi) = A(r)B(\theta)C(\phi)$. Then

$$W_3 A(r)B(\theta)C(\phi) = sA(r)B(\theta)C(\phi)$$

gives rise to

$$(1+r^2)\cos\theta A'BC - \frac{\sin\theta}{r}AB'C + ABC' = 2isABC$$

where A' is the derivative of A and so on, and by dividing both sides by ABC we have

$$(1+r^2)\cos\theta\frac{A'(r)}{A(r)} - \frac{\sin\theta}{r}\frac{B'(\theta)}{B(\theta)} + \frac{C'(\phi)}{C(\phi)} = 2is.$$

Now, the third term on the left-hand side must be a constant, because the remaining terms are functions independent of ϕ. Therefore, this term is written as $C'(\phi)/C(\phi) = ik$ and thus $C(\phi) = e^{ik\phi}$ up to an arbitrary constant factor. Since $C(\phi + 2\pi) = C(\phi)$ this implies that the constant k must be an integer. The other two functions satisfy

$$r(1+r^2)\cos\theta A'B - \sin\theta AB' + ir(k-2s)AB = 0. \tag{3.66}$$

If there exist solutions with real functions A and B, then necessarily $k = 2s$ so that the eigenvalue s can be any integer or half integer, and equation (3.66) can be separated in the form:

$$r(1+r^2)\frac{A'(r)}{A(r)} = \frac{\sin\theta}{\cos\theta}\frac{B'(\theta)}{B(\theta)} = p = \text{constant}, \tag{3.67}$$

where, up to constant factors, the general solution is

$$A(r) = \left(\frac{r^2}{1+r^2}\right)^{p/2}, \quad B(\theta) = (\sin\theta)^p.$$

By acting on this solution $F_s^s \equiv A(r)B(\theta)C(\phi)$, with W_+, since $W_+F_s^s = 0$, it gives:

$$r(1+r^2)\sin^2\theta A'B + (\sin\theta\cos\theta + ir\sin\theta)AB' - 2s(ir\cos\theta + 1)AB = 0.$$

By dividing all terms by AB, taking into account (3.67), we get the condition $(p-2s)(1 + ir\cos\theta) = 0$. Then there exist real solutions in separate variables whenever $p = 2s = k$. They are given, up to a constant factor, by

$$F_s^s(r,\theta,\phi) = \left(\frac{r^2}{1+r^2}\right)^s (\sin\theta)^{2s} e^{i2s\phi}. \tag{3.68}$$

For $s = 1/2$ and after the action of W_- we obtain the two orthogonal spinors

$$\Psi_{1/2}^{1/2} = \frac{r}{\sqrt{1+r^2}}\sin\theta\, e^{i\phi}, \qquad W_-\Psi_{1/2}^{1/2} = \Psi_{1/2}^{-1/2} = \frac{r\cos\theta + i}{\sqrt{1+r^2}},$$

that produce a two-dimensional representation of the rotation group. We can similarly check that $W_-\Psi_{1/2}^{-1/2} = 0$.

By inspection of the structure of $W_\pm$ operators, if we take the complex conjugate of expression $W_+F_s^s = 0$ we get $-W_-(F_s^s)^* = 0$ and therefore $(F_s^s)^* \sim G_s^{-s}$ so that taking the complex conjugate spinors of the above representation we obtain another pair of orthogonal $s = 1/2$ spinors,

$$\widetilde{\Psi}_{1/2}^{1/2} = \frac{r\cos\theta - i}{\sqrt{1+r^2}}, \qquad \widetilde{\Psi}_{1/2}^{-1/2} = \frac{r}{\sqrt{1+r^2}}\sin\theta\, e^{-i\phi}.$$

The remaining representations for higher spins can thus be obtained by the same method, or by taking tensor products of the above two-dimensional representations. For instance, for $s = 1$ we can obtain the following three orthogonal representations. From (3.68) with $s = 1$ and acting with the W_- operator we get

$$\begin{aligned}
\Psi_1^1 = (\Psi_{1/2}^{1/2})^2 &= \frac{r^2}{1+r^2}\,\sin^2\theta\, e^{i2\phi}, \\
\Psi_1^0 = (\Psi_{1/2}^{1/2})(\Psi_{1/2}^{-1/2}) &= \frac{r}{1+r^2}\,\sin\theta\,(i + r\cos\theta)\, e^{i\phi}, \\
\Psi_1^{-1} = (\Psi_{1/2}^{-1/2})^2 &= \frac{(i + r\cos\theta)^2}{1+r^2},
\end{aligned}$$

that can also be obtained as the tensor product $\Psi \otimes \Psi$.

If we work in the normal or canonical representation of the rotation group, where the parameters are $\boldsymbol{\alpha} = \alpha \boldsymbol{n}$, this amounts to replacing the variable $r = \tan(\alpha/2)$ in terms of parameter α and expressing the differential operator $\partial/\partial r$ in terms of $\partial/\partial\alpha$, and then the spin operators are given by

$$W_1 = \frac{1}{2i}\left[2\sin\theta\,\cos\phi\,\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta\,\cos\phi}{\tan(\alpha/2)} - \sin\phi\right)\frac{\partial}{\partial\theta} - \left(\frac{\sin\phi}{\tan(\alpha/2)\sin\theta} + \frac{\cos\theta\,\cos\phi}{\sin\theta}\right)\frac{\partial}{\partial\phi}\right],$$

$$W_2 = \frac{1}{2i}\left[2\sin\theta\,\sin\phi\,\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta\,\sin\phi}{\tan(\alpha/2)} + \cos\phi\right)\frac{\partial}{\partial\theta} - \left(\frac{\cos\theta\,\sin\phi}{\sin\theta} - \frac{\cos\phi}{\tan(\alpha/2)\sin\theta}\right)\frac{\partial}{\partial\phi}\right],$$

$$W_3 = \frac{1}{2i}\left[2\cos\theta\,\frac{\partial}{\partial\alpha} - \frac{\sin\theta}{\tan(\alpha/2)}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\right],$$

$$W^2 = -\left[\frac{\partial^2}{\partial\alpha^2} + \frac{1}{\tan(\alpha/2)}\frac{\partial}{\partial\alpha} + \frac{1}{4\sin^2(\alpha/2)}\left\{\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\right],$$

$$W_+ = \frac{e^{i\phi}}{2i}\left[2\sin\theta\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta}{\tan(\alpha/2)} + i\right)\frac{\partial}{\partial\theta} - \left(\frac{\cos\theta\tan(\alpha/2) - i}{\tan(\alpha/2)\sin\theta}\right)\frac{\partial}{\partial\phi}\right],$$

$$W_- = \frac{e^{-i\phi}}{2i}\left[2\sin\theta\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta}{\tan(\alpha/2)} - i\right)\frac{\partial}{\partial\theta} - \left(\frac{\cos\theta\tan(\alpha/2) + i}{\tan(\alpha/2)\sin\theta}\right)\frac{\partial}{\partial\phi}\right]$$

and the orthogonal spinors of the two two-dimensional representations can be written as

$$\Psi_{1/2}^{1/2} = i\sin\frac{\alpha}{2}\sin\theta\, e^{i\phi}, \qquad \Psi_{1/2}^{-1/2} = \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\cos\theta \tag{3.69}$$

and

$$\widetilde{\Psi}_{1/2}^{1/2} = \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\cos\theta, \qquad \widetilde{\Psi}_{1/2}^{-1/2} = -i\sin\frac{\alpha}{2}\sin\theta\, e^{-i\phi}. \tag{3.70}$$

We have mentioned that the different spinors are orthogonal. To endow the group manifold with a Hilbert space structure it is necessary to define a hermitian, definite positive, scalar product. The Jacobian matrix of variables $\boldsymbol{\rho}'$ in terms of variables $\boldsymbol{\rho}$ given in (3.39), has the determinant

$$\det\left(\frac{\partial\rho'^i}{\partial\rho^j}\right) = \frac{(1+\mu^2)^2}{(1 - \boldsymbol{\mu}\cdot\boldsymbol{\rho})^4},$$

and thus the transformation of the volume element

$$d^3\rho' = \frac{(1+\mu^2)^2}{(1 - \boldsymbol{\mu}\cdot\boldsymbol{\rho})^4}\, d^3\rho.$$

We also get from (3.39) that

$$1+\rho'^2 = \frac{(1+\mu^2)}{(1-\boldsymbol{\mu}\cdot\boldsymbol{\rho})^2}\,(1+\rho^2)$$

and then the measure

$$\frac{d^3\rho'}{(1+\rho'^2)^2} = \left(\frac{(1-\boldsymbol{\mu}\cdot\boldsymbol{\rho})^2}{(1+\mu^2)(1+\rho^2)}\right)^2 \frac{(1+\mu^2)^2}{(1-\boldsymbol{\mu}\cdot\boldsymbol{\rho})^4} d^3\rho = \frac{d^3\rho}{(1+\rho^2)^2}$$

is in fact an invariant measure.

In spherical coordinates it is written as

$$\frac{r^2\sin\theta}{(1+r^2)^2}\; dr d\theta d\phi$$

and in the normal representation is

$$\sin^2(\alpha/2)\sin\theta d\alpha d\theta d\phi.$$

Since the rotation group is a double-connected group, the above measure must be defined on a simply connected manifold, *i.e.*, on the universal covering group of $SO(3)$, which is $SU(2)$. The $SU(2)$ group manifold in the normal representation is given by the three-dimensional sphere of radius 2π and where points on the surface of this sphere represent a unique $SU(2)$ element, namely the 2×2 unitary matrix $-\mathbb{I}$. The normalized invariant measure becomes

$$d\mu_N(\alpha,\theta,\phi) \equiv \frac{1}{4\pi^2}\,\sin^2(\alpha/2)\sin\theta\, d\alpha\, d\theta\, d\phi. \tag{3.71}$$

Therefore, the hermitian scalar product will be defined as

$$< f|g >= \frac{1}{4\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\; f^*(\alpha,\theta,\phi) g(\alpha,\theta,\phi)\sin^2(\alpha/2)\sin\theta, \tag{3.72}$$

where f^* is the complex conjugate function of f.

All the previous computed spinors are orthogonal vectors with respect to the group invariant measure (3.71). In particular, the normalized $s=1/2$ spinors are those given in (3.69)-(3.70), multiplied by $\sqrt{2}$.

The spin projection operators on the body axis $\boldsymbol{e}_i$ linked to the particle, are given in (3.58) in the $\boldsymbol{\rho}$ parametrization, and we have seen that they differ from the spin operators $\boldsymbol{W}$ only in the change of $\boldsymbol{\rho}\to-\boldsymbol{\rho}$, and a global change of sign. In the normal parametrization this corresponds to the change $\alpha\to-\alpha$, followed by a global change of sign.

It can be checked as mentioned before, that

$$[T_i, T_k] = -i\epsilon_{ikl}\, T_l, \tag{3.73}$$

$$[W_i, T_k] = 0. \tag{3.74}$$

Since $W^2 = T^2$ we can find simultaneous eigenvectors of the operators W^2, W_3 and T_3, which will be denoted by $D^{(s)}_{mn}(\boldsymbol{\alpha})$ in such a way that

$$\begin{aligned} W^2 D^{(s)}_{mn}(\boldsymbol{\alpha}) &= s(s+1) D^{(s)}_{mn}(\boldsymbol{\alpha}), \\ W_3 D^{(s)}_{mn}(\boldsymbol{\alpha}) &= m D^{(s)}_{mn}(\boldsymbol{\alpha}), \\ T_3 D^{(s)}_{mn}(\boldsymbol{\alpha}) &= n D^{(s)}_{mn}(\boldsymbol{\alpha}). \end{aligned}$$

Since $W_3(\alpha)D^{(s)}_{mn}(\alpha) = mD^{(s)}_{mn}(\alpha)$, by producing the change $\alpha \to -\alpha$ we get $W_3(-\alpha)D^{(s)}_{mn}(-\alpha) = mD^{(s)}_{mn}(-\alpha)$ and the subsequent global change of sign it reduces to

$$-W_3(-\alpha)D^{(s)}_{mn}(-\alpha) = T_3(\alpha)D^{(s)}_{mn}(-\alpha) = -mD^{(s)}_{mn}(-\alpha),$$

so that the above spinors (3.69)-(3.70) are also eigenvectors of T_3.

With this notation, the four normalized spinors, denoted by the corresponding eigenvalues $|s, m, n >$, are

$$\Phi_1 = |1/2, 1/2, 1/2 > = \sqrt{2}(\cos(\alpha/2) + i\cos\theta\sin(\alpha/2)), \tag{3.75}$$

$$\Phi_2 = |1/2, -1/2, 1/2 > = i\sqrt{2}\sin(\alpha/2)\,\sin\theta e^{-i\phi}, \tag{3.76}$$

$$\Phi_3 = |1/2, 1/2, -1/2 > = i\sqrt{2}\sin(\alpha/2)\,\sin\theta e^{i\phi}. \tag{3.77}$$

$$\Phi_4 = |1/2, -1/2, -1/2 > = \sqrt{2}(\cos(\alpha/2) - i\cos\theta\sin(\alpha/2)), \tag{3.78}$$

They form an orthonormal set with respect to the normalized invariant measure (3.71) and with the scalar product defined in (3.72). We can check that the lowering operators $W_-\Phi_1 = \Phi_2$, $W_-\Phi_2 = 0$, $W_-\Phi_3 = \Phi_4$, $W_-\Phi_4 = 0$, and similarly $T_-\Phi_1 = 0$, $T_-\Phi_3 = \Phi_1$, $T_-\Phi_2 = 0$, and $T_-\Phi_4 = \Phi_2$, and the corresponding up relations when acting with the rising operators W_+ and T_+, respectively. Remark that because the opposite sign in the commutation relations of the T_i operators, here the $T_\pm$ operate in the reverse direction.

The important feature is that if the system has spin 1/2, although the $s = 1/2$ irreducible representations of the rotation group are two-dimensional, to describe the spin part of the wave function we need a function defined in the above four-dimensional complex Hilbert space, because to describe orientation we attach some local frame to the particle, and therefore in addition to the spin values in the laboratory frame we also have as additional observables the spin projections on the body axes, which can be included within the set of commuting operators.

3.5.5 Matrix representation of internal observables

The matrix representation of any observable A that acts on the orientation variables or in this internal four-dimensional space spanned by these spin 1/2 wave functions Φ_i, is obtained as $A_{ij} =< \Phi_i|A\Phi_j >$, $i, j = 1, 2, 3, 4$. Once these four normalized basis vectors are fixed, when acting on the subspace they span, the differential operators W_i and T_i have the 4×4 block matrix representation

$$\boldsymbol{S} \equiv \boldsymbol{W} = \frac{\hbar}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{3.79}$$

$$T_1 = \frac{\hbar}{2}\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad T_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & i\mathbb{I} \\ -i\mathbb{I} & 0 \end{pmatrix}, \quad T_3 = \frac{\hbar}{2}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \tag{3.80}$$

where $\boldsymbol{\sigma}$ are the three Pauli matrices and $\mathbb{I}$ represents the 2×2 unit matrix. We have included Planck's constant into the angular momentum operators.

If we similarly compute the matrix elements of the nine components of the unit vectors $(\boldsymbol{e}_i)_j$, $i, j = 1, 2, 3$ we obtain the nine traceless hermitian matrices

$$\boldsymbol{e}_1 = \frac{1}{3}\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},\ \boldsymbol{e}_2 = \frac{1}{3}\begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix},\ \boldsymbol{e}_3 = \frac{1}{3}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}. \tag{3.81}$$

We can check that the $T_i = \boldsymbol{S}\cdot\boldsymbol{e}_i = \boldsymbol{e}_i\cdot\boldsymbol{S}$. We see that the different components of the unit vectors $\boldsymbol{e}_i$, in general do not commute. The eigenvalues of every component e_{ij}, in this matrix representation of definite spin, are $\pm 1/3$. However, the matrix representation of the square of any component is $(e_{ij})^2 = \mathbb{I}/3$, so that the magnitude squared of each vector $\boldsymbol{e}_i^2 = \sum_j (e_{ij})^2 = \mathbb{I}$ when acting on these wave functions. The eigenvalues of the squared operator $(e_{ij})^2$ are not

the squared eigenvalues of e_{ij}. This is because the function $e_{ij}\Phi_k$ does not belong in general to the same space spanned by the Φ_k, $k = 1, \ldots, 4$ although this space is invariant space for operators W_i and T_j. In fact, each function $e_{ij}\Phi_k$ is a linear combination of a spin 1/2 and a spin 3/2 wave function.

We do not understand why any component of a classical unit vector e_{ij} of a Cartesian frame, can have as eigenvalues $\pm 1/3$ in the quantum case and its square $(e_{jj})^2 = \mathbb{I}/3$ instead of $\mathbb{I}/9$.

3.5.6 Peter-Weyl theorem for compact groups

The above spinors can also be obtained by making use of an important theorem for representations of compact groups, known as the Peter-Weyl theorem, [12] which is stated without proof that can be read in any of the mentioned references.

> **Theorem.-** Let $D^{(s)}(g)$ be a complete system of non-equivalent, unitary, irreducible representations of a compact group G, labeled by the parameter s. Let d_s be the dimension of each representation and $D^{(s)}_{ij}(g)$, $1 \leq i, j \leq d_s$ the corresponding matrix elements. Then, the functions
>
> $$\sqrt{d_s}\, D^{(s)}_{ij}(g), \quad 1 \leq i, j \leq d_s$$
>
> form a complete orthonormal system on G, with respect to some normalized invariant measure $\mu_N(g)$ defined on this group, *i.e.*,
>
> $$\int_G \sqrt{d_s}\, D^{(s)*}_{ij}(g)\, \sqrt{d_r}\, D^{(r)}_{kl}(g)\, d\mu_N(g) = \delta^{sr}\delta_{ik}\delta_{jl}. \tag{3.82}$$

That the set is complete means that every square integrable function defined on G, $f(g)$, admits a series expansion, convergent in norm, in terms of the above orthogonal functions $D^{(s)}_{ij}(g)$, in the form

$$f(g) = \sum_{s,i,j} a^{(s)}_{ij}\, \sqrt{d_s}\, D^{(s)}_{ij}(g),$$

where the coefficients, in general complex numbers $a^{(s)}_{ij}$, are obtained by

$$a^{(s)}_{ij} = \int_G \sqrt{d_s}\, D^{(s)*}_{ij}(g)\, f(g) d\mu_N(g).$$

In our case $SU(2)$, as a group manifold, is the simply connected three-dimensional sphere of radius 2π, with the normalized measure as seen before (3.71),

$$d\mu_N(\alpha, \theta, \phi) = \frac{1}{4\pi^2} \sin\theta \sin(\alpha/2)^2\, d\alpha d\theta d\phi.$$

In the normal parametrization, the two-dimensional representation of $SU(2)$ corresponds to the eigenvalue $s = 1/2$ of S^2 and the matrix representation is given by

$$D^{(1/2)}(\boldsymbol{\alpha}) = \cos(\alpha/2)\mathbb{I} - i \sin(\alpha/2)(\boldsymbol{u} \cdot \boldsymbol{\sigma}),$$

---

[12] N. Ja. Vilenkin, *Fonctions spéciales et Théorie de la représentation des groupes*, Dunod, Paris (1969), p. 39.
A.O. Barut and R. Raczka, *Theory of group representations and applications*, PWN-Polish Scientific Publishers, Warszawa (1980), p. 174.
F. Peter and H. Weyl, *Math. Ann.* **7**, 735 (1927).

i.e.,

$$D^{(1/2)}(\boldsymbol{\alpha}) = \begin{pmatrix} \cos(\alpha/2) - i\cos\theta\sin(\alpha/2) & -i\sin\theta\sin(\alpha/2)\, e^{-i\phi} \\ -i\sin\theta\sin(\alpha/2)\, e^{i\phi} & \cos(\alpha/2) + i\cos\theta\sin(\alpha/2) \end{pmatrix}.$$

If we compare these four matrix components with the four orthogonal spinors given in (3.75)-(3.78) we see that

$$D^{(1/2)}(\boldsymbol{\alpha}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi_4 & -\Phi_2 \\ -\Phi_3 & \Phi_1 \end{pmatrix} \tag{3.83}$$

In the three-dimensional representation of $SO(3)$, considered as a representation of SU(2)

$$D^{(1)}_{ij}(\boldsymbol{\alpha}) = \delta_{ij}\cos\alpha + u_i u_j(1-\cos\alpha) + \epsilon_{ikj}u_k \sin\alpha \equiv e_{j\,i}$$

we get another set of nine orthogonal functions. Multiplied by $\sqrt{3}$ they form another orthonormal set orthogonal to the previous four spinors. It is a good exercise to check this orthogonality among these functions.

3.5.7 General spinors

In the case that the zitterbewegung content of the spin is not vanishing we can also obtain spin 1/2 wave-functions as the irreducible representations contained in the tensor product of integer and half-integer spin states coming from the $U(\boldsymbol{u})$ and $V(\boldsymbol{\rho})$ part of the general wave function (3.60).

The total spin operator of the system is of the form

$$\boldsymbol{S} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W} = \boldsymbol{Z} + \boldsymbol{W},$$

where $\boldsymbol{Z} = -i\hbar\nabla_u$ and $\boldsymbol{W}$ is given in (3.53). Spin projections on the body axes, *i.e.*, operators $T_i = \boldsymbol{e}_i \cdot \boldsymbol{W}$, are described in (3.58). They satisfy the commutation relations

$$[\boldsymbol{Z}, \boldsymbol{Z}] = i\boldsymbol{Z}, \quad [\boldsymbol{W}, \boldsymbol{W}] = i\boldsymbol{W}, \quad [\boldsymbol{T}, \boldsymbol{T}] = i\boldsymbol{T},$$

$$[\boldsymbol{Z}, \boldsymbol{W}] = 0, \quad [\boldsymbol{Z}, \boldsymbol{T}] = 0, \quad [\boldsymbol{W}, \boldsymbol{T}] = 0.$$

These commutation relations are invariant under the change $\boldsymbol{\rho}$ by $-\boldsymbol{\rho}$ in the definition of the operators $\boldsymbol{W}$ and $\boldsymbol{T}$, because they are changed into each other. The expression of the body frame unit vectors $\boldsymbol{e}_i$ is given in (3.55) and (3.56).

We can see that these unit vector components and spin operators W_i and T_j satisfy the following properties:

1) $e_{ij}(-\alpha, \theta, \phi) = -e_{j\,i}(\alpha, \theta, \phi)$.
2) $\boldsymbol{e}_i \cdot \boldsymbol{W} \equiv \sum_j e_{ij} W_j = T_i$.
3) $\sum_j \boldsymbol{e}_j T_j = \boldsymbol{W}$.
4) For all i, j, the action $W_i e_{j\,i} = 0$, with no addition on index i.
5) For all i, j, the action $T_i e_{ij} = 0$, with no addition on index i.
6) For all i, j, k, with $i \neq j$, we have that $W_i e_{kj} + W_j e_{ki} = 0$, and in the case that $i = j$, it leads to property 4.
7) For all i, j, k, with $i \neq j$, we have that $T_i e_{j\,k} + T_j e_{ik} = 0$, and similarly as before in the case $i = j$ it leads to property 4.

This implies that $\boldsymbol{e}_i \cdot \boldsymbol{W} = \boldsymbol{W} \cdot \boldsymbol{e}_i = T_i$, because of property 4, since when acting on an arbitray function f,

$$(\boldsymbol{W} \cdot \boldsymbol{e}_i) f \equiv \sum_j W_j(e_{ij} f) = f \sum_j W_j(e_{ij}) + \sum_j e_{ij} W_j(f) = T_i(f),$$

because $\sum_j W_j e_{ij} = 0$.

In the same way $\sum_j \boldsymbol{e}_j T_j \equiv \sum_j T_j \boldsymbol{e}_j = \boldsymbol{W}$.

Now we fix the value of spin. Particles of different values of spin can be described. Let us consider systems that take the lowest admissible spin values. For spin $1/2$ particles, if we take first for simplicity eigenfunctions $V(\boldsymbol{\rho})$ of W^2 with eigenvalue $1/2$, and then since the total spin has to be $1/2$, the orbital $\boldsymbol{Z}$ part can only contribute with spherical harmonics of value $z = 0$ and $z = 1$.

If there is no zitterbewegung spin, $z = 0$, and Wigner's functions can be taken as simultaneous eigenfunctions of the three commuting W^2, W_3, and T_3 operators, and the normalized eigenvectors $|w, w_3, t_3>$ are explicitly given by the functions (3.75-3.78).

If we have a zitterbewegung spin of value $z = 1$, then the $U(\boldsymbol{u})$ part contributes with the spherical harmonics described in (3.65)

$$Y_1^1(\beta,\lambda) \equiv |1,1> = -\sin(\beta)e^{i\lambda}\sqrt{\frac{3}{8\pi}}, \tag{3.84}$$

$$Y_1^0(\beta,\lambda) \equiv |1,0> = \cos(\beta)\sqrt{\frac{3}{4\pi}}, \tag{3.85}$$

$$Y_1^{-1}(\beta,\lambda) \equiv |1,-1> = \sin(\beta)e^{-i\lambda}\sqrt{\frac{3}{8\pi}}, \tag{3.86}$$

normalized with respect to the measure

$$\int_0^{\pi}\int_0^{2\pi} \sin(\beta) d\beta d\lambda,$$

which are the indicated eigenfunctions $|z, z_3>$ of $\boldsymbol{Z}^2$ and Z_3, and where the variables β and λ determine the orientation of the velocity $\boldsymbol{u}$.

The tensor product representation of the rotation group constructed from the two irreducible representations $\mathbf{1}$ associated to the spherical harmonics (3.84)-(3.86) and $\mathbf{1/2}$ given in (3.75)-(3.78) is split into the direct sum $\mathbf{1} \otimes \mathbf{1/2} = \mathbf{3/2} \oplus \mathbf{1/2}$.

The following functions of five variables β, λ, α, θ and ϕ, where variables β and λ correspond to the ones of the spherical harmonics Y_l^m, and the remaining α, θ and ϕ, to the previous spinors Φ_i, are normalized spin $1/2$ functions $|s, s_3, t_3>$ that are eigenvectors of total spin S^2, and S_3 and T_3 operators

$$\Psi_1 \equiv |1/2, 1/2, 1/2> = \frac{1}{\sqrt{3}}\left(Y_1^0\Phi_1 - \sqrt{2}Y_1^1\Phi_2\right), \tag{3.87}$$

$$\Psi_2 \equiv |1/2, -1/2, 1/2> = \frac{1}{\sqrt{3}}\left(-Y_1^0\Phi_2 + \sqrt{2}Y_1^{-1}\Phi_1\right), \tag{3.88}$$

$$\Psi_3 \equiv |1/2, 1/2, -1/2> = \frac{1}{\sqrt{3}}\left(Y_1^0\Phi_3 - \sqrt{2}Y_1^1\Phi_4\right), \tag{3.89}$$

$$\Psi_4 \equiv |1/2, -1/2, -1/2> = \frac{1}{\sqrt{3}}\left(-Y_1^0\Phi_4 + \sqrt{2}Y_1^{-1}\Phi_3\right), \tag{3.90}$$

such that $\Psi_2 = S_-\Psi_1$ and similarly $\Psi_4 = S_-\Psi_3$, and also that $\Psi_3 = T_-\Psi_1$, and $\Psi_4 = T_-\Psi_2$. They are no longer eigenfunctions of the W_3 operator, although they span an invariant vector space for S^2, S_3 and T_3 operators. In the above basis (3.87)-(3.90) formed by orthonormal vectors Ψ_i, the normalized invariant measure is

$$\frac{1}{16\pi^3}\int_0^{\pi}\sin\beta d\beta\int_0^{2\pi}d\lambda\int_0^{2\pi}\sin^2(\alpha/2)d\alpha\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi = 1,$$

and the matrix representation of the spin is

$$\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W} = \frac{\hbar}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}, \tag{3.91}$$

while the matrix representation of the $\boldsymbol{Z}$ and $\boldsymbol{W}$ part is

$$\boldsymbol{Z} = \frac{2\hbar}{3}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \boldsymbol{W} = \frac{-\hbar}{6}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{3.92}$$

which do not satisfy commutation relations of angular momentum operators because the vector space spanned by the above basis Ψ_j, is not an invariant space for these operators $\boldsymbol{Z}$ and $\boldsymbol{W}$.

It must be remarked that $\boldsymbol{Z}$ has the same orientation than the spin $\boldsymbol{S}$, because it is a positive multiple of it, while $\boldsymbol{W}$ has the opposite orientation, as suggested by the picture of the front page of these Notes.

$$S^2 = \frac{3}{4}\mathbb{I}, \quad S = \frac{\sqrt{3}}{2}\mathbb{I}, \quad Z^2 = \frac{4}{3}\mathbb{I}, \quad Z = \frac{2}{\sqrt{3}}\mathbb{I}, \quad W^2 = \frac{1}{12}\mathbb{I}, \quad W = \frac{1}{2\sqrt{3}}\mathbb{I}.$$

The absolute value of $\boldsymbol{S}$, S is $\sqrt{3}/2$, while that of $\boldsymbol{Z}$ is just $2/\sqrt{3}$, only $4/3$ of the other, meanwhile for $\boldsymbol{W}$ its absolute value is $1/2\sqrt{3}$, just $1/3$ of the absolute value of $\boldsymbol{S}$ but in the opposite direction. Therefore, because $\boldsymbol{Z}$ is opposite to $\boldsymbol{W}$, the modulus of $\boldsymbol{S}$ is $S = Z - W = \sqrt{3}/2$. This justifies, from the quantum point of view, the geometrical representation of those operators in the front page, with $\boldsymbol{Z}$ in the same direction than $\boldsymbol{S}$, and $\boldsymbol{W}$ in the opposite direction.

If we pay attention to the spinors Ψ_i, they are eigenvectors of Z^2 with eigenvalue $1(1+1) = 2$, and of W^2 with eigenvalue $1/2(1/2+1) = 3/4$, but they are not eigenvectors of Z_3 and W_3. In fact, the action of these operators on these vectors, take them out of this four-dimensional Hilbert space. It is not a representation space of an irreducible representation of the algebra generated by the operators Z_i and W_i, but it is a vector space of a closed representation of the operators S_i. It is a direct sum of two irreducible representations of spin $s = 1/2$.

The spin projection of the $\boldsymbol{W}$ part on the body axis, *i.e.*, the $\boldsymbol{T}$ operator, takes the same form as before (3.80)

$$T_1 = \frac{\hbar}{2}\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad T_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & i\mathbb{I} \\ -i\mathbb{I} & 0 \end{pmatrix}, \quad T_3 = \frac{\hbar}{2}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \tag{3.93}$$

because Ψ_1 and Ψ_2 functions are eigenfunctions of T_3 with eigenvalue $1/2$, while Ψ_3 and Ψ_4 are of eigenvalue $-1/2$, and thus the spinors Ψ_i span an invariant space for S_i and T_j operators. In fact the basis is formed by simultaneous eigenfunctions of total spin S^2, S_3 and T_3, and the ket representation is the same as in the case of the Φ_i given in (3.75)-(3.78).

The expression in this basis of the components of the unit vectors $\boldsymbol{e}_i$ are represented by

$$\boldsymbol{e}_1 = -\frac{1}{9}\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \; \boldsymbol{e}_2 = -\frac{1}{9}\begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix}, \; \boldsymbol{e}_3 = -\frac{1}{9}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}. \tag{3.94}$$

3.6 Summary of Classical and Quantum Mechanics

We shall summarize very general aspects of classical and quantum mechanical elementary particles.

Classical Mechanics

States: Each point $x \in X$ of the kinematical space X.

Elementary particle: X is a homogeneous space of the kinematical group G.

Observables: Every function of the kinematical variables and their time derivatives.

Transformation of the state: $x' = gx \equiv f(x,g),\ g \in G$.

Elementary particle: $\widetilde{L}_0 = T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}} + \boldsymbol{U}\cdot\dot{\boldsymbol{u}} + \boldsymbol{W}\cdot\boldsymbol{\omega}$.

Transformation of the Lagrangian: $\widetilde{L}'(x',\dot{x}') = \widetilde{L}(x,\dot{x}) + d\alpha(g,x)/d\tau$

Interaction: $\widetilde{L}_I = -eA_0(t,\boldsymbol{r})\dot{t} + e\boldsymbol{A}(t,\boldsymbol{r})\cdot\dot{\boldsymbol{r}}$.

Noether Constants (non-rel.) $\mathcal{G}\otimes SO(3)_L$**:**

$$H = -T - \boldsymbol{u}\cdot\frac{d\boldsymbol{U}}{dt},\quad \boldsymbol{P} = m\boldsymbol{u} - \frac{d\boldsymbol{U}}{dt},\quad \boldsymbol{K} = m\boldsymbol{r} - \boldsymbol{P}t - \boldsymbol{U},\quad \boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{u}\times\boldsymbol{U} + \boldsymbol{W},$$

$$T_i = \boldsymbol{W}\cdot\boldsymbol{e}_i,\quad i = 1,2,3.$$

Noether Constants (relat.) $\mathcal{P}\otimes SO(3)_L$**:**

$$H = -T - \boldsymbol{u}\cdot\frac{d\boldsymbol{U}}{dt},\quad \boldsymbol{P} = \boldsymbol{R} - \frac{d\boldsymbol{U}}{dt},\quad \boldsymbol{K} = H\boldsymbol{r}/c^2 - \boldsymbol{P}t - \boldsymbol{S}\times\boldsymbol{u}/c^2,\quad \boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{u}\times\boldsymbol{U} + \boldsymbol{W},$$

$$T_i = \boldsymbol{W}\cdot\boldsymbol{e}_i,\quad i = 1,2,3.$$

Invariants (no relat.)

$$m,\quad H - \frac{\boldsymbol{P}^2}{2m} = 0,\quad \boldsymbol{S}_{CM}^2 = \left(\boldsymbol{J} - \frac{1}{m}\boldsymbol{K}\times\boldsymbol{P}\right)^2,\quad \boldsymbol{T}^2$$

Invariants (relat.)

$$p_\mu p^\mu = (H/c)^2 - \boldsymbol{P}^2 = m^2c^2,\quad w_\mu w^\mu = (\boldsymbol{P}\cdot\boldsymbol{S}_{CM})^2 - (H\boldsymbol{S}_{CM}/c)^2 = -m^2c^2S^2,\quad \boldsymbol{T}^2.$$

$$\boldsymbol{S}_{CM} = \boldsymbol{J} - \boldsymbol{q}\times\boldsymbol{P},\quad H\boldsymbol{S}_{CM}/c = H\boldsymbol{J}/c^2 - \boldsymbol{K}\times\boldsymbol{P},\quad \boldsymbol{K} = H\boldsymbol{q}/c^2 - \boldsymbol{P}t.$$

Quantum Mechanics

States: Each normalized vector $|\phi>$, $\phi(x) \in \mathbb{L}^2(X)$ of the Hilbert space $\mathbb{L}^2(X)$.

Elementary Particle: $\mathbb{L}^2(X)$ is the representation space of a projective unitary irreducible representation of the kinematical group G.

Observables: Every selfadjoint operator acting on the Hilbert space.

Transformation of the state:

$$|\phi'>= U(g)|\phi>, \phi'(x) = U(g)\phi(x) = \phi(g^{-1}x) \exp\left\{\frac{-i}{\hbar}\alpha(g^{-1};x)\right\},$$

and the unitary operators and their infinitesimal generators are

$$U(g) = \exp\left\{\frac{-i}{\hbar} g^\sigma X_\sigma\right\}, \quad X_\sigma = \frac{\hbar}{i} u^i_\sigma(x)\frac{\partial}{\partial x_i} - \lambda_\sigma(x), \quad \lambda_\sigma(x) = \left.\frac{\partial \alpha(g,x)}{\partial g^\sigma}\right|_{g=0}.$$

Generators (non-relat.) $\mathcal{G} \otimes SO(3)_L$**:**

$$H = i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{P} = \frac{\hbar}{i}\nabla, \qquad \boldsymbol{K} = m\boldsymbol{r} - \boldsymbol{P}t - \boldsymbol{U}, \quad \boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{u}\times\boldsymbol{U} + \boldsymbol{W}.$$

$$\boldsymbol{U} = \frac{\hbar}{i}\nabla_u, \quad \boldsymbol{W} = \frac{\hbar}{2i}\left(\nabla_\rho + \boldsymbol{\rho}\times\nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)\right), \quad \boldsymbol{S} = \boldsymbol{u}\times\boldsymbol{U} + \boldsymbol{W},$$

$$T_i = \boldsymbol{W}\cdot\boldsymbol{e}_i, \quad i = 1,2,3. \quad \boldsymbol{T} = \frac{\hbar}{2i}\left(\nabla_\rho - \boldsymbol{\rho}\times\nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)\right),$$

Generators (relat.) $\mathcal{P} \otimes SO(3)_L$**:**

$$H = i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{P} = \frac{\hbar}{i}\nabla, \qquad \boldsymbol{K} = H\boldsymbol{r}/c^2 - \boldsymbol{P}t - \boldsymbol{S}\times\boldsymbol{u}/c^2, \quad \boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{u}\times\boldsymbol{U} + \boldsymbol{W}.$$

$$\boldsymbol{U} = \frac{\hbar}{i}\nabla_u, \quad \boldsymbol{W} = \frac{\hbar}{2i}\left(\nabla_\rho + \boldsymbol{\rho}\times\nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)\right), \quad \boldsymbol{S} = \boldsymbol{u}\times\boldsymbol{U} + \boldsymbol{W}.$$

$$T_i = \boldsymbol{W}\cdot\boldsymbol{e}_i, \quad i = 1,2,3. \quad \boldsymbol{T} = \frac{\hbar}{2i}\left(\nabla_\rho - \boldsymbol{\rho}\times\nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)\right),$$

Invariants-Casimir Operators (non-relat.)

$$m, \quad H - \frac{\boldsymbol{P}^2}{2m} = 0, \quad \boldsymbol{S}^2_{CM} = \left(\boldsymbol{J} - \frac{1}{m}\boldsymbol{K}\times\boldsymbol{P}\right)^2, \quad \boldsymbol{T}^2 = \boldsymbol{W}^2$$

Invariants-Casimir Operators (relat.)

$$p_\mu p^\mu = (H/c)^2 - \boldsymbol{P}^2 = m^2c^2, \quad w_\mu w^\mu = (\boldsymbol{P}\cdot\boldsymbol{S}_{CM})^2 - (H\boldsymbol{S}_{CM}/c)^2 = -m^2c^2s(s+1)\hbar^2, \quad \boldsymbol{T}^2 = \boldsymbol{W}^2.$$

Dirac equation

$$H - \boldsymbol{P}\cdot\boldsymbol{u} - \frac{1}{c^2}\boldsymbol{S}\cdot\left(\frac{d\boldsymbol{u}}{dt}\times\boldsymbol{u}\right) = 0.$$

Chapter 4

Dirac particle

4.1 Quantization of the $u = c$ model

For Luxons we have the nine-dimensional homogeneous space of the Poincaré group, spanned by the ten variables $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha})$, but now $\boldsymbol{u}$ is restricted to $u = c$. For this particle, since $\boldsymbol{u}\cdot\dot{\boldsymbol{u}} = 0$ and $\dot{\boldsymbol{u}} \neq 0$, we are describing particles with a circular internal orbital motion at the constant speed c.

In the center of mass frame, (see Fig.4.1) the center of charge describes a circle of radius $R_0 = S/mc$ at the constant speed c, the spin being orthogonal to the charge trajectory plane and a constant of the motion in this frame. Let us consider the quantization of this $u = c$ model whose dynamical equation is given by (2.167).

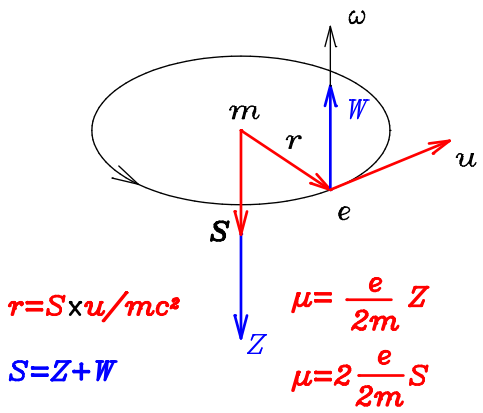


Figure 4.1: **Motion of the center of charge of the particle** $(H > 0)$**, in the C.M. frame.**

If we analyse this particle in the centre of mass frame it becomes a system of three degrees of freedom. These are the x and y coordinates of the point charge on the plane and the phase α of the rotation of the body axis with angular velocity ω. But this phase is the same as the phase of the orbital motion, as we shall see later, and because this motion is a circle of constant radius only one degree of freedom is left, for instance the x coordinate. In the centre of mass frame the particle is equivalent to a one-dimensional harmonic oscillator of angular frequency $\omega = mc^2/S$ in its ground state.

Identification of the ground energy of the one-dimensional harmonic oscillator $\hbar\omega/2$ with the rest energy of the system in the center of mass frame $+mc^2$, for $H > 0$ particles, implies that

the classical constant parameter $S = \hbar/2$. All Lagrangian systems defined with this kinematical space, irrespective of the particular Lagrangian we choose, have this behaviour and represent spin 1/2 particles when quantized.

4.2 Dirac equation

The kinematical variables of this system transform under $\mathcal{P}$ according to

$$t'(\tau) = \gamma t(\tau) + \gamma(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{r}(\tau))/c^2 + b, \tag{4.1}$$

$$\boldsymbol{r}'(\tau) = R(\boldsymbol{\mu})\boldsymbol{r}(\tau) + \gamma\boldsymbol{v}t(\tau) + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{r}(\tau))\boldsymbol{v} + \boldsymbol{a}, \tag{4.2}$$

$$\boldsymbol{u}'(\tau) = \frac{R(\boldsymbol{\mu})\boldsymbol{u}(\tau) + \gamma\boldsymbol{v} + (\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{u}(\tau))\boldsymbol{v}\gamma^2/(1+\gamma)c^2}{\gamma(1+\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{u}(\tau)/c^2)}, \tag{4.3}$$

$$\boldsymbol{\rho}'(\tau) = \frac{\boldsymbol{\mu} + \boldsymbol{\rho}(\tau) + \boldsymbol{\mu}\times\boldsymbol{\rho}(\tau) + \boldsymbol{F}_c(\boldsymbol{v},\boldsymbol{\mu};\boldsymbol{u}(\tau),\boldsymbol{\rho}(\tau))}{1 - \boldsymbol{\mu}\cdot\boldsymbol{\rho}(\tau) + G_c(\boldsymbol{v},\boldsymbol{\mu};\boldsymbol{u}(\tau),\boldsymbol{\rho}(\tau))}, \tag{4.4}$$

where the functions $\boldsymbol{F}_c$ and G_c are given in (2.132) and (2.133), respectively. When quantized, the wave function of the system is a function $\Phi(t,\boldsymbol{r},\boldsymbol{u},\boldsymbol{\rho})$ of these kinematical variables. For the Poincaré group all exponents and thus all gauge functions on homogeneous spaces are equivalent to zero, and the Lagrangians for free particles can thus be taken strictly invariant. Projective representations reduce to true representations so that the ten generators on the Hilbert space, taking into account (4.1)-(4.4) and (3.15) are given by:

$$H = i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{P} = \frac{\hbar}{i}\nabla, \quad \boldsymbol{K} = \boldsymbol{r}\,\frac{i\hbar}{c^2}\frac{\partial}{\partial t} - t\,\frac{\hbar}{i}\nabla - \frac{1}{c^2}\boldsymbol{S}\times\boldsymbol{u}, \tag{4.5}$$

$$\boldsymbol{J} = \boldsymbol{r}\times\frac{\hbar}{i}\nabla + \boldsymbol{S}, \tag{4.6}$$

where as we shall see, the angular momentum operator $\boldsymbol{S}$ with respect to the center of charge, represents Dirac's spin operator and is given by the differential operator

$$\boldsymbol{S} = \boldsymbol{u}\times\frac{\hbar}{i}\nabla_u + \frac{\hbar}{2i}\left\{\nabla_\rho + \boldsymbol{\rho}\times\nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho}\cdot\nabla_\rho)\right\} = \boldsymbol{u}\times\boldsymbol{U} + \boldsymbol{W}, \tag{4.7}$$

and where the differential operators ∇_u and ∇_ρ are the corresponding gradient operators with respect to the $\boldsymbol{u}$ and $\boldsymbol{\rho}$ variables as in the Galilei case. The operator $\boldsymbol{S}$, satisfies $d\boldsymbol{S}/dt = \boldsymbol{P}\times\boldsymbol{u}$, and is not a constant of the motion even for the free particle.

To obtain the complete commuting set of observables we start with the Casimir invariant operator, or Klein-Gordon operator

$$H^2 - c^2\boldsymbol{P}^2 = m^2c^4. \tag{4.8}$$

In the above representation, H and $\boldsymbol{P}$ only differentiate the wave function with respect to time t and position $\boldsymbol{r}$, respectively. Since the spin operator $\boldsymbol{S}$ operates only on the velocity and orientation variables, it commutes with the Klein-Gordon operator (4.8). Thus, we can find simultaneous eigenfunctions of the three operators (4.8), S^2 and S_3. This allows us to try solutions in separate variables so that the wave function can be written as

$$\Phi(t,\boldsymbol{r},\boldsymbol{u},\boldsymbol{\rho}) = \sum_i \psi_i(t,\boldsymbol{r})\chi_i(\boldsymbol{u},\boldsymbol{\rho}), \tag{4.9}$$

where $\psi_i(t,\boldsymbol{r})$ are the space-time components and the $\chi_i(\boldsymbol{u},\boldsymbol{\rho})$ represent the internal spin structure. Consequently

$$(H^2 - c^2\boldsymbol{P}^2 - m^2c^4)\,\psi_i(t,\boldsymbol{r}) = 0, \tag{4.10}$$

i.e., space-time components satisfy the Klein-Gordon equation, while the internal structure part satisfies

$$S^2\chi_i(\boldsymbol{u},\boldsymbol{\rho}) = s(s+1)\hbar^2\chi_i(\boldsymbol{u},\boldsymbol{\rho}), \tag{4.11}$$

$$S_3\chi_i(\boldsymbol{u},\boldsymbol{\rho}) = m_s\hbar\chi_i(\boldsymbol{u},\boldsymbol{\rho}). \tag{4.12}$$

Eigenfunctions of the above type have been found in Section 3.5. In particular we are interested in solutions that give rise to spin 1/2 particles. These solutions, which are also eigenvectors of the spin projection on the body axis T_3, become a four-component wave function.

For spin 1/2 particles, if we take first for simplicity eigenfunctions $\chi(\boldsymbol{\rho})$ of S^2 with eigenvalue 1/2, then since the total spin has to be 1/2, the orbital zitterbewegung part $\boldsymbol{Z} = \boldsymbol{u}\times\boldsymbol{U}$ can only contribute with spherical harmonics of value $z = 0$ and $z = 1$. This means that we can find at least two different kinds of elementary particles of spin 1/2, one characterized by the singlet $z = 0$ (lepton?) and another by $z = 1$ (quark?) in three possible states according to the component z_3. If we call to the spin part Z the **colour**, we can have colourless and coloured systems of spin 1/2. The three different colours z_3 are unobservable because the Ψ_i states (3.87-3.90) are eigenstates of S_3 and T_3 but not eigenstates of Z_3. Nevertheless this interpretation of this spin part $\boldsymbol{Z}$ as representing the colour, as in the standard model, is still unclear and will be discussed elsewhere.

For $z = 0$, the spin 1/2 functions $\chi_i(\boldsymbol{\rho})$ are linear combinations of the four Φ_i functions (3.75)-(3.78) and in the case $z = 1$ they are linear combinations of the four Ψ_i of (3.87)-(3.90), such that the factor function in front of the spherical harmonics is 1 because for this model $u = c$ is a constant. It turns out that the Hilbert space that describes the internal structure of a Dirac[1], particle is isomorphic to the four-dimensional Hilbert space $\mathbb{C}^4$.

If we have two arbitrary directions in space characterized by the unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively, and $S_{\boldsymbol{u}}$ and $S_{\boldsymbol{v}}$ are the corresponding angular momentum projections $S_{\boldsymbol{u}} = \boldsymbol{u}\cdot\boldsymbol{S}$ and $S_{\boldsymbol{v}} = \boldsymbol{v}\cdot\boldsymbol{S}$, then $S_{-\boldsymbol{u}} = -S_{\boldsymbol{u}}$, and $[S_{\boldsymbol{u}}, S_{\boldsymbol{v}}] = i\hbar S_{\boldsymbol{u}\times\boldsymbol{v}}$. In the case of the opposite sign commutation relations of operators T_i, we have for instance for the spin projections $[T_1, T_2] = -i\hbar T_3$, thus suggesting that $\boldsymbol{e}_1\times\boldsymbol{e}_2 = -\boldsymbol{e}_3$, and any cyclic permutation $1\to 2\to 3$, and thus $\boldsymbol{e}_i$ vectors linked to the body, not only have as eigenvalues $\pm 1/3$, but also behave in the quantum case as a left-handed system. In this case $\boldsymbol{e}_i$ vectors are not arbitrary vectors in space, but rather vectors linked to the rotating body and thus they are not compatible observables, so that any measurement to determine, say the components of $\boldsymbol{e}_i$, will produce some interaction with the body that will mask the measurement of the others. We shall use this interpretation of a left-handed system for particles later, when we analyse the chirality in section 4.2.8. For antiparticles it will behave as a right handed one.

Operators S_i and T_i have the matrix representation obtained before in the two possible basic states, either (3.75)-(3.78) or in (3.87)-(3.90), which is just

$$\boldsymbol{S}\equiv\boldsymbol{W} = \frac{\hbar}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}, \tag{4.13}$$

$$T_1 = \frac{\hbar}{2}\begin{pmatrix}0 & \mathbb{I}\\ \mathbb{I} & 0\end{pmatrix},\quad T_2 = \frac{\hbar}{2}\begin{pmatrix}0 & i\mathbb{I}\\ -i\mathbb{I} & 0\end{pmatrix},\quad T_3 = \frac{\hbar}{2}\begin{pmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{pmatrix}, \tag{4.14}$$

where we represent by $\boldsymbol{\sigma}$ the three Pauli matrices and $\mathbb{I}$ is the 2×2 unit matrix.

[1] **Paul Adrien Maurice Dirac** Born the 8-th August 1902 in Bristol, Gloucestershire, England and dies the 20-th October 1984 in Tallahassee, Florida, USA. He graduates as an electric engenering in Bristol and afterwards derives into mathematics at the St John's College of Cambridge. His articles of 1925 and the two of 1928 about the structure of the electron, and the equation which bears his name and the subsequent publication of the book *The principles of Quantum Mechanics* in 1930 awarded him the Nobel Prize in 1933. He is considered as one of the founders of the quantum formalism.

Similarly, the matrix elements of the nine components of the unit vectors $(\boldsymbol{e}_i)_j$, $i,j=1,2,3$ give rise to the two alternative sets of representations depending on whether the zitterbewegung contribution is $z=0$ or $z=1$. In the first case we get

$$\boldsymbol{e}_1=\frac{1}{3}\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\ \boldsymbol{e}_2=\frac{1}{3}\begin{pmatrix}0 & i\boldsymbol{\sigma}\\ -i\boldsymbol{\sigma} & 0\end{pmatrix},\ \boldsymbol{e}_3=\frac{1}{3}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & -\boldsymbol{\sigma}\end{pmatrix}, \tag{4.15}$$

while in the $z=1$ case the representation is

$$\boldsymbol{e}_1=-\frac{1}{9}\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\ \boldsymbol{e}_2=-\frac{1}{9}\begin{pmatrix}0 & i\boldsymbol{\sigma}\\ -i\boldsymbol{\sigma} & 0\end{pmatrix},\ \boldsymbol{e}_3=-\frac{1}{9}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & -\boldsymbol{\sigma}\end{pmatrix}. \tag{4.16}$$

It must be remarked that the different components of the observables $\boldsymbol{e}_i$ are not compatible in general, because they are represented by non-commuting operators.

We finally write the wave function for spin 1/2 particles in the following form for $z=0$

$$\Phi_{(0)}(t,\boldsymbol{r},\boldsymbol{u},\boldsymbol{\alpha})=\sum_{i=1}^{i=4}\psi_i(t,\boldsymbol{r})\Phi_i(\alpha,\theta,\phi), \tag{4.17}$$

independent of the $\boldsymbol{u}$ variables, and in the case $z=1$ by

$$\Phi_{(1)}(t,\boldsymbol{r},\boldsymbol{u},\boldsymbol{\alpha})=\sum_{i=1}^{i=4}\psi_i(t,\boldsymbol{r})\Psi_i(\beta,\lambda;\alpha,\theta,\phi). \tag{4.18}$$

where β and λ represent the direction of vector $\boldsymbol{u}$. Then, once the Φ_i or Ψ_j functions that describe the internal structure (given in the appendix in (4.93-4.96) or ((4.97-4.100)), respectively) are identified with the four orthogonal unit vectors of the internal Hilbert space $\mathbb{C}^4$, the wave function becomes a four-component space-time wave function, and the six spin components S_i and T_j and the nine vector components $(\boldsymbol{e}_i)_j$, together the 4×4 unit matrix, completely exhaust this 16 linearly independent 4×4 hermitian matrices. They form a vector basis of Dirac's algebra, such that any other translation invariant internal observable that describes internal structure, for instance internal velocity and acceleration, angular velocity, etc., must necessarily be expressed as a real linear combination of the mentioned 16 hermitian matrices. We shall see in Sec. 4.3 that the internal orientation completely characterizes its internal structure.

The velocity operator in the basis Ψ_i will be calculated in terms of its components in polar spherical coordinates

$$u_1=c\sin\beta\cos\lambda,\quad u_2=c\sin\beta\sin\lambda,\quad u_3=c\cos\beta.$$

Its matrix representation in this basis vanishes because these vectors are eigenvectors of the operators S^2, S_3 and T_3 and in these states the expectation value of the velocity operator is zero with a great uncertainty.

The spin operator with respect to the center of charge $\boldsymbol{S}=\boldsymbol{u}\times\boldsymbol{U}+\boldsymbol{W}$ which, as seen in (3.91) and (4.13), coincides with the usual matrix representation of Dirac's spin operator.

4.2.1 Dirac operator

If we consider the expression of the kinematical momentum for free $u=c$ particles (2.158)

$$\boldsymbol{K}=\frac{H}{c^2}\boldsymbol{r}-t\boldsymbol{P}-\frac{1}{c^2}\boldsymbol{S}\times\boldsymbol{u}.$$

Taking the time derivative of this expression followed by the scalar product with $\boldsymbol{u}$, it leads to the Poincaré invariant operator **(Dirac operator)**:

$$H-\boldsymbol{P}\cdot\boldsymbol{u}-\frac{1}{c^2}\left(\frac{d\boldsymbol{u}}{dt}\times\boldsymbol{u}\right)\cdot\boldsymbol{S}=0. \tag{4.19}$$

When Dirac operator acts on a general wave function $\Phi_{(0)}$ or $\Phi_{(1)}$, we know that H and $\boldsymbol{P}$ have the differential representation given by (4.5) and the spin by the differential representation (4.7), or the equivalent matrix representation (4.13), but we do not know how to represent the action of the velocity $\boldsymbol{u}$ and the $(d\boldsymbol{u}/dt)\times\boldsymbol{u}$ observable. However, we know that for this particle $\boldsymbol{u}$ and $d\boldsymbol{u}/dt$ are orthogonal vectors and together with vector $\boldsymbol{u}\times d\boldsymbol{u}/dt$ they form an orthogonal right-handed system, and in the center of mass frame the particle describes a circle of radius $R_0 = \hbar/2mc$ in the plane spanned by $\boldsymbol{u}$ and $d\boldsymbol{u}/dt$.

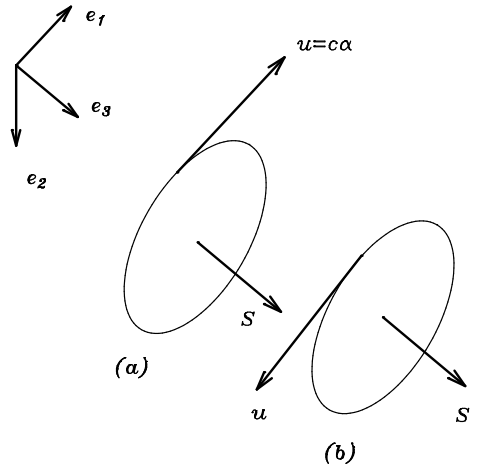


Figure 4.2: **Representation of the local body frame and the different observables for the (a) $H > 0$ solution and (b) $H < 0$ solution. This orientation produces Dirac equation in the Pauli-Dirac representation**

4.2.2 Pauli-Dirac representation

Let us consider first the case $z = 0$. Since $\boldsymbol{u}$ and $d\boldsymbol{u}/dt$ are translation invariant observables they will be elements of Dirac's algebra, and it turns out that we can relate these three vectors with the left-handed orthogonal system formed by vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ with representation (4.15). Then, as shown in part (a) of Figure 4.2 for the $H > 0$ system, we have $\boldsymbol{u} = a\boldsymbol{e}_1$ and $d\boldsymbol{u}/dt \times \boldsymbol{u} = b\boldsymbol{e}_3$, where a and b are constant positive real numbers. Then the third term in Dirac operator is $(b/c^2)\boldsymbol{e}_3 \cdot \boldsymbol{S} = (b/c^2)T_3$, and (4.19) operator becomes

$$H - a\boldsymbol{P}\cdot\boldsymbol{e}_1 - \frac{b}{c^2}T_3 = 0. \tag{4.20}$$

If we make the identification with the $H < 0$ solution of part (b) of Figure 4.2, the relation of the above observables is opposite to the previous one but now with the coefficients $-a$ and $-b$, respectively, *i.e.*, we get

$$H + a\boldsymbol{P}\cdot\boldsymbol{e}_1 + \frac{b}{c^2}T_3 = 0, \tag{4.21}$$

which clearly corresponds to the change $H \to -H$ in equation (4.20). Explicitely eq.(4.20) looks like:

$$H - \frac{a}{3}\begin{pmatrix} 0 & \boldsymbol{P}\cdot\boldsymbol{\sigma} \\ \boldsymbol{P}\cdot\boldsymbol{\sigma} & 0 \end{pmatrix} - \frac{b\hbar}{2c^2}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} = 0, \quad \boldsymbol{P}\cdot\boldsymbol{\sigma} = \begin{pmatrix} P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_3 \end{pmatrix}.$$

Multiplying (4.21) by (4.20) we obtain an expression which is satisfied by both particle and antiparticle

$$H^2 - \frac{a^2}{9}\boldsymbol{P}^2\mathbb{I} - \frac{b^2\hbar^2}{4c^4}\mathbb{I} = 0, \tag{4.22}$$

and which is an algebraic relation between H^2 and P^2. By identification of this expression with the Klein-Gordon operator (4.8), which also contains both $H > 0$ and $H < 0$ solutions, leads to $a = 3c$ and $b = 2mc^4/\hbar = c^3/R_0$ and by substitution in (4.20) we obtain Dirac operator:

$$H - c\boldsymbol{P}\cdot\boldsymbol{\alpha} - \beta mc^2 = 0, \tag{4.23}$$

where Dirac's hermitian matrices $\boldsymbol{\alpha}$ and β are represented by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}. \tag{4.24}$$

Sometimes Dirac equation is written in another form. If in(4.23) we multiply both sides from the left by the matrix β, since $\beta^2 = \mathbb{I}$, we arrive to

$$\beta H - c\boldsymbol{P}\cdot\beta\boldsymbol{\alpha} - mc^2 = 0, \quad \gamma^0 P_0 - \gamma^i P^i - mc = 0, \quad \gamma^\mu P_\mu - mc = 0,$$

and the gamma matrices γ^μ are

$$\gamma^0 \equiv \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \boldsymbol{\gamma} \equiv \gamma^0\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}. \tag{4.25}$$

This is called the **Pauli-Dirac representation** where $3\boldsymbol{e}_1$ plays the role of a unit vector in the direction of the velocity. The matrix γ^0 is hermitian and the γ^i antihermitian.

If these coefficients a and b should be substituted in the equation corresponding to the antiparticle (4.21) this produces an equivalent representation which corresponds to the change $\beta \to -\beta$ and $\boldsymbol{\alpha} \to -\boldsymbol{\alpha}$. As we sall see in the section **4.2.8** when analyzing the chirality, we should have arrived to the same expression of Dirac equation than in (4.23) with the same matrices (4.24) if the considered model of the antiparticle (b) but with the opposite axis $\boldsymbol{e}_i$ to those of the figure 4.2, i.e., a right handed local system.

Dirac equation (4.23) represents the relationship between the mechanical temporal momentum or energy H, as the sum of two terms. One related to the motion of the center of mass $c\boldsymbol{P}\cdot\boldsymbol{\alpha}$, or energy of translation, and the term related to the spin, or rotation energy. This expression is valid for the free electron, but if the electron is under the interaction with an external electromagnetic field, this relationship has to be hold for the mechanical properties, according to the Atomic Principle. According to this principle the internal structure is not modified and therefore the total energy and linear momentum are $H = H_m + e\phi$ y $\boldsymbol{P} = \boldsymbol{P}_m + e\boldsymbol{A}$, where H_m and $\boldsymbol{P}_m$ are the mechanical observables which still satisfy (4.23), and thus for the total observables we get

$$H = e\phi + c\,(\boldsymbol{P} - e\boldsymbol{A})\cdot\boldsymbol{\alpha} + \beta mc^2, \tag{4.26}$$

where ϕ and $\boldsymbol{A}$ are the external scalar and vector potential, respectively.

When acting with Dirac operator on some arbitrary spinor, this equation looks in the free case as

$$(i\hbar[\gamma^\mu]_{ij}\partial_\mu - mc\,\delta_{ij})\,\psi_j(t,\boldsymbol{r}) = 0,$$

and when interacting with some external electromagnetic field

$$((i\hbar\partial_\mu - eA_\mu(t,\boldsymbol{r}))[\gamma^\mu]_{ij} - mc\,\delta_{ij})\,\psi_j(t,\boldsymbol{r}) = 0.$$

For the center of mass observer ($\boldsymbol{P} = 0$), and in (4.23) the operator H is reduced to

$$H = \beta mc^2 = mc^2 \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}.$$

Therefore, those spinors ψ with nonvanishing upper components ψ_1 and ψ_2 are eigenvectors of H with eigenvalue $+mc^2$ while those with nonvanishing components ψ_3 and ψ_4 are eigenvectors of H with eigenvalue $-mc^2$.

We can also obtain Dirac equation when the zitterbewegung part of the spin is $z = 1$, by using the set of matrices (4.16) instead of the set (4.15), because they are multiples of each other and the only difference are some constant intermediate factors. Explicitely, the equation equivalent to (4.20) is now

$$H + \frac{a}{9}\begin{pmatrix} 0 & \boldsymbol{P}\cdot\boldsymbol{\sigma} \\ \boldsymbol{P}\cdot\boldsymbol{\sigma} & 0 \end{pmatrix} + \frac{b\hbar}{6c^2}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} = 0, \quad \boldsymbol{P}\cdot\boldsymbol{\sigma} = \begin{pmatrix} P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_3 \end{pmatrix}.$$

By multiplication by itself with H replaced by $-H$, we obtain, like in the previous case, after identification with the Klein-Gordon equation

$$H^2 - \frac{a^2}{81}\boldsymbol{P}^2\mathbb{I} - \frac{b^2\hbar^2}{36c^4}\mathbb{I} = 0, \quad a = 9c, \quad b = 6\frac{mc^4}{\hbar},$$

i.e., like in (4.23) with the same matrices $\boldsymbol{\alpha}$ and β:

$$H - c\boldsymbol{P}\cdot\boldsymbol{\alpha} - \beta mc^2 = 0.$$

The four basic states Φ_i, (4.93)-(4.96) or (4.97)-(4.100) in the Pauli-Dirac representation, are

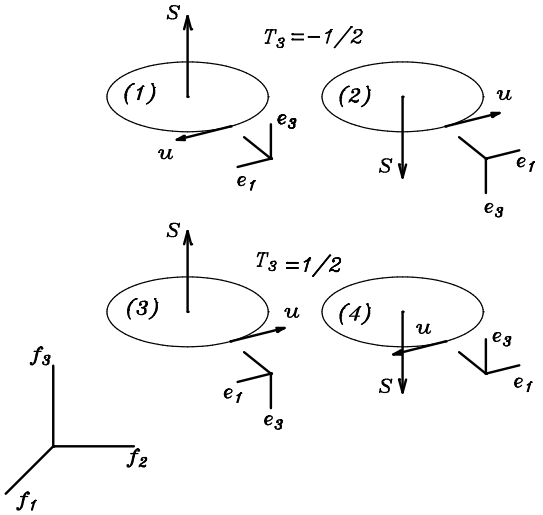


Figure 4.3: **Orientation of the body axis in Pauli-Dirac representation, corresponding to the four basic states Φ_i of (4.93)-(4.96), respectively, with the vector e_2 along the direction of the acceleration and e_1 and $-e_1$ along the direction of the velocity. The local system of axis in the cases (1) and (2) is left-handed, while it is right-handed in (3) and (4), which correspond to the antiparticle.**

related to the four states represented in the figure **4.3**, where the $\boldsymbol{f}_i$ are the unit vectors in the laboratory reference frame. The vectors $\boldsymbol{e}_i$ are the unit vectors linked to the point $\boldsymbol{r}$, with the vector $\boldsymbol{e}_2$ in the direction of the acceleration. In the states (1) and (2) the projection of the spin along the axis $\boldsymbol{e}_3$, T_3 is $+1/2$, while in the two lower states (3) and (4) correspond to $T_3 = -1/2$, and both with the two possibilities of $S_3 = \pm 1/2$ in the laboratory axis. It must be remarked that the two lower states are states of the antiparticle.

Weyl Representation

The possible identification of the internal observables with the different linear combinations of the hermitian matrices $\boldsymbol{e}_i$ leads to different equivalent representations of Dirac matrices, and therfore to different, but equivalent, expressions of Dirac equation.

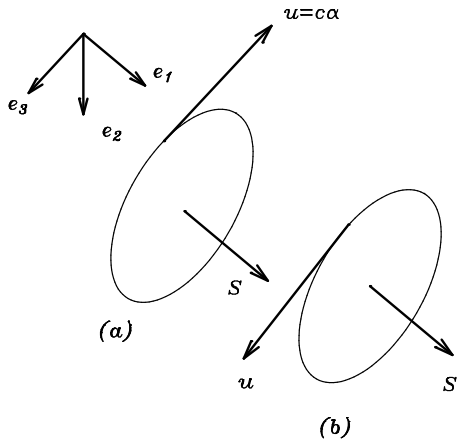


Figure 4.4: **Orientation of the local axis in the Weyl representation.**

For example, if we make the identification suggested in the Figure 4.4 for the particle, the velocity $\boldsymbol{u} = -a\boldsymbol{e}_3$ and the observable acceleration $d\boldsymbol{u}/dt \times \boldsymbol{u} = b\boldsymbol{e}_1$ in terms of the real positive constants a and b, we get by the same method the hermitian matrices

$$\beta_W = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \sim T_1, \qquad \boldsymbol{\alpha}_W = \begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \sim -\boldsymbol{e}_3, \tag{4.27}$$

and thus the corresponding gamma matrices are

$$\gamma^0_W \equiv \beta = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma}_W \equiv \gamma^0\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma^5_W = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}. \tag{4.28}$$

This is called the **Weyl representation** of Dirac equation. The interest of this representation lies in the description of weak interactions, where fermions violate the parity conservation. Weak interaction distinguishes the chyral part (Left) from the part (Right) defined by

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}\psi, \quad \psi_R = \frac{1}{2}(1+\gamma^5)\psi = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}\psi, \quad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

and now the upper components of the spinor are Left while the lower are Right. In this representation, the generators of the Lorentz group are

$$\boldsymbol{J} = \frac{i}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \boldsymbol{K} = \frac{1}{2}\begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

and the Lorentz group representation acting on the spinors is

$$D(\Lambda) = \begin{pmatrix} D^{(1/2,0)}(\Lambda) & 0 \\ 0 & D^{(0,1/2)}(\Lambda) \end{pmatrix}$$

and therefore upper and lower components transform independently of each other with the two, non-equivalent, 2×2 irreducible representations of $SL(2,\mathbb{C})$.

When we compare with the previous representation we see that Weyl representation is obtained from Pauli-Dirac representation if we rotate the local axis by means of a rotation of value $\pi/2$ around the axis $\boldsymbol{e}_2$. In this way the velocity operator $c\boldsymbol{\alpha}$ has the opposite direction to the vector $\boldsymbol{e}_3$. Therfore, the rotation operator is

$$R(\pi/2, \boldsymbol{e}_2) = \exp(\frac{i}{\hbar}\frac{\pi}{2}\boldsymbol{e}_2 \cdot \boldsymbol{W}) = \exp(\frac{i}{\hbar}\frac{\pi}{2}T_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{I} & -\mathbb{I} \\ \mathbb{I} & \mathbb{I} \end{pmatrix}.$$

We can check that $R\gamma^\mu_{PD}R^\dagger = \gamma^\mu_W$, where γ^μ_{PD} and γ^μ_W are, respectively, the gamma matrices in the Pauli-Dirac and Weyl representations.

Supersymmetric Representation

It is that representation where Dirac matrices are

$$\beta_S = \begin{pmatrix} 0 & -i\mathbb{I} \\ i\mathbb{I} & 0 \end{pmatrix} \sim -T_2, \quad \boldsymbol{\alpha}_S = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \sim \boldsymbol{e}_1. \tag{4.29}$$

and the gamma matrices γ^μ are

$$\gamma^0_S = \begin{pmatrix} 0 & -i\mathbb{I} \\ i\mathbb{I} & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}_S = \begin{pmatrix} -i\boldsymbol{\sigma} & 0 \\ 0 & i\boldsymbol{\sigma} \end{pmatrix}, \quad \gamma^5_S = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \tag{4.30}$$

It is related to the Pauli-Dirac representation by means of the unitary transformation $T_S\gamma^\mu_{PD}T^\dagger_S = \gamma^\mu_S$. This transformation corresponds to a rotation of angle $\pi/2$ around the axis $\boldsymbol{e}_1$, in the Pauli-Dirac representation

$$T_S = R(\pi/2, \boldsymbol{e}_1) = \exp(\frac{i}{\hbar}\frac{\pi}{2}\boldsymbol{e}_1 \cdot \boldsymbol{W}) = \exp(\frac{i}{\hbar}\frac{\pi}{2}T_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{I} & i\mathbb{I} \\ i\mathbb{I} & \mathbb{I} \end{pmatrix}.$$

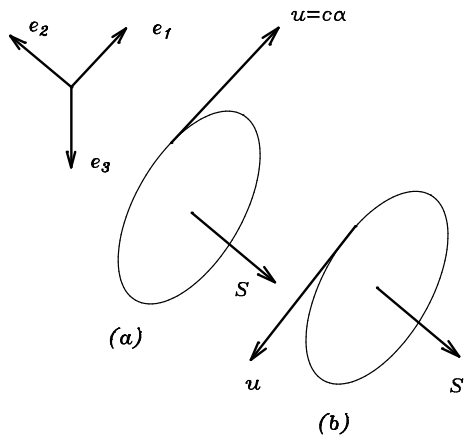


Figure 4.5: **Orientation of the local axis in the Supersymmetric representation.**

Majorana Representation

If we write Dirac equation as a system of partial differential equations we get

$$(i\hbar\gamma^\mu\partial_\mu - mc)\,\psi_i(t, \boldsymbol{r}) = 0.$$

It is a linear differential operator with constant coefficients γ^μ which are complex 4×4 matrices, acting on a four component vector, with components which are complex functions of t and $\boldsymbol{r}$. If we look for a representation where the γ^μ matrices have all its components pure imaginary complex numbers, then the above system of partial differential equations all coefficients are real constants and therefore we can try to find solutions where the four unknown functions ψ_i can be chosen as real functions. This is called the **Majorana representation**. In this representation, the hermitian matrices are

$$\beta_M = \begin{pmatrix} 0 & i\mathbb{I} \\ -i\mathbb{I} & 0 \end{pmatrix} \sim T_2, \quad \alpha_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \tag{4.31}$$

and therefore the gamma matrices are pure imaginary

$$\gamma^0_M \equiv \beta = \begin{pmatrix} 0 & i\mathbb{I} \\ -i\mathbb{I} & 0 \end{pmatrix}, \quad \gamma^1_M = \begin{pmatrix} 0 & -i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix}, \quad \gamma^2_M = \begin{pmatrix} i\mathbb{I} & 0 \\ 0 & -i\mathbb{I} \end{pmatrix}, \quad \gamma^3_M = \begin{pmatrix} 0 & -i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix}. \tag{4.32}$$

We can transform Pauli-Dirac representation to Majorana representation by means of the unitary transformation

$$T_M = \frac{1}{2}\begin{pmatrix} \mathbb{I}+i\sigma_2 & \mathbb{I}-i\sigma_2 \\ -i\mathbb{I}+\sigma_2 & i\mathbb{I}+\sigma_2 \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ -i & -i & i & -i \\ i & -i & i & i \end{pmatrix}$$

in the form $T_M\gamma^\mu_{PD}T^\dagger_M = \gamma^\mu_M$.

For the center of mass observer where $H = \beta mc^2$, the spinors of the form

$$\Psi_1 = \begin{pmatrix} i\psi \\ 0 \\ \psi \\ 0 \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} 0 \\ i\psi \\ 0 \\ \psi \end{pmatrix}, \quad H\Psi_i = mc^2\Psi_i, \quad i = 1,2$$

are states of $H > 0$, while the following linearly independent spinors

$$\Psi_3 = \begin{pmatrix} -i\psi \\ 0 \\ \psi \\ 0 \end{pmatrix}, \quad \Psi_4 = \begin{pmatrix} 0 \\ -i\psi \\ 0 \\ \psi \end{pmatrix}, \quad H\Psi_i = -mc^2\Psi_i, \quad i = 3,4$$

are states of $H < 0$.

In the literature we can find another Majorana representation where gamma matrices, which are also pure imaginary, are:

$$\gamma^0_M \equiv \beta = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^1_M = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \quad \gamma^2_M = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^3_M = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}.$$

As a summary, Pauli-Dirac representation leads to the hermitian matrix β to be proportional to the matrix $T_3 = \boldsymbol{e}_3 \cdot \boldsymbol{W}$, and hermitian matrices $\boldsymbol{\alpha}$ to $\boldsymbol{e}_1$. Weyl representation is that representation where matrix β es proportional to $T_1 = \boldsymbol{e}_1 \cdot \boldsymbol{W}$ and the matrices $\boldsymbol{\alpha}$ to $-\boldsymbol{e}_3$. In the supersymmetric representation the β is proporcional to $-T_2 = -\boldsymbol{e}_2 \cdot \boldsymbol{W}$, and the matrices $\boldsymbol{\alpha}$ to $\boldsymbol{e}_1$. Majorana representation is a unitary transformation which combines particle and antiparticle states evenly.

4.2.3 Gauge Transformation

Let us consider a change of the phase of the part of the wave function corresponding to the particle but not to the antiparticle, in the Pauli-Dirac representation. Acting on the Dirac spinor we have the unitary transformation,

$$G\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_3\end{pmatrix}=\begin{pmatrix}e^{i\alpha}\psi_1\\ e^{i\alpha}\psi_2\\ \psi_3\\ \psi_3\end{pmatrix},\quad G=\begin{pmatrix}e^{i\alpha}\mathbb{I} & 0\\ 0 & \mathbb{I}\end{pmatrix},\quad G^{-1}=G^{\dagger}=\begin{pmatrix}e^{-i\alpha}\mathbb{I} & 0\\ 0 & \mathbb{I}\end{pmatrix},$$

which affects only to the upper components of the wave function. In the Pauli-Dirac represen-

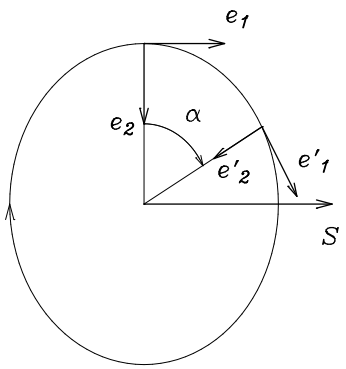


Figure 4.6: **The change of phase α of the particle corresponds to a rotation of value α of the center of charge in the direction of its internal motion. The axis vector $\boldsymbol{e}_3$ and the spin $\boldsymbol{S}$ are not affected by this transformation.**

tation this means that when acting on the basic observables with this unitary transformation we get,

$$G\boldsymbol{e}_1G^{\dagger}=\begin{pmatrix}0 & e^{i\alpha}\boldsymbol{\sigma}\\ e^{-i\alpha}\boldsymbol{\sigma} & 0\end{pmatrix}=\begin{pmatrix}0 & (\cos\alpha+i\sin\alpha)\boldsymbol{\sigma}\\ (\cos\alpha-i\sin\alpha)\boldsymbol{\sigma} & 0\end{pmatrix}=\cos\alpha\boldsymbol{e}_1+\sin\alpha\boldsymbol{e}_2,$$

$$G\boldsymbol{e}_2G^{\dagger}=\begin{pmatrix}0 & ie^{i\alpha}\boldsymbol{\sigma}\\ -ie^{-i\alpha}\boldsymbol{\sigma} & 0\end{pmatrix}=\begin{pmatrix}0 & (-\sin\alpha+i\cos\alpha)\boldsymbol{\sigma}\\ (-\sin\alpha-i\cos\alpha)\boldsymbol{\sigma} & 0\end{pmatrix}=$$

$$=-\sin\alpha\boldsymbol{e}_1+\cos\alpha\boldsymbol{e}_2,$$

$$G\boldsymbol{e}_3G^{\dagger}=\boldsymbol{e}_3,\quad G\boldsymbol{S}G^{\dagger}=\boldsymbol{S},$$

$$GT_1G^{\dagger}=\cos\alpha T_1+\sin\alpha T_2,\quad GT_2G^{\dagger}=-\sin\alpha T_1+\cos\alpha T_2,\quad GT_3G^{\dagger}=T_3.$$

This transformation corresponds in the figure **4.6** to the displacement of an angle α of the position of the center of charge of the particle. If the change of phase would affect to only one of the components of the Dirac spinor, this would represent a rotation of the center of charge of the corresponding basic vector of the figure **4.3**.

4.2.4 Dynamics of observables

In the Heisenberg representation, the time derivative of any observable A is

$$\frac{dA}{dt} = \frac{i}{\hbar}[H, A] + \frac{\partial A}{\partial t}. \tag{4.33}$$

The wave function depends on the kinematical variables. Among them we find the time variable. The time evolution of the particle corresponds to a time translation generated by the Hamiltonian H, in such a way that $\psi(t, x_1, \ldots, x_n) = \exp(-iHt/\hbar)\psi(0, x_1, \ldots, x_n)$. The expectation value, at instant t of an observable A, when the system is on the state $\psi(t)$ is

$$< A(t) >=< \psi(t)|A|\psi(t) >=< \psi(0)|e^{iHt/\hbar}\, A\, e^{-iHt/\hbar}|\psi(0) > .$$

This amounts to take the expectation value of a different operator, $e^{iHt/\hbar}\, A\, e^{-iHt/\hbar}$ but on the state at the initial time $t = 0$. At the instant $t + \Delta t$

$$< A(t+\Delta t) >=< \psi(t+\Delta t)|A|\psi(t+\Delta t) >=< \psi(0)|e^{iH(t+\Delta t)/\hbar}\, A\, e^{-iH(t+\Delta t)/\hbar}|\psi(0) > .$$

and the difference

$$< A(t+\Delta t) > - < A(t) >= \frac{i}{\hbar} < \psi(0)|HA - AH|\psi(0) > \Delta t,$$

and therefore

$$\frac{d < A(t) >}{dt} = \frac{i}{\hbar} < \psi(0)|[H, A]|\psi(0) > .$$

In this way, the calculation of the time variation of an expectation value is equivalent to the expectation value of the observable $[H, A]$ with respect to the initial state $\psi(0)$. It is easy to see that if the observable A is explicitely time dependent, then the observable we have to consider is the one defined in (4.33), with respect to the initial state $\psi(0)$ of the system.

We can see that for Dirac, the point $\boldsymbol{r}$ is moving at the speed c. In fact, the commutator $[H, \boldsymbol{r}]$ is different from zero because $[P_i, x_j] = -i\hbar\delta_{ij}$, and therefore, the velocity of the point $\boldsymbol{r}$ is

$$\boldsymbol{u} = \frac{d\boldsymbol{r}}{dt} = \frac{i}{\hbar}[H, \boldsymbol{r}] = \frac{i}{\hbar}[c\,(\boldsymbol{P} - e\boldsymbol{A}) \cdot \boldsymbol{\alpha}, \boldsymbol{r}] = c\boldsymbol{\alpha},$$

even under any electromagnetic interaction (4.26). The eigenvalues of the matrices α_i are ± 1, and therefore if any component of the velocity vector is measured without dispersion, only can take the values $\pm c$.

The Pauli-Dirac representation is compatible with the acceleration $d\boldsymbol{u}/dt$ lying along the vector $\boldsymbol{e}_2$. In fact, in the center of mass frame and in the Heisenberg representation, Dirac's Hamiltonian reduces to $H = \beta mc^2$, and the time derivative of the velocity observable $\boldsymbol{u} = c\boldsymbol{\alpha}$ is

$$\frac{d\boldsymbol{u}}{dt} = \frac{i}{\hbar}[mc^2\beta, c\boldsymbol{\alpha}] = \frac{2mc^3}{\hbar}\begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix} = \frac{c^2}{R_0} 3\boldsymbol{e}_2, \tag{4.34}$$

c^2/R_0 being the constant modulus of the acceleration in this frame, and where $3\boldsymbol{e}_2$ plays the role of a unit vector along that direction.

The time derivative of this Cartesian system is

$$\frac{d\boldsymbol{e}_1}{dt} = \frac{i}{\hbar}[\beta mc^2, \boldsymbol{e}_1] = \frac{c}{R_0}\boldsymbol{e}_2, \tag{4.35}$$

$$\frac{d\boldsymbol{e}_2}{dt} = \frac{i}{\hbar}[\beta mc^2, \boldsymbol{e}_2] = -\frac{c}{R_0}\boldsymbol{e}_1, \tag{4.36}$$

$$\frac{d\boldsymbol{e}_3}{dt} = \frac{i}{\hbar}[\beta mc^2, \boldsymbol{e}_3] = 0, \tag{4.37}$$

since $\boldsymbol{e}_3$ is orthogonal to the trajectory plane and does not change, and where $c/R_0 = \omega$ is the angular velocity of the internal orbital motion. This time evolution of the observables $\boldsymbol{e}_i$ is the correct one if assumed to be a rotating left-handed system of vectors as shown in Figure 4.2-(a).

It is for this reason that we considered at the beginning of this chapter that the body frame rotates with the same angular velocity as the orbital motion of the charge.

We should arrive to the same conclusion if we use the Weyl representation, where Dirac matrices are given in (4.27). In fact, in this representation

$$\frac{d\boldsymbol{e}_1}{dt} = \frac{i}{\hbar}[\beta_W mc^2, \boldsymbol{e}_1] = 0, \tag{4.38}$$

$$\frac{d\boldsymbol{e}_2}{dt} = \frac{i}{\hbar}[\beta_W mc^2, \boldsymbol{e}_2] = \frac{c}{R_0}\boldsymbol{e}_3, \tag{4.39}$$

$$\frac{d\boldsymbol{e}_3}{dt} = \frac{i}{\hbar}[\beta_W mc^2, \boldsymbol{e}_3] = -\frac{c}{R_0}\boldsymbol{e}_2. \tag{4.40}$$

To be consistent with the above consideration as $3\boldsymbol{e}_i$ as unit vectors, this means that the spin in the center of mass frame should be along $3\boldsymbol{e}_3$. This is the case for the upper components while for the lower components (which in this representation correspond to $H < 0$ states) the orientation is the opposite. This means that for particles the corresponding set of axis forms a left handed system while for antiparticles they behave as a right handed system, showing a clear chirality difference between particles and antiparticles.

In general

$$\frac{d\boldsymbol{S}}{dt} = \frac{i}{\hbar}[H, \boldsymbol{S}] = \frac{i}{\hbar}[c\boldsymbol{P}\cdot\boldsymbol{\alpha} + \beta mc^2, \boldsymbol{S}] = c\boldsymbol{P}\times\boldsymbol{\alpha} \equiv \boldsymbol{P}\times\boldsymbol{u},$$

is not a constant of the motion, but for the center of mass observer, this spin operator $\boldsymbol{u}\times\boldsymbol{U}+\boldsymbol{W}$ is the same with respect to any point and is constant in this frame:

$$\frac{d\boldsymbol{S}}{dt} = \frac{i}{\hbar}[\beta mc^2, \boldsymbol{S}] = 0. \tag{4.41}$$

Only the T_3 spin component on the body axis remains constant while the other two T_1 and T_2 change because of the rotation of the corresponding axis,

$$\frac{dT_1}{dt} = \frac{i}{\hbar}[\beta mc^2, T_1] = \frac{c}{R_0}T_2, \tag{4.42}$$

$$\frac{dT_2}{dt} = \frac{i}{\hbar}[\beta mc^2, T_2] = -\frac{c}{R_0}T_1, \tag{4.43}$$

$$\frac{dT_3}{dt} = \frac{i}{\hbar}[\beta mc^2, T_3] = 0. \tag{4.44}$$

Nevertheless it is the component T_1 which remains constant in the Weyl representation

$$\frac{dT_1}{dt} = \frac{i}{\hbar}[\beta_W mc^2, T_1] = 0, \tag{4.45}$$

$$\frac{dT_2}{dt} = \frac{i}{\hbar}[\beta_W mc^2, T_2] = \frac{c}{R_0}T_3, \tag{4.46}$$

$$\frac{dT_3}{dt} = \frac{i}{\hbar}[\beta_W mc^2, T_3] = -\frac{c}{R_0}T_2. \tag{4.47}$$

When analyzed from the point of view of an arbitrary observer, the classical motion is a helix and the acceleration is not of constant modulus c^2/R_0, and the spin operator $\boldsymbol{S}$ is no longer a constant of the motion, because it is the total angular momentum $\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{S}$ that is conserved.

4.2.5 Probability Conservation

Quantum mechanical Dirac equation is:

$$i\hbar\frac{\partial\Phi}{\partial t} - \frac{\hbar}{i}\boldsymbol{u}\cdot\nabla\Phi - mc^2\beta\Phi = 0, \tag{4.48}$$

where Φ is either the spinor (4.17) or the spinor (4.18) which is a four component spinor.

$$\Phi(t,\boldsymbol{r},\boldsymbol{u},\boldsymbol{\alpha}) = \begin{pmatrix}\psi_1(t,\boldsymbol{r})\\ \psi_2(t,\boldsymbol{r})\\ \psi_3(t,\boldsymbol{r})\\ \psi_4(t,\boldsymbol{r})\end{pmatrix} \tag{4.49}$$

once the dependence on the kinematical variables $\boldsymbol{u}$ and $\boldsymbol{\alpha}$ has been substituted by a linear combination in terms of the four basic spinors. The velocity operator $\boldsymbol{u} = c\boldsymbol{\alpha}$ is Dirac velocity operator written in terms of the hermitian 4×4 Dirac matrices $\boldsymbol{\alpha}$, and $\beta = \gamma^0$ is the hermitian 4×4 Dirac matrix, related to the spin projection along the body axis. Each spacetime function $\psi_i(t,\boldsymbol{r})$, $i = 1,2,3,4$, is solution of the Klein-Gordon equation, $(\partial^\mu\partial_\mu + m^2c^2/\hbar^2)\psi_i = 0$.

If we now take the complex conjugate and transpose of the above expression we get

$$-i\hbar\frac{\partial\Phi^*}{\partial t} + \frac{\hbar}{i}\nabla\Phi^*\cdot\boldsymbol{u} - mc^2\Phi^*\beta = 0, \tag{4.50}$$

where Φ^* represents the file spinor, complex conjugate and transpose of the spinor (4.49). Now the first equation (4.48) is multiplied on the left hand side by the row vector Φ^*, and the expression (4.50) by the column vector Φ on the right hand side, and substract the second from the first, we arrive to

$$\frac{\partial(\Phi^*\Phi)}{\partial t} + \nabla(\Phi^*\boldsymbol{u}\Phi) = 0.$$

If we call $\Phi^*\Phi = \sum\psi_i^*\psi_i = \rho(t,\boldsymbol{r})$, it is a scalar and definite positive function which can be interpreted as the probability density of presence of the electron and $\Phi^*\boldsymbol{u}\Phi = \boldsymbol{j}(t,\boldsymbol{r})$, as the current probability density, so that Dirac equation leads to the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0. \tag{4.51}$$

There exist a local conservation of the probability at any point $(t,\boldsymbol{r}) \in \mathbb{R}^4$, of spacetime. This conservation law implies that the integral at constant t, $\int_V \rho dV$ is conserved for any integration volume V. If this volume is the whole three-dimensional space, this integral is 1, as it corresponds to a normalized wave function.

The current density $\boldsymbol{j} = \Phi^*\boldsymbol{u}\Phi = c\psi^*\gamma^0\boldsymbol{\gamma}\psi$, and since $(\gamma^0)^2 = \mathbb{I}$, we can define the conjugate spinor, as the row vector $\bar\psi = \psi^*\gamma^0$, this allows us to write the continuity equation in a covariant form in terms of the four vector $j^\mu = c\bar\psi\gamma^\mu\psi \equiv (c\rho,\boldsymbol{j})$, as

$$\partial_\mu j^\mu = 0. \tag{4.52}$$

If the four vector j^μ is multiplied by the value of the charge e, we obtain the electric current density four vector, which also satisfies the same continuity equation. This reinforces the idea that the wave function, as a function of t and $\boldsymbol{r}$, what represents is how the charge is distributed around the point $\boldsymbol{r}$, which represents the location of the electric charge, as is assumed in the classical model.

Because the electric current density four vector is $j^\mu = ec\bar\psi\gamma^\mu\psi$, the interaction with an external field is written in the form of a minimal coupling, in terms of the external potentials, which are functions of $(t,\boldsymbol{r})$:

$$j^\mu(t,\boldsymbol{r})A_\mu(t,\boldsymbol{r}).$$

In quantum electrodynamics there is no other coupling than the coupling between the external fields and the particle current, which also reinforces the idea that, efectively, from the classical point of view what we have is the interaction of a charged point, the center of charge, with the external potentials and no further multipoles.

4.2.6 PCT Invariance

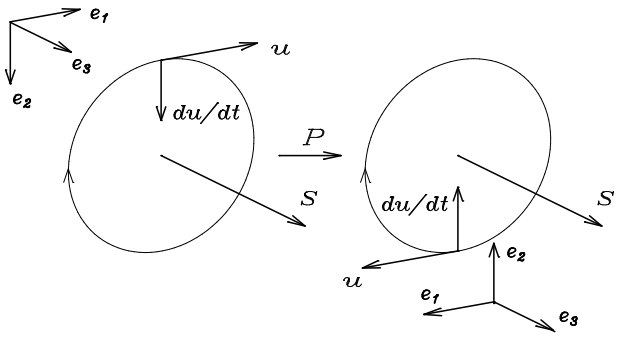


Figure 4.7: **Space reversal of the electron in the center of mass frame is equivalent to a rotation of value π along $\boldsymbol{S}$. The local system of axis remain left-handed because the particle (or antiparticle) character does not change.**

In the Figure **4.7** we represent the parity reversal P of the description of the electron as given by this model of luxon which is circling around the center of mass at the velocity c and in the center of mass frame it changes the variables according to

$$P:\{\boldsymbol{r}\to-\boldsymbol{r},\boldsymbol{u}\to-\boldsymbol{u},d\boldsymbol{u}/dt\to-d\boldsymbol{u}/dt,\boldsymbol{S}\to\boldsymbol{S},H\to H,\boldsymbol{e}_1\to-\boldsymbol{e}_1,\boldsymbol{e}_2\to-\boldsymbol{e}_2,\boldsymbol{e}_3\to\boldsymbol{e}_3,\}.$$

In the Pauli-Dirac representation as we see in Figure 4.2, this amounts to a rotation of value π around axis $\boldsymbol{e}_3$ and thus

$$P\equiv R(\pi,\boldsymbol{e}_3)=\exp(i\pi\boldsymbol{e}_3\cdot\boldsymbol{S}/\hbar)=\exp(i\pi T_3/\hbar)=i\gamma_0,$$

which is one of the possible representations of the parity operator $\pm\gamma_0$ or $\pm i\gamma_0$. In Weyl's representation this is a rotation of value π around $\boldsymbol{e}_1$ which gives again $P\equiv i\gamma_0$. Parity reversal does not afect to the character particle or antiparticle and also to the left-handed or righ-handed local system. We can check with $P\equiv i\gamma_0$, we get

$$PHP^\dagger=mc^2P\beta P^\dagger=H,\quad P\boldsymbol{S}P^\dagger=\boldsymbol{S},\quad P\boldsymbol{e}_1P^\dagger=-\boldsymbol{e}_1,\quad P\boldsymbol{e}_2P^\dagger=-\boldsymbol{e}_2,\quad P\boldsymbol{e}_3P^\dagger=\boldsymbol{e}_3.$$

In the Figure **4.8** we represent the time reversal operation T also in the center of mass frame

$$T:\{\boldsymbol{r}\to\boldsymbol{r},\boldsymbol{u}\to-\boldsymbol{u},d\boldsymbol{u}/dt\to d\boldsymbol{u}/dt,\boldsymbol{S}\to\boldsymbol{S},H\to-H,\boldsymbol{e}_1\to-\boldsymbol{e}_1,\boldsymbol{e}_2\to\boldsymbol{e}_2,\boldsymbol{e}_3\to\boldsymbol{e}_3\},$$

but this corresponds to a particle of $H<0$ such that the relative orientation of spin, velocity and position, given by equation (2.169) agrees with the motion depicted in this figure. The local system of axis is left-handed. The time reversal operator is given in this reference[2] and can also be considered as a rotation of value π around the axis $\boldsymbol{e}_2$,

$$T=R(\pi,\boldsymbol{e}_2)=\exp(i\pi\boldsymbol{e}_2\cdot\boldsymbol{S}/\hbar)=\exp(i\pi T_2/\hbar)=\begin{pmatrix}0&-\mathbb{I}\\ \mathbb{I}&0\end{pmatrix},\quad T^{-1}=T^\dagger.$$

[2]B. Thaller, The Dirac equation, Springer, Berlin (1992), p 75.

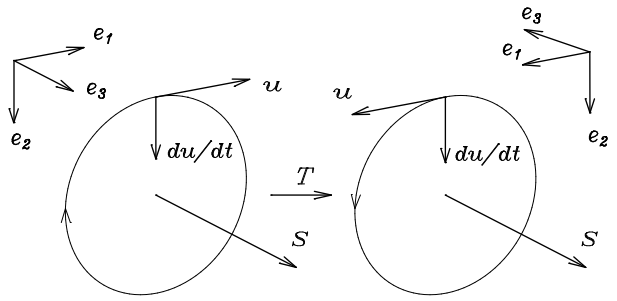


Figure 4.8: **Time reversal of the electron produces a particle of negative energy.**

$$THT^\dagger = mc^2 T\beta T^\dagger = -H, \quad T\boldsymbol{S}T^\dagger = \boldsymbol{S}, \quad T\boldsymbol{e}_1 T^\dagger = -\boldsymbol{e}_1, \quad T\boldsymbol{e}_2 T^\dagger = \boldsymbol{e}_2, \quad T\boldsymbol{e}_3 T^\dagger = -\boldsymbol{e}_3.$$

The PT transformation is given by the matrix

$$TP = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad PT = \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix},$$

and when transforming the gamma matrices γ^μ produce

$$P\gamma^0 P^{-1} = \gamma^0, \quad P\gamma^i P^{-1} = -\gamma^i, \quad T\gamma^0 T^{-1} = -\gamma^0, \quad T\gamma^i T^{-1} = \gamma^i,$$

$$(PT)\gamma^\mu (PT)^{-1} = -\gamma^\mu,$$

because the four matrices γ transform like the components of a fourvector

$$(PT)x^\mu (PT)^{-1} = -x^\mu, \quad (PT)\partial^\mu (PT)^{-1} = -\partial^\mu,$$

This is the reason why the Dirac equation for the free particle is invariant under the complete Poincaré group including inversions.

A Dirac particle is a mechanical system whose intrinsic attributes are mass $m > 0$ and spin $\hbar/2$. We also see that the sign of H is also Poincaré invariant and it is also an intrinsic property which establishes two different systems of the same value of m and S. The system with $H > 0$ is called the particle and the other with $H < 0$ the antiparticle. The value of the mass attribute is introduced by hand. To characterize its interaction with an external electromagnetic field, we also introduce by hand another intrinsic property the electric charge e, located at the point $\boldsymbol{r}$. This implies that in addition to the mechanical properties m and S the system has electromagnetic properties like the electric charge e and because the charge location is separated from its center of mass and its motion is at the speed of light, we also have an electric dipole moment $\boldsymbol{d}$ and a magnetic moment $\boldsymbol{\mu}$, respectively. The electric charge can also have either a positive or negative sign. If we consider the charge conjugation transformation C,

$$P\begin{pmatrix} S \\ m \\ H \\ e \\ \boldsymbol{\mu} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} S \\ m \\ H \\ e \\ \boldsymbol{\mu} \\ -\boldsymbol{d} \end{pmatrix}, \quad T\begin{pmatrix} S \\ m \\ H \\ e \\ \boldsymbol{\mu} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} S \\ m \\ -H \\ -e \\ -\boldsymbol{\mu} \\ \boldsymbol{d} \end{pmatrix}, \quad C\begin{pmatrix} S \\ m \\ H \\ e \\ \boldsymbol{\mu} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} S \\ m \\ -H \\ -e \\ -\boldsymbol{\mu} \\ -\boldsymbol{d} \end{pmatrix}$$

and the global PCT transformation gives,

$$PCT \begin{pmatrix} S \\ m \\ H \\ e \\ \boldsymbol{\mu} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} S \\ m \\ H \\ e \\ \boldsymbol{\mu} \\ \boldsymbol{d} \end{pmatrix}.$$

The PCT transformation transforms the particle into the antiparticle and conversely, reversing also the local frame, while keeping invariant the mechanical attributes m and S and the electromagnetic attributes $\boldsymbol{\mu}$ and $\boldsymbol{d}$. The PCT invariance of the system establishes a relationship between the sign of H and the sign of e, although an indeterminacy exists in the selection of the sign of the charge of the particle. The product eH is PCT invariant. Dirac equation is PCT invariant because this transformation maps $H > 0$ solutions into $H < 0$ solutions and conversely, because this equation describes both types of elementary particles.

This implies that particle and antiparticle, in the center of mass frame, have a magnetic moment and an oscillating electric dipole in a plane orthogonal to the spin. Once the spin direction is fixed, the magnetic moment of both have the same relative orientation with the spin, either parallel or antiparallel, according to the selection of the sign of the electric charge. The electric dipole moment oscillates leftwards for particles and rightwards for antiparticles which shows a difference between them which is called *chirality*. If as usual we call the electron to the system of negative electric charge *the particle*, the above PCT transformation transforms the system (a) of figure 4.9 into the system (b). If what we call the particle is of positive electric charge, then the spin and magnetic moment are opposite to each other for both particle and antiparticle.

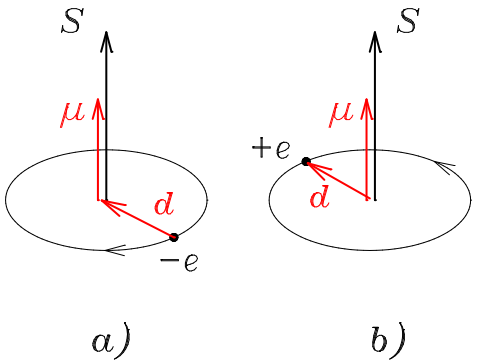


Figure 4.9: **Electromagnetic attributes $\boldsymbol{\mu}$ and $\boldsymbol{d}$ for (a) a negatively charged particle and its PCT transformed (b), the positive charged antiparticle, and their relative orientation with respect to the spin, in the center of mass frame. The electric dipole of the particle oscillates leftwards and rightwards for the antiparticle.**

However, to our knowledge no explicit direct measurement of the relative orientation between spin and magnetic moment of the free electron, can be found in the literature although very high precision experiments are performed to obtain the absolute value of g, the gyromagnetic ratio.

4.2.7 Two plausible experiments

A plausible indirect experiment [3] has been proposed to measure the relative orientation between spin and magnetic moment for one electron atoms in the outer shell, like Rb or Cs.

Rb^{87} atoms have one electron at the level $5s$. Its nucleus has spin 3/2 and the ground state of the atom has a total spin 1, and therefore the outer electron has its spin in the opposite direction to the spin of the nucleus. The magnetic moment of the atom is basically the magnetic moment of this outer electron because the inner shells are full and the magnetic moment of the nucleus is relatively smaller.

Ultracold Rb^{87} atoms in an external magnetic field will be oriented with their magnetic moments pointing along the field direction. If in this direction we send a beam of circularly polarized photons of sufficient energy ~ 6.8GHz to produce the corresponding hyperfine transition to flip the electron spin in the opposite direction and thus leaving the atom in a spin 2 state, only those photons with the spin opposite to the spin of the outer electron will be absorbed. Measuring the spin orientation of the circularly polarized beam will give us the spin orientation of the electron thus showing its relationship with the magnetic moment orientation. Now the task is to check also the relative orientation for positrons.

The indefiniteness in the sign of the charge of matter is also present in Dirac's formalism. This prediction is consistent with the known structures formed by a particle and the corresponding antiparticle. As a matter of fact, the positronium (electron-positron bound sytem) has a ground state of spin 0 and magnetic moment 0. This means that the spins of both electron and positron are antiparallel to each other and the same thing happens to the corresponding magnetic moments. Therefore, for the electron and positron there should exist the same relative orientation between the spin and magnetic moment.

Another example is the neutral pion π^0 which is a linear combination of the quark-antiquark bound systems $u\bar{u}$, $d\bar{d}$ and sometimes the pair $s\bar{s}$ is also included. It is a system of 0 spin and 0 magnetic moment. Because each of the above quarks have different masses and charges, and thus different magnetic moments, the possibility is that each quark-antiquark pair is a system of 0 spin and 0 magnetic moment, and, therefore each quark and the corresponding antiquark must have the same relative orientation between the spin and the magnetic moment.

Another experiment is the measurement of the precession direction of the spin of e^+ and e^- and of μ^+ and μ^- in a storage ring. If e^+ and e^- and μ^+ and μ^- have the same relative orientation between spin and magnetic moment, then the torque and thus the precession will be the same.

$$\boldsymbol{\mu} \times \boldsymbol{B} = \frac{d\boldsymbol{S}}{dt}$$

Nevertheless, if we inject into the accelerator particles and antiparticles with the spin up, and because the magnetic field of the ring has to be reversed for the antiparticle, then the precession direction of both beams will be opposite to each other. If it is possible to detect the precession direction this will confirm the prediction and also the relative orientation between spin and magnetic moment.

4.2.8 Chirality

The classical model which satisfies Dirac's equation when quantized gives rise to two possible physical systems of $H > 0$ and $H < 0$. The $H > 0$ is usually called the particle. According to the previous analysis the internal motion of the charge takes place on a plane orthogonal to the spin direction and in a leftward sense when we fix as positive the spin direction. For the antiparticle the motion is rightwards. For particles, the local orientable frame of unit vectors

[3] M.Rivas, *Are the electron spin and magnetic moment parallel or antiparallel vectors?*, ArXiv:physics/0112057.

$\boldsymbol{e}_i$ behaves as a left handed system rotating with an angular velocity in the opposite direction to the spin, while for antiparticles it can be considered as a right handed one.

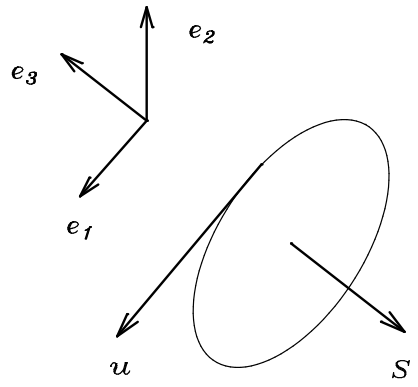


Figure 4.10: **Relative orientation of the body axis for the antiparticle that leads to Pauli-Dirac representation. It behaves as a rotating right handed Cartesian frame around the spin direction.**

If we should have started the analysis by considering first the antiparticle, then in order to get the same Pauli-Dirac representation as before we have to consider the body axis as the ones depicted in figure 4.10, *i.e.*, in the opposite direction to the ones we chose before and this leads by the same arguments that the γ^μ matrices have to replaced by the $-\gamma^\mu$, so that the Hamiltonian in the center of mass frame is $-\beta mc^2$. In this way the motion of the body frame, instead of (4.35-4.37) is

$$\frac{d\boldsymbol{e}_1}{dt} = \frac{i}{\hbar}[-\beta mc^2, \boldsymbol{e}_1] = -\omega \boldsymbol{e}_2, \tag{4.53}$$

$$\frac{d\boldsymbol{e}_2}{dt} = \frac{i}{\hbar}[-\beta mc^2, \boldsymbol{e}_2] = \omega \boldsymbol{e}_1, \tag{4.54}$$

$$\frac{d\boldsymbol{e}_3}{dt} = \frac{i}{\hbar}[-\beta mc^2, \boldsymbol{e}_3] = 0, \tag{4.55}$$

with $\omega = c/R_0$, which clearly corresponds to a rotating right handed system with an angular velocity around the spin direction.

Matter is left handed and antimatter is right handed in this kind of models as far as the charge motion and the rotation of the local body frame are concerned, so that particles and antiparticles show a clear chirality.

Although the local motion of the charge, which takes place in a region of order of Compton's wavelength, is probably physically unobservable, this motion corresponds nevertheless to the oscillation of the instantaneous electric dipole moment, which oscillates at very high frequency, but its direction of motion, once the spin direction is fixed, reflects this difference between particle and antiparticle. This electric dipole motion is independent of whether the particle is positively or negatively charged.

Finally, when we compare the spin operator and the vector $\boldsymbol{e}_3$ we see

$$\boldsymbol{S} = \frac{\hbar}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \boldsymbol{e}_3 = \frac{1}{3}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}.$$

that the two upper components of the Dirac spinor correspond to positive energy solutions and therefore the upper components of these operators are related by $\boldsymbol{S} \sim \boldsymbol{e}_3$, while the lower components correspond to negative energy solutions and for this components these operators behave as $\boldsymbol{S} \sim -\boldsymbol{e}_3$, and thus the spin projection operator $\boldsymbol{T} \sim \boldsymbol{e}_3$ in both cases, a vector relationship which is clearly depicted in the figures 4.2 and 4.10, respectively.

4.3 Dirac algebra

The three spatial spin components S_i, the three spin projections on the body frame T_j and the nine components of the body frame $(\boldsymbol{e}_i)_j$, $i,j = 1,2,3$, whose matrix representations are given in the $z = 0$ case in (4.15) or in (4.16) in the $z = 1$ case, together with the 4×4 unit matrix $\mathbb{I}$, form a set of 16 linearly independent hermitian matrices. They are a linear basis of Dirac's algebra, and satisfy the following commutation relations:

$$[S_i, S_j] = i\hbar\epsilon_{ijk} S_k, \qquad [T_i, T_j] = -i\hbar\epsilon_{ijk} T_k, \qquad [S_i, T_j] = 0, \tag{4.56}$$

$$[S_i, (\boldsymbol{e}_j)_k] = i\hbar\epsilon_{ikr} (\boldsymbol{e}_j)_r, \qquad [T_i, (\boldsymbol{e}_j)_k] = -i\hbar\epsilon_{ijr} (\boldsymbol{e}_r)_k, \tag{4.57}$$

and the scaled $3\boldsymbol{e}_i$ vectors in the $z = 0$ case

$$[(3\boldsymbol{e}_i)_k, (3\boldsymbol{e}_j)_l] = \frac{4i}{\hbar} \left(\delta_{ij}\epsilon_{klr} S_r - \delta_{kl}\epsilon_{ijr} T_r \right), \tag{4.58}$$

showing that the $\boldsymbol{e}_i$ operators transform like vectors under rotations but they are not commuting observables. In the case $z = 1$, the scaled $-9\boldsymbol{e}_i$, satisfy the same relations.

If we fix the pair of indexes i, and j, then the set of four operators S^2, S_i, T_j and $(\boldsymbol{e}_j)_i$ form a complete commuting set. In fact, the wave functions Φ_i, $i = 1, \ldots, 4$, given before (3.75)-(3.78), are simultaneous eigenfunctions of S^2, S_3, T_3 and $(\boldsymbol{e}_3)_3$ with eigenvalues $s = 1/2$ and for s_3, t_3, and e_{33}.

The basic observables satisfy the following anticommutation relations:

$$\{S_i, S_j\} = \{T_i, T_j\} = \frac{\hbar^2}{2} \delta_{ij} \mathbb{I}, \tag{4.59}$$

$$\{S_i, T_j\} = \frac{\hbar^2}{2} (3\boldsymbol{e}_j)_i, \tag{4.60}$$

$$\{S_i, (3\boldsymbol{e}_j)_k\} = 2\,\delta_{ik} T_j, \qquad \{T_i, (3\boldsymbol{e}_j)_k\} = 2\,\delta_{ij} S_k, \tag{4.61}$$

$$\{(\boldsymbol{e}_i)_j, (\boldsymbol{e}_k)_l\} = \frac{2}{9}\,\delta_{ik}\delta_{jl}\mathbb{I} + \frac{2}{3}\epsilon_{ikr}\epsilon_{jls} (\boldsymbol{e}_r)_s. \tag{4.62}$$

If we define the dimensionless normalized matrices:

$$a_{ij} = 3(\boldsymbol{e}_i)_j, \ (\text{or}\, a_{ij} = -9(\boldsymbol{e}_i)_j), \qquad s_i = \frac{2}{\hbar} S_i, \qquad t_i = \frac{2}{\hbar} T_i, \tag{4.63}$$

together with the 4×4 unit matrix $\mathbb{I}$, they form a set of 16 matrices Γ_λ, $\lambda = 1, \ldots, 16$ that are hermitian, unitary, linearly independent and of unit determinant. They are the orthonormal basis of the corresponding Dirac's Clifford algebra.

The set of 64 unitary matrices of determinant $+1$, $\pm\Gamma_\lambda$, $\pm i\Gamma_\lambda$, $\lambda = 1, \ldots, 16$ form a finite subgroup of $SU(4)$. Its composition law can be obtained from:

$$a_{ij}\, a_{kl} = \delta_{ik}\delta_{jl}\mathbb{I} + i\delta_{ik}\epsilon_{jlr}\, s_r - i\delta_{jl}\epsilon_{ikr}\, t_r + \epsilon_{ikr}\epsilon_{jls}\, a_{rs}, \tag{4.64}$$

$$a_{ij}\, s_k = i\epsilon_{jkl}\, a_{il} + \delta_{jk}\, t_i, \tag{4.65}$$

$$a_{ij}\, t_k = -i\epsilon_{ikl}\, a_{lj} + \delta_{ik}\, s_j, \tag{4.66}$$

$$
\begin{aligned}
s_i\, a_{jk} &= i\epsilon_{ikl}\, a_{jl} + \delta_{ik} t_j, && (4.67)\\
s_i\, s_j &= i\epsilon_{ijk}\, s_k + \delta_{ij}\mathbb{I}, && (4.68)\\
s_i\, t_j &= t_j\, s_i = a_{ji}, && (4.69)\\
t_i\, a_{jk} &= -i\epsilon_{ijl}\, a_{lk} + \delta_{ij} s_k, && (4.70)\\
t_i\, t_j &= -i\epsilon_{ijk}\, t_k + \delta_{ij}\mathbb{I}, && (4.71)
\end{aligned}
$$

and similarly we can use these expressions to derive the commutation and anticommutation relations (4.56-4.62).

Dirac algebra is generated by the four Dirac gamma matrices γ^μ, $\mu = 0, 1, 2, 3$ that satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}, \tag{4.72}$$

$\eta^{\mu\nu}$ being Minkowski's metric tensor.

Similarly it can be generated by the following four observables, for instance: S_1, S_2, T_1 and T_2. In fact by (4.68) and (4.71) we obtain S_3 and T_3 respectively and by (4.69), the remaining elements.

Classically, the internal orientation of an electron is characterized by the knowledge of the components of the body frame $(\boldsymbol{e}_i)_j$, $i, j = 1, 2, 3$ that altogether constitute an orthogonal matrix. To completely characterize in a unique way this orthogonal matrix we need at least four of these components. In the quantum version, the knowledge of four $(\boldsymbol{e}_i)_j$ matrices and by making use of (4.64)-(4.71), allows us to recover the remaining elements of the complete Dirac algebra. It is in this sense that the **internal orientation** of the electron completely characterizes its internal structure. Dirac's algebra of translation invariant observables of the electron can be algebraically generated by any four of the orientation operators.

4.4 Additional spacetime symmetries

The kinematical variables of this classical Dirac particle are reduced to time t, position $\boldsymbol{r}$, velocity $\boldsymbol{u}$ and orientation $\boldsymbol{\alpha}$, but the velocity is always $u = c$. It is always 1 in natural units. If the particle has mass $m \neq 0$ and spin $s \neq 0$, we can also define a natural unit of length s/mc and a natural unit of time s/mc^2. The unit of length is the radius of the zitterbewegung motion of figure 2.9, and the unit of time is the time employed by the charge, in the centre of mass frame, during a complete turn. This implies that the whole set of kinematical variables and their time derivatives can be taken dimensionless, and the classical formalism is therefore invariant under spacetime dilations which do not modify the speed of light.

It turns out that although we started with the Poincaré group as the basic spacetime symmetry group, this kind of massive spinning Dirac particles, has a larger symmetry group. It also contains at least spacetime dilations with generator D. The action of this transformation on the kinematical variables is

$$t' = e^\lambda t, \quad \boldsymbol{r}' = e^\lambda \boldsymbol{r}, \quad \boldsymbol{u}' = \boldsymbol{u}, \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha}.$$

The new conserved Noether observable takes the form

$$D = tH - \boldsymbol{r}\cdot\boldsymbol{P}. \tag{4.73}$$

Let $R(\boldsymbol{\beta})$ be an arbitrary rotation which changes observer's axes. The action of this arbitrary rotation $R(\boldsymbol{\beta})$ on the kinematical variables is

$$t' = t, \quad \boldsymbol{r}' = R(\boldsymbol{\beta})\boldsymbol{r}, \quad \boldsymbol{u}' = R(\boldsymbol{\beta})\boldsymbol{u}, \quad R(\boldsymbol{\alpha}') = R(\boldsymbol{\beta})R(\boldsymbol{\alpha}),$$

and this is the reason why the generators $\boldsymbol{J}$ of rotations involve differential operators with respect to all these variables, the time excluded.

The orientation of the particle, represented by the variables $\boldsymbol{\alpha}$, or the equivalent orthogonal rotation matrix $R(\boldsymbol{\alpha})$, is interpreted as the orientation of an hypothetical Cartesian frame of unit axis $\boldsymbol{e}_i$, $i = 1, 2, 3$, located at point $\boldsymbol{r}$. It has no physical reality but can be interpreted as the corresponding Cartesian frame with origin at that point. But the selection of this frame is completely arbitrary so that the formalism is independent of its actual value. This means that, in addition to the above rotation group between inertial observers, there will be another rotation group of elements $R(\boldsymbol{\gamma})$ which modifies only the orientation variables $\boldsymbol{\alpha}$, without modifying the variables $\boldsymbol{r}$ and $\boldsymbol{u}$, i.e., the rotation only of the body frame:

$$t' = t, \quad \boldsymbol{r}' = \boldsymbol{r}, \quad \boldsymbol{u}' = \boldsymbol{u}, \quad R(\alpha') = R(\boldsymbol{\gamma})R(\boldsymbol{\alpha}), \tag{4.74}$$

The generators of this new rotation group, which affects only the orientation variables, will be the projection of the angular momentum generators $\boldsymbol{W}$ onto the body axes. It is clear that the operations of the rotation of the observer frame and the rotation of the body frame commute with each other. This last rotation represents an active rotation of the body axis. From Noether's theorem the corresponding classical conserved observables are

$$T_i = \boldsymbol{W} \cdot \boldsymbol{e}_i, \tag{4.75}$$

where $\boldsymbol{e}_i$ are the three orthogonal unit vectors which define the body axis.

If $R(\boldsymbol{\alpha})$ is the orthogonal rotation matrix which describes the orientation of the particle, when considered by columns these columns describe the components of the three orthogonal unit vectors $\boldsymbol{e}_i$, $i = 1, 2, 3$. Equations (4.74) correspond to the transformation $\boldsymbol{e}'_i = R(\boldsymbol{\gamma})\boldsymbol{e}_i$ of the body frame.

The W_i operators represent the components of the angular momentum operators associated to the change of orientation of the particle and projected in the laboratory frame. The corresponding conserved quantities (4.75) are that components of the angular momentum operators projected onto the body frame $T_i = \boldsymbol{e}_i \cdot \boldsymbol{W}$. When quantizing the system they are given by the differential operators (4.90)-(4.92) of the appendix below and satisfy

$$T^2 = W^2, \quad [T_i, T_j] = -i\epsilon_{ijk}T_k,$$

$$[T_i, K_j] = [T_i, J_j] = [T_i, H] = [T_i, D] = [T_i, P_j] = 0.$$

We can see that the self-adjoint operators T_i generate another $SU(2)$ group which is the representation of the local rotation group $SO(3)_L$ which modifies only the orientation variables, commutes with the rotation group generated by the J_j, and with the whole enlarged Poincaré group, including spacetime dilations.

Since we expect that the formalism is independent of the orientation variables we have another $SO(3)$ group of spacetime symmetries of the particle.

4.4.1 Analysis of the enlarged symmetry group

Let H, $\boldsymbol{P}$, $\boldsymbol{K}$ and $\boldsymbol{J}$ be the generators of the Poincaré group $\mathcal{P}$. With the usual identification of $p^\mu \equiv (H/c, \boldsymbol{P})$ as the four-momentum operators and $w^\mu \equiv (\boldsymbol{P} \cdot \boldsymbol{J}, H\boldsymbol{J}/c - \boldsymbol{K} \times \boldsymbol{P})$ as the Pauli-Lubanski four-vector operator, the two Casimir operators of the Poincaré group are

$$C_1 = p_\mu p^\mu, \quad C_2 = -w_\mu w^\mu.$$

These two Casimir operators, if measured in the centre of mass frame where $\boldsymbol{P} = \boldsymbol{K} = 0$, in natural units $c = 1$, $\hbar = 1$, reduce respectively in an irreducible representation to $C_1 = m^2$ and

$C_2 = H^2 J^2 = m^2 s(s+1)$. The two parameters m and s, which characterize every irreducible representation of the Poincaré group, represent the intrinsic properties of a Poincaré invariant elementary particle.

Let us consider the additional spacetime dilations of generator D. The action of this transformation on the kinematical variables is

$$t' = e^{\lambda} t, \quad \boldsymbol{r}' = e^{\lambda} \boldsymbol{r}, \quad \boldsymbol{u}' = \boldsymbol{u}, \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha}.$$

Let us denote this enlargement of the Poincaré group, sometimes called the Weyl group, by $\mathcal{W}$. In the quantum representation, this new generator when acting on the above wavefunctions, has the form:

$$D = it\partial/\partial t + i\boldsymbol{r} \cdot \nabla. \tag{4.76}$$

It satisfies

$$[D, p^{\mu}] = -ip^{\mu}, \quad [D, J^{\mu\nu}] = 0.$$

This enlarged group has only one Casimir operator[4] which, for massive systems where the operator $C_1 \neq 0$ is invertible, is reduced to

$$C = C_2 C_1^{-1} = C_1^{-1} C_2 \equiv C_2/C_1 = s(s+1).$$

In the centre of mass frame this operator is reduced to $C = S^2$, the square of the spin operator.

By assuming also the spacetime dilation invariance this implies that the mass is not an intrinsic property. It is the spin which is the only intrinsic property of this elementary particle. In fact, since the radius of the internal motion is $R = s/mc$, a change of length and time scale corresponds to a change of mass while keeping s and c constant. By this transformation the elementary particle of spin 1/2 modifies its internal radius and therefore its mass and goes into another mass state.

The structure of the differential operator $\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{Z} + \boldsymbol{W}$, where the spin part $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$ has only $s = 1/2$ eigenvalue for the above model, implies that the eigenvalue of the W^2 corresponds to $w = 1/2$ while for the $\boldsymbol{Z}^2$ part can be reduced to the two possibilities $z = 0$ or $z = 1$.

In addition to the group $\mathcal{W}$ we also consider the representation of the local rotation group generated by the T_i with eigenvalue $w = t = 1/2$. We have thus a larger spacetime symmetry group with an additional $SU(2)$ structure when quantized.

The generators T_i commute with all generators of the group $\mathcal{W}$, and this new symmetry group can be written as $\mathcal{W} \otimes SU(2)_T$.

This new group has only two Casimir operators S^2 and T^2 of eigenvalues 1/2. This justifies that our wavefunction will be written as a four-component wavefunction. When choosing the complete commuting set of operators to classify its states we take the operator $T^2 = S^2$, the S_3 and T_3 which can take the values $\pm 1/2$ and for instance the $p^{\mu} p_{\mu}$ and the p^{μ}. In this way we can separate in the wavefunction the orientation and velocity variables from the spacetime variables,

$$\psi(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha}) = \sum_{i=1}^{i=4} \phi_i(t, \boldsymbol{r}) \chi_i(\boldsymbol{u}, \boldsymbol{\alpha})$$

where the four $\chi_i(\boldsymbol{v}, \boldsymbol{\alpha})$ can be classified according to the eigenvalues $|s_3, t_3>$. The functions $\phi_i(t, \boldsymbol{r})$ can be chosen as eigenfunctions of the Klein-Gordon operator

$$p_{\mu} p^{\mu} \phi_i(t, \boldsymbol{r}) = m_i^2 \phi_i(t, \boldsymbol{r}).$$

[4] Abellanas L and Martinez Alonso L 1975 *J. Math. Phys.* **16** 1580

Because this operator $p_\mu p^\mu$ does not commute with the D observable, the mass eigenvalue m_i is not an intrinsic property and the corresponding value depends on the particular state ϕ_i we consider.

For the classification of the $\chi_i(\boldsymbol{u},\boldsymbol{\alpha})$ states we have also to consider the $\boldsymbol{Z}$ angular momentum operators. Because $[Z^2,S^2]=[Z^2,T^2]=[Z^2,p^\mu]=0$, we can choose Z^2 as an additional commuting observable. It can only take integer eigenvalues when acting on functions of the velocity variables, because it has the structure of an orbital angular momentum. But because the total spin $\boldsymbol{S}=\boldsymbol{Z}+\boldsymbol{W}$, and the S^2 has eigenvalue $1/2$, the possible eigenvalues of Z^2 can be $z=0$ or $z=1$. See the appendix below for the possible clasification of the $\chi_i(\boldsymbol{u},\boldsymbol{\alpha})$ part, according to $z=0$ which gives rise to the (3.75-3.78) eigenfunctions, and the $z=1$ eigenfunctions (4.97-4.100). In this last case the eigenfunctions cannot be simultaneously eigenfunctions of Z_3. Nevertheless the expectation value of Z_3 in the $z=0$ basis vectors Φ_i is 0, while its expectation value in the $z=1$ basis Ψ_i is $\pm 2/3$.

4.4.2 Enlargement of the kinematical space

Once the kinematical group has been enlarged by including spacetime dilations, we have a new dimensionless group parameter asociated to this one-parameter subgroup which can also be used as a new kinematical variable, to produce a larger homogeneous space of the group. In fact, if we take the time derivative of the constant of the motion (4.73) we get

$$H=\boldsymbol{P}\cdot\boldsymbol{u}.$$

If we compare this with the equation (4.19), one term is lacking. This implies that we need, from the classical point of view, an additional kinematical variable, a dimensionless scale β, such that under the action of this new transformation the enlarged kinematical variables transform

$$t'=e^\lambda t,\quad \boldsymbol{r}'=e^\lambda\boldsymbol{r},\quad \boldsymbol{u}'=\boldsymbol{u},\quad \boldsymbol{\alpha}'=\boldsymbol{\alpha},\quad \beta'=\lambda+\beta.$$

From the group theoretical point of view this new dimensionless variable corresponds to the normal dimensionles group parameter of the transformation generated by D.

From the Lagrangian point of view, the new Lagrangian has also to depend on β and $\dot\beta$, with a general structure

$$\widetilde{L}=\dot tT+\dot{\boldsymbol{r}}\cdot\boldsymbol{R}+\dot{\boldsymbol{u}}\cdot\boldsymbol{U}+\boldsymbol{\omega}\cdot\boldsymbol{W}+\dot\beta B,$$

with $B=\partial\widetilde{L}/\partial\dot\beta$. The constant of the motion associated to the invariance of the dynamical equations under this new transformation implies that

$$D=tH-\boldsymbol{r}\cdot\boldsymbol{P}-B,$$

and the new generator in the quantum version takes the form

$$D=it\partial/\partial t+i\boldsymbol{r}\cdot\nabla+i\frac{\partial}{\partial\beta}.$$

In this way the last term of (4.19) is related to the time derivative of this last term

$$\frac{dB}{dt}=\frac{1}{c^2}\boldsymbol{S}\cdot\left(\frac{d\boldsymbol{u}}{dt}\times\boldsymbol{u}\right).$$

This new observable B, with dimensions of action, has a positive time derivative for particles and a negative time derivative for antiparticles. This sign is clearly related to the sign of H. In

the center of mass frame $\boldsymbol{P} = 0$, $H = \pm mc^2 = dB/dt$, with solution $B(t) = B(0) \pm mc^2 t$. In units of $\hbar$ this observable represents half the phase of the internal motion

$$B(t) = B(0) \pm \frac{1}{2}\hbar\omega t.$$

Because the additional local rotations generated by the T_i commute with the $\mathcal{W}$ group, the above kinematical variables also span a homogeneous space of the whole $\mathcal{W} \otimes SU(2)_T$ group and, therefore, they represent the kinematical variables of an elementary system which has this new group as its kinematical group of spacetime symmetries.

4.4.3 Relationship with the standard model

We have analyzed the spacetime symmetry group of a relativistic model of a Dirac particle. Matter described by this model ($H > 0$ states), is left handed while antimatter ($H < 0$), is right handed, as far as the relative orientation between the spin and the motion of the charge, is concerned. For matter, once the spin direction is fixed, the motion of the charge is counterclockwise when looking along the spin direction. It is contained in a plane orthogonal to the spin direction, with the usual sign convention for multivectors in the geometric algebra. The motion is clockwise for antimatter.

This particle has as symmetry group of the Lagrangian $\mathcal{W} \otimes SO(3)_L$ and $\mathcal{W} \otimes SU(2)_T$ in its quantum description, which is larger than the Poincaré group we started with as the initial kinematical group of the model. It contains in its quantum description, in addition to the Poincaré transformations, a $U(1)$ group which is a unitary representation of the spacetime dilations and also a $SU(2)_T$ group which is the unitary representation of the symmetry group of local rotations of the body frame. The whole group has two Casimir operators S^2, the Casimir of $\mathcal{W}$ and T^2 the Casimir of $SU(2)_T$, which take the eigenvalues $s = t = 1/2$ for the Dirac particle considered here.

Some of the features we get have a certain resemblance to the standard model of elementary particles, as far as kinematics is concerned. If we interpret the generators T_i of the unitary representation of the local rotations as describing isospin and the angular momentum operators $\boldsymbol{Z}$ related to the zitterbewegung as describing colour, an elementary particle described by this formalism is a massive system of spin 1/2, isospin 1/2, of undetermined mass and charge. It can be in a $s_3 = \pm 1/2$ spin state and also in a $t_3 = \pm 1/2$ isospin state. There are two nonequivalent irreducible representations according to the value of the zitterbewegung part of the spin z. It can only be a colourless particle $z = 0$ (lepton?) or a coloured one $z = 1$ in any of three possible colour states $z_3 = 1, 0, -1$, (quark?) but no greater z value is allowed. The basic states can thus also be taken as eigenvectors of $\boldsymbol{Z}^2$ but not of Z_3, so that the corresponding colour is unobservable. There is no possibility of transitions between the coloured and colourless particles because of the orthogonality of the corresponding irreducible representations, Φ_i and Ψ_j, $< \Phi_i | \Psi_j >= 0$.

Because the eigenvalues of Z_3 are unobservable we also have an additional unitary group of transformations $SU(3)$ which transforms the three Z_3 eigenvectors Y_i^j of (4.101) among themselves and which do not change the $z = 1$ value of the eigenstates Ψ_i. Nevertheless, the relationship between this new $SU(3)$ internal group, which is not a spacetime symmetry group, and $\mathcal{W} \otimes SU(2)_T$ is not as simple as a direct product and its analysis is left to a subsequent research. In another context, the $z = 0$ states corresponds to the motion of the charge pasing through the centre of mass and therefore no closed current loop and thus no magnetic moment.

This formalism is pure kinematical. We have made no mention to any electromagnetic, weak or strong interaction among the different models. So that, if we find this comparison with the standard model a little artificial, the mentioned model of Dirac particle just represents a massive system of spin 1/2, spin projection on the body frame 1/2, of undetermined mass and

charge. It can be in a $s_3 = \pm 1/2$ spin state and also in a $t_3 = \pm 1/2$ when the spin is projected on the body axis. There are two different models of these Dirac particles according to the value of the orbital or zitterbewegung spin, $z = 0$ or $z = 1$, in any of the three possible orbital spin states $z_3 = 1, 0, -1$, which are unobservable, but no particle of greater z value is allowed. It is the spin, with its twofold structure orbital and rotational, the only intrinsic attribute of this Dirac elementary particle.

4.5 An interaction Lagrangian for two Dirac particles

An elementary particle can be annihilated by the interaction with the corresponding antiparticle, but if it is not destroyed, we made the assumption that the structure of an elementary particle is not modified by any interaction so that its intrinsic properties, the spin S and the spin projection on the body frame T cannot be altered by the interaction with an external field or by the presence in its neigbourhood of any other particle.

Let us consider a compound system formed by two spinning particles with the same kind of kinematical variables. We shall use a subscript $a = 1, 2$ to distinguish the variables corresponding to each particle. Then the kinematical space of the compound system is spanned by the variables $(t_a, \boldsymbol{r}_a, \boldsymbol{u}_a, \boldsymbol{\alpha}_a, \beta_a), a = 1, 2$. The Lagrangian of the system will be written as

$$L = L_1 + L_2 + L_I$$

where the L_a, $a = 1, 2$, are the free Lagrangians of each particle and L_I is the interaction Lagrangian we are looking for. Both L_a are invariant under the enlarged group $\mathcal{S}$ and we are going to find an interaction Lagrangian L_I also invariant under $\mathcal{S}$. The general structure of the free Lagrangian L_a of each particle, which only depends on the corresponding kinematical variables of particle a, is

$$\widetilde{L}_a = T_a \dot{t}_a + \boldsymbol{R}_a \cdot \dot{\boldsymbol{r}}_a + \boldsymbol{U}_a \cdot \dot{\boldsymbol{u}}_a + \boldsymbol{W}_a \cdot \boldsymbol{\omega}_a + B_a \dot{\beta}_a$$

where $T_a = \partial \widetilde{L}_a / \partial \dot{t}_a$, $\boldsymbol{R}_a = \partial \widetilde{L}_a / \partial \dot{\boldsymbol{r}}_a$, $\boldsymbol{U}_a = \partial \widetilde{L}_a / \partial \dot{\boldsymbol{u}}_a$, $\boldsymbol{W}_a = \partial \widetilde{L}_a / \partial \boldsymbol{\omega}_a$ and $B_a = \partial \widetilde{L}_a / \partial \dot{\beta}_a$, because of the homogeneity of each $\widetilde{L}_a$ in terms of the $\tau-$derivatives of the corresponding kinematical variables. The spin and the spin projection on the body frame for each particle, are

$$\boldsymbol{S}_a = \boldsymbol{u}_a \times \boldsymbol{U}_a + \boldsymbol{W}_a, \quad T_{ai} = \boldsymbol{e}_{ai} \cdot \boldsymbol{W}_a$$

where $\boldsymbol{e}_{ai}$, $i = 1, 2, 3$ are three orthogonal unit vectors with origin at point $\boldsymbol{r}_a$.

The interaction Lagrangian between these two particles $\widetilde{L}_I$ will be in general a function of the kinematical variables of both particles and of their $\tau-$derivatives. If both intrinsic properties S_a and T_a of each particle are not modified by any interaction then the interaction Lagrangian cannot be a function of the derivatives of the kinematical variables $\dot{\boldsymbol{u}}_a$ and $\boldsymbol{\omega}_a$, $a = 1, 2$. Otherwise the functions $\boldsymbol{U}_a$ and $\boldsymbol{W}_a$ will be different than in the free case. In this case the functions $\boldsymbol{U}_a$ and $\boldsymbol{W}_a$, which give rise to the definition of the spin, are obtained only from the corresponding free Lagrangian $\widetilde{L}_a$.

Then, as far as the $\tau-$derivatives of the kinematical variables are concerned, the interaction Lagrangian $\widetilde{L}_I$ will only depend on the variables $\dot{t}_a$, $\dot{\boldsymbol{r}}_a$ and $\dot{\beta}_a$, $a = 1, 2$. In addition to this, it will also be a function of the kinematical variables t_a, $\boldsymbol{r}_a$, $\boldsymbol{u}_a$ and β_a, but not of $\boldsymbol{\alpha}_a$ because of the invariance under the local rotation group $SO(3)_L$. Spacetime dilation invariance implies that the Lagrangian is a function of the phase difference $\beta_1 - \beta_2$, and of $\dot{\beta}_1 - \dot{\beta}_2$, but being both phases completely arbitrary and independent of each other it means that must be independent of these variables.

Because of the constraint $\boldsymbol{u}_a = \dot{\boldsymbol{r}}_a / \dot{t}_a$, the interaction Lagrangian will thus be finally a function

$$\widetilde{L}_I = \widetilde{L}_I(t_a, \boldsymbol{r}_a, \dot{t}_a, \dot{\boldsymbol{r}}_a),$$

and a homogeneous function of first degree of the derivatives $\dot{t}_a, \dot{\boldsymbol{r}}_a$, $a = 1, 2$.

If we call as usual the Minkowski four-vector $x_a^\mu \equiv (t_a, \boldsymbol{r}_a)$, translation invariance implies that the Lagrangian must be a function of $x_1^\mu - x_2^\mu$. The following two terms $\eta_{\mu\nu}\dot{x}_1^\mu \dot{x}_2^\nu$ and $\eta_{\mu\nu}(x_1^\mu - x_2^\mu)(x_2^\nu - x_1^\nu)$, where $\eta_{\mu\nu}$ is Minkowski's metric tensor, are Poincaré invariant. If we consider that the evolution parameter τ is dimensionless, these terms have both dimensions of length squared. It therefore implies that its quotient is dimensionless and therefore invariant under spacetime dilations. The other requirement is that the Lagrangian is a homogeneous function of first degree of the $\tau-$derivatives of the kinematical variables. The squared root will do the job. A final discrete symmetry will be assumed because when the two particles are the same, and therefore indistinguishable, the interaction Lagrangian must be invariant under the interchange $1 \leftrightarrow 2$ between the labels of the two particles. We thus arrive to the $\mathcal{S}$ group invariant Lagrangian

$$\widetilde{L}_I = g\sqrt{\frac{\eta_{\mu\nu}\dot{x}_1^\mu \dot{x}_2^\nu}{\eta_{\mu\nu}(x_1^\mu - x_2^\mu)(x_2^\nu - x_1^\nu)}} = g\sqrt{\frac{c^2\dot{t}_1\dot{t}_2 - \dot{\boldsymbol{r}}_1 \cdot \dot{\boldsymbol{r}}_2}{(\boldsymbol{r}_2 - \boldsymbol{r}_1)^2 - c^2(t_2 - t_1)^2}}$$

where g is a coupling constant.

Alternative Lagrangians which fulfill these requirements can be constructed. For instance, $\widetilde{L} = g(\dot{x}_1 - \dot{x}_2)^\mu (x_1 - x_2)_\mu/(x_1 - x_2)^2$, but this one is a total $\tau-$derivative of the function $\log(x_1 - x_2)^2$. Another could be $\widetilde{L} = g(\dot{x}_1 + \dot{x}_2)^\mu (x_1 - x_2)_\mu/(x_1 - x_2)^2$, also dimensionless and linear in the derivatives of the kinematical variables, but it reverses its sign under the interchange $1 \leftrightarrow 2$, and thus all interaction observables, like the interaction energy are reversed, which is physically meaningless for two alike particles.

Another possibility would be

$$\widetilde{L}_I = g\frac{\eta_{\mu\nu}\dot{x}_1^\mu \dot{x}_2^\nu}{\eta_{\mu\nu}(\dot{x}_1^\mu - \dot{x}_2^\mu)(x_2^\nu - x_1^\nu)} = g\frac{c^2\dot{t}_1\dot{t}_2 - \dot{\boldsymbol{r}}_1 \cdot \dot{\boldsymbol{r}}_2}{(\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2) \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1) - c^2(\dot{t}_1 - \dot{t}_2)(t_2 - t_1)},$$

which fullfils the requirements of homogeneity and scale invariance. In a sinchronous description it looks like

$$L_I = g\frac{c^2 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}{(\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1)} \tag{4.77}$$

The interaction between two Dirac particles is not unique. We know that among leptons and quarks there are short range interactions like the weak and strong interactions and a short and long range one like the electromagnetic interaction. The proposed Lagrangian has the advantage of describing an interaction which is scale invariant and thus it is valid as a long and short range interaction and which has a Coulomb-like behaviour when the spin is supressed, as we shall see in the next section. In this way it suplies a kind of generalization of an action at a distance electromagnetic interaction. The novelty is that this interaction Lagrangian has been obtained by assuming a spacetime symmetry group larger than the Poincaré group.

4.5.1 Synchronous description

Once an inertial observer is fixed we shall consider a synchronous time description, i.e. to use as evolution parameter the own observer's time t which is the same as the two time variables t_1 and t_2. In this case, $t = t_1 = t_2$, $\dot{t}_1 = \dot{t}_2 = 1$, and thus

$$L_I = g\sqrt{\frac{c^2 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}{(\boldsymbol{r}_2 - \boldsymbol{r}_1)^2}} = g\frac{\sqrt{c^2 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}}{r} \tag{4.78}$$

where $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ is the instantaneous separation between the corresponding charges in this frame and $\boldsymbol{u}_a = d\boldsymbol{r}_a/dt$ the velocity of the charge of particle a.

An average over the charge position and velocity in the centre of mass of particle 1 implies that $< \boldsymbol{r}_1 >= \boldsymbol{q}_1$ and $< \boldsymbol{u}_1 >= 0$, so that the interaction becomes the instantaneous Coulomb interaction, between the centre of mass of the first particle and the charge position of the other. The average over the other then corresponds to the instantaneous Coulomb interaction of two spinless point particles because when neglecting the zitterbewegung we are suppressing the spin structure.

It is suggesting, in the international system of units, that the coupling constant $gc = \pm e^2/4\pi\epsilon_0$ in terms of the electric charges of each particle and where the sign $\pm$ depends on the charges, whether they are opposite or of the same sign. If the coupling constant has dimensions of action, we write as $g = \alpha\hbar$, it results that the dimensionless constant α, is precisely the fine structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}.$$

In this way, the interaction is independent of the masses of the particles and in the synchronous description looks like

$$L_I = \alpha\hbar\frac{\sqrt{c^2 - \boldsymbol{u}_1\cdot\boldsymbol{u}_2}}{r}.$$

The Lagrangian of the two particle system $\widetilde{L} = \widetilde{L}_1 + \widetilde{L}_2 + \widetilde{L}_I$ is not invariant under scale transformation, because the free Lagrangians $\widetilde{L}_i$, $i = 1, 2$ are not, and a change of space-time scale what describes is the interaction, with the same $\widetilde{L}_I$ of two other particles of the same charge and spin, but of different masses as the previous particles. If initially we are describing the interaction between two electrons, the scale transformation suggests that the rescaled material system corresponds to the interaction, with the same interaction Lagrangian $\widetilde{L}_I$, of two muons or two tau-particles. If we use dimensionless variables by using as a unit of lenght the internal radius $R_0 = \hbar/2mc$ and as a unit of velocity the universal constant c, the interaction Lagrangian is

$$L_I = \frac{\alpha\hbar c}{\hbar/2mc}\frac{\sqrt{1 - \boldsymbol{u}_1\cdot\boldsymbol{u}_2}}{r} = 2\alpha mc^2\frac{\sqrt{1 - \boldsymbol{u}_1\cdot\boldsymbol{u}_2}}{r},$$

where all variables are dimensionless. If L_1 and L_2 depend on the mass of the particles, by rescaling the whole Lagrangian by the factor mc^2, we obtain a dimensionless Lagrangian which in the synchronous description is

$$L = L_1 + L_2 + 2\alpha\frac{\sqrt{1 - \boldsymbol{u}_1\cdot\boldsymbol{u}_2}}{r}. \tag{4.79}$$

It is independent of the masses of the particles and the only physical parameter is the fine structure constante α.

The Lagrangian (4.77) becomes in the synchronous description of the form

$$L_I = \alpha\hbar\frac{c^2 - \boldsymbol{u}_1\cdot\boldsymbol{u}_2}{(\boldsymbol{u}_1 - \boldsymbol{u}_2)\cdot(\boldsymbol{r}_2 - \boldsymbol{r}_1)},$$

with the same coupling constant. If we make the same scaling as before and the final factor by mc^2, it becomes

$$L = L_1 + L_2 + 2\alpha\frac{1 - \boldsymbol{u}_1\cdot\boldsymbol{u}_2}{(\boldsymbol{u}_1 - \boldsymbol{u}_2)\cdot(\boldsymbol{r}_2 - \boldsymbol{r}_1)}, \tag{4.80}$$

in terms of the same dimensionless physical constant α, and where all variables are dimensionless.

A first conclusion for this synchronous description, for both (4.79) and (4.80) is that to have interaction between the particles it is necessary that always $\boldsymbol{u}_1 \neq \boldsymbol{u}_2$. Otherwise there

will be no interaction because the interacting Lagrangian vanishes. A mechanical system of two electrons at rest with their spins parallel and moving with the same velocity $\boldsymbol{u}_1 = \boldsymbol{u}_2$, do not interact, independently of their separation of their centers of charge. The same happens for an electron-positron system with opposite spins where we can have $\boldsymbol{u}_1 = \boldsymbol{u}_2$, and the interaction Lagrangian will be zero.

The formalism is Poincaré invariant while the interaction is invariant under the enlarged group $\mathcal{S}$, which also contains scale transformations and local rotations. The free part L_0 would describe particles of spin 1/2 and different masses, but the interaction woud be independent of the masses of the particles. All leptons and quarks have different masses, but their interaction would be described bay the same $\widetilde{L}_I$.

The variational derivative of L with respect to the variables $\boldsymbol{r}_a$ produces from the part L_a the term $-d\boldsymbol{p}_a/dt$,

$$-\frac{d\boldsymbol{p}_a}{dt} = -\frac{d}{dt}\left(\frac{\partial L_a}{\partial \boldsymbol{u}_a}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_a}{\partial \boldsymbol{a}_a}\right),$$

because L_a is independent of $\boldsymbol{r}_a$, and from the part L_I we get

$$\frac{\partial L_I}{\partial \boldsymbol{r}_a} - \frac{d}{dt}\left(\frac{\partial L_I}{\partial \boldsymbol{u}_a}\right) = \boldsymbol{F}_a$$

which finally gives rise to the dynamical equations

$$\frac{d\boldsymbol{p}_a}{dt} = \boldsymbol{F}_a, \quad a = 1, 2.$$

The variational derivative of L with respect to the orientation variables, is the same like in the free case because L_I is independentof these variables and thus

$$\frac{d\boldsymbol{W}_a}{dt} = \boldsymbol{\omega}_a \times \boldsymbol{W}_a, \quad a = 1, 2.$$

Because the mechanical linear momentum of each particle can be writeen in terms of the velocity of its center of mass, $\boldsymbol{p}_a = \gamma(v_a) m \boldsymbol{v}_a$, $a = 1, 2$. The first equation will define the acceleration of the center of mass of each particle, and this leads to define this center of mass as

$$\boldsymbol{q}_a = \boldsymbol{r}_a + \frac{(\boldsymbol{q}_a - \boldsymbol{r}_a)^2}{c^2 - \boldsymbol{v}_a \cdot \boldsymbol{u}_a}\frac{d^2\boldsymbol{r}_a}{dt^2}.$$

This amounts that the second order differential equation for the center of charge of each particle in terms of the position and velocity of the center of mass is:

$$\frac{d^2\boldsymbol{r}_a}{dt^2} = \frac{c^2 - \boldsymbol{v}_a \cdot \boldsymbol{u}_a}{(\boldsymbol{q}_a - \boldsymbol{r}_a)^2}(\boldsymbol{q}_a - \boldsymbol{r}_a), \quad a = 1, 2.$$

We shall make use of this form to describe the interaction between two electrons, which leads to the existence of bound states up to some upper relative velocity, in the section **6.6**.

4.6 Appendix: The group $\mathcal{W} \otimes SU(2)_T \otimes U(1)_Q$

Under infinitesimal time and space translations of parameters $\delta\tau$ and $\delta\boldsymbol{b}$, respectively, the kinematical variables transform as

$$t' = t + \delta\tau, \quad \boldsymbol{r}' = \boldsymbol{r} + \delta\boldsymbol{b}, \quad \boldsymbol{u}' = \boldsymbol{u}, \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha}, \quad \beta' = \beta$$

so that the self-adjoint generators of translations are

$$H = i\frac{\partial}{\partial t}, \quad \boldsymbol{P} = -i\nabla, \qquad [H, \boldsymbol{P}] = 0.$$

Under an infinitesimal spacetime dilation of normal parameter $\delta\lambda$, they transform in the way:

$$t' = t + t\delta\lambda, \quad \boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{r}\delta\lambda, \quad \boldsymbol{v}' = \boldsymbol{v}, \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha}, \quad \beta' = \beta + \delta\lambda$$

so that the generator takes the form ($\hbar = 1$)

$$D = it\frac{\partial}{\partial t} + i\boldsymbol{r}\cdot\nabla + i\frac{\partial}{\partial\beta} = tH - \boldsymbol{r}\cdot\boldsymbol{P} - B, \qquad [D, H] = -iH, \quad [D, P_j] = -iP_j.$$

To describe orientation we can represent every element of the rotation group by the three-vector $\boldsymbol{\alpha} = \alpha\boldsymbol{n}$, where α is the rotated angle and $\boldsymbol{n}$ is a unit vector along the rotation axis. This is the normal or canonical parameterization. Alternatively we can represent every rotation by the three-vector $\boldsymbol{\rho} = \tan(\alpha/2)\boldsymbol{n}$. In this case, every rotation matrix takes the form

$$R(\boldsymbol{\rho})_{ij} = \frac{1}{1+\rho^2}\left((1-\rho^2)\delta_{ij} + 2\rho_i\rho_j + 2\epsilon_{ikj}\rho_k\right).$$

The advantage of this parameterization is that the composition of rotations $R(\boldsymbol{\rho}') = R(\boldsymbol{\mu})R(\boldsymbol{\rho})$ takes the simple form

$$\boldsymbol{\rho}' = \frac{\boldsymbol{\mu} + \boldsymbol{\rho} + \boldsymbol{\mu}\times\boldsymbol{\rho}}{1 - \boldsymbol{\mu}\cdot\boldsymbol{\rho}}.$$

Under an infinitesimal rotation of parameter $\delta\boldsymbol{\mu} = \delta\boldsymbol{\alpha}/2$, in terms of the normal parameter, the kinematical variables transform as

$$\begin{aligned}
\delta t &= 0, \quad \delta\beta = 0 \\
\delta r_i &= -2\epsilon_{ijk} r_j \delta\mu_k \\
\delta u_i &= -2\epsilon_{ijk} u_j \delta\mu_k \\
\delta\rho_i &= \left(\delta_{ik} + \rho_i\rho_k + \epsilon_{ikl}\rho_l\right)\delta\mu_k,
\end{aligned}$$

so that the variation of the kinematical variables per unit of normal rotation parameter $\delta\alpha_k$ is

$$\begin{aligned}
\delta t_k &= 0, \quad \delta\beta_k = 0 \\
\delta r_{ik} &= -\epsilon_{ijk} r_j \\
\delta u_{ik} &= -\epsilon_{ijk} u_j \\
\delta\rho_{ik} &= \frac{1}{2}\left[\delta_{ik} + \rho_i\rho_k + \epsilon_{ikl}\rho_l\right],
\end{aligned}$$

and the self-adjoint generators J_k, are

$$J_k = i\epsilon_{ijk} r_j \frac{\partial}{\partial r_i} + i\epsilon_{ijk} u_j \frac{\partial}{\partial u_i} + \frac{1}{2i}\left(\frac{\partial}{\partial\rho_k} + \rho_k\rho_i\frac{\partial}{\partial\rho_i} + \epsilon_{ikl}\rho_l\frac{\partial}{\partial\rho_i}\right).$$

They can be separated into three parts, according to the differential operators involved, with respect to the three kinds of kinematical variables $\boldsymbol{r}$, $\boldsymbol{u}$ and $\boldsymbol{\rho}$, respectively:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{Z} + \boldsymbol{W},$$

$$L_k = i\epsilon_{ijk} r_j \frac{\partial}{\partial r_i},$$

$$Z_k = i\epsilon_{ijk} u_j \frac{\partial}{\partial u_i}, \quad W_k = \frac{1}{2i}\left(\frac{\partial}{\partial \rho_k} + \rho_k \rho_i \frac{\partial}{\partial \rho_i} + \epsilon_{ikl}\rho_l \frac{\partial}{\partial \rho_i}\right). \tag{4.81}$$

They satisfy the angular momentum commutation rules and commute among themselves:

$$[L_j, L_k] = i\epsilon_{jkl} L_l, \quad [Z_j, Z_k] = i\epsilon_{jkl} Z_l, \quad [W_j, W_k] = i\epsilon_{jkl} W_l,$$

$$[\boldsymbol{L}, \boldsymbol{Z}] = [\boldsymbol{L}, \boldsymbol{W}] = [\boldsymbol{Z}, \boldsymbol{W}] = 0.$$

and thus

$$[J_j, J_k] = i\epsilon_{jkl} J_l, \quad [\boldsymbol{J}, H] = [\boldsymbol{J}, D] = 0, \quad [J_j, P_k] = i\epsilon_{jkl} P_l.$$

The above orientation variable $\boldsymbol{\rho}$, under a general boost of velocity $\boldsymbol{v}$, transforms as

$$\boldsymbol{\rho}' = \frac{\boldsymbol{\rho} + \boldsymbol{F}(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\rho})}{1 + G(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\rho})},$$

where

$$\boldsymbol{F}(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\rho}) = \frac{\gamma(v)}{1+\gamma(v)} \left(\boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{v}(\boldsymbol{u} \cdot \boldsymbol{\rho}) + (\boldsymbol{u} \times \boldsymbol{\rho}) \times \boldsymbol{v}\right),$$

$$G(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\rho}) = \frac{\gamma(v)}{1+\gamma(v)} \left(\boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{\rho})\right), \quad \gamma(v) = (1 - v^2)^{-1/2}.$$

Finally, under an infinitesimal boost of value $\delta\boldsymbol{v}$, $\gamma(v) \approx 1$, the kinematical variables transform as

$$\begin{aligned}
\delta t &= \boldsymbol{r} \cdot \delta\boldsymbol{v} \\
\delta \boldsymbol{r} &= t\delta\boldsymbol{v} \\
\delta \boldsymbol{u} &= \delta\boldsymbol{v} - \boldsymbol{u}(\boldsymbol{u} \cdot \delta\boldsymbol{v}) \\
\delta \boldsymbol{\rho} &= -\left[\boldsymbol{\rho}(\boldsymbol{u} \cdot \delta\boldsymbol{v}) + \boldsymbol{\rho}\left((\boldsymbol{u} \times \boldsymbol{\rho}) \cdot \delta\boldsymbol{v}\right) - \boldsymbol{u} \times \delta\boldsymbol{v} - \delta\boldsymbol{v}(\boldsymbol{u} \cdot \boldsymbol{\rho}) - \right. \\
&\quad \left. (\boldsymbol{u} \times \boldsymbol{\rho}) \times \delta\boldsymbol{v}\right]/2, \\
\delta\beta &= 0,
\end{aligned}$$

and the variation of these variables per unit of infinitesimal velocity parameter δv_j is

$$\begin{aligned}
\delta t_j &= r_j \\
\delta r_{ij} &= t\delta_{ij} \\
\delta v_{ij} &= \delta_{ij} - u_i u_j \\
\delta \rho_{ij} &= -\frac{1}{2}\left[\rho_j u_i + \rho_i \epsilon_{jkl} u_k \rho_l - \epsilon_{ikj} u_k - \delta_{ij} u_k \rho_k\right], \\
\delta \beta_j &= 0,
\end{aligned}$$

so that the boost generators K_j have the form

$$\begin{aligned}
K_j &= i r_j \frac{\partial}{\partial t} + it \frac{\partial}{\partial r_j} + i\left(\frac{\partial}{\partial u_j} - u_j u_i \frac{\partial}{\partial u_i}\right) + \\
&\quad \frac{1}{2i}\left(\rho_j u_i \frac{\partial}{\partial \rho_i} + \rho_i \epsilon_{jkl} u_k \rho_l \frac{\partial}{\partial \rho_i} - \epsilon_{ikj} u_k \frac{\partial}{\partial \rho_i} - u_k \rho_k \frac{\partial}{\partial \rho_j}\right)
\end{aligned}$$

Similarly, the generators K_j can be decomposed into three parts, according to the differential operators involved and we represent them with the same capital letters as in the case of the $\boldsymbol{J}$ operators but with a tilde:

$$\boldsymbol{K} = \widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{Z}} + \widetilde{\boldsymbol{W}}, \quad \widetilde{L}_j = ir_j \frac{\partial}{\partial t} + it \frac{\partial}{\partial r_j}, \quad \widetilde{Z}_j = i\left(\frac{\partial}{\partial u_j} - u_j u_i \frac{\partial}{\partial u_i}\right),$$

$$\widetilde{W}_j = \frac{1}{2i}\left(\rho_j u_i \frac{\partial}{\partial \rho_i} + \rho_i \epsilon_{jkl} u_k \rho_l \frac{\partial}{\partial \rho_i} + \epsilon_{jki} u_k \frac{\partial}{\partial \rho_i} - u_k \rho_k \frac{\partial}{\partial \rho_j}\right)$$

They satisfy the commutation rules:

$$[\widetilde{L}_j, \widetilde{L}_k] = -i\epsilon_{jkl} L_l, \quad [\widetilde{Z}_j, \widetilde{Z}_k] = -i\epsilon_{jkl} Z_l, \quad [\widetilde{\boldsymbol{L}}, \widetilde{\boldsymbol{Z}}] = [\widetilde{\boldsymbol{L}}, \widetilde{\boldsymbol{W}}] = 0,$$

and also

$$[K_j, K_k] = -i\epsilon_{jkl} J_l.$$

We can check that

$$\widetilde{\boldsymbol{Z}} = \boldsymbol{u} \times \boldsymbol{Z}, \quad \widetilde{\boldsymbol{W}} = \boldsymbol{u} \times \boldsymbol{W}.$$

If we define the spin operator $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$, and the part of the kinematical momentum $\widetilde{\boldsymbol{S}} = \widetilde{\boldsymbol{Z}} + \widetilde{\boldsymbol{W}} = \boldsymbol{u} \times \boldsymbol{S}$, they satisfy

$$[S_j, S_k] = i\epsilon_{jkl} S_l, \quad [S_j, \widetilde{S}_k] = i\epsilon_{jkl} \widetilde{S}_l, \quad [\widetilde{S}_j, \widetilde{S}_k] = -i\epsilon_{jkl} S_l,$$

where in the last expression we have used the constraint $u^2 = 1$. They generate the Lie algebra of a Lorentz group which commutes with spacetime translations $[\boldsymbol{S}, p^\mu] = [\widetilde{\boldsymbol{S}}, p^\mu] = 0$.

With respect to the part $SU(2)_T$, let us calculate its generators. This group, with infinitesimal parameters $\delta\nu_k$, when acting on the kinematical variables in the form:

$$\delta t = 0, \quad \delta\beta = 0, \quad \delta r_i = 0, \quad \delta u_i = 0, \quad \delta\rho_i = (\delta_{ik} + \rho_i\rho_k + \epsilon_{ikl}\rho_l)\,\delta\nu_k,$$

In the $\boldsymbol{\rho}$ parameterization of the rotation group, the unit vectors of the body frame $\boldsymbol{e}_i$, $i = 1, 2, 3$ have the following components:

$$(\boldsymbol{e}_i)_j = R(\boldsymbol{\rho})_{ji},$$

so that the $T_k = \boldsymbol{e}_k \cdot \boldsymbol{W}$ operators of projecting the rotational angular momentum $\boldsymbol{W}$ onto the body frame are given by

$$T_k = \frac{1}{2i}\left(\frac{\partial}{\partial \rho_k} + \rho_k\rho_i \frac{\partial}{\partial \rho_i} - \epsilon_{ikl}\rho_l \frac{\partial}{\partial \rho_i}\right). \tag{4.82}$$

They differ from the W_k in (4.81) by the change of $\boldsymbol{\rho}$ by $-\boldsymbol{\rho}$, followed by a global change of sign. They satisfy the commutation relations

$$[T_j, T_k] = -i\epsilon_{jkl} T_l. \tag{4.83}$$

The minus sign on the right hand side of (4.83) corresponds to the difference between the active and passive point of view of transformations. The rotation of the laboratory axis (passive rotation) has as generators the $\boldsymbol{J}$, which satisfy $[J_j, J_k] = i\epsilon_{jkl} J_l$. The T_i correspond to the generators of rotations of the particle axis (active rotation), so that, the generators $-T_i$ will also be passive generators of rotations and satisfy $[-T_j, -T_k] = i\epsilon_{jkl}(-T_l)$.

In the normal parameterization of rotations $\boldsymbol{\alpha} = \alpha\boldsymbol{n}$, if we describe the unit vector $\boldsymbol{n}$ along the rotation axis by the usual polar and azimuthal angles θ and ϕ, respectively, so that $\boldsymbol{n} \equiv (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the above W_i generators take the form

$$\begin{aligned} W_1 \;=\; & \frac{1}{2i}\left[2\sin\theta\,\cos\phi\,\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta\,\cos\phi}{\tan(\alpha/2)} - \sin\phi\right)\frac{\partial}{\partial\theta} - \right. \\ & \left. \left(\frac{\sin\phi}{\tan(\alpha/2)\sin\theta} + \frac{\cos\theta\,\cos\phi}{\sin\theta}\right)\frac{\partial}{\partial\phi}\right], \end{aligned} \tag{4.84}$$

$$W_2 = \frac{1}{2i}\left[2\sin\theta\,\sin\phi\,\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta\,\sin\phi}{\tan(\alpha/2)} + \cos\phi\right)\frac{\partial}{\partial\theta} - \left(\frac{\cos\theta\,\sin\phi}{\sin\theta} - \frac{\cos\phi}{\tan(\alpha/2)\sin\theta}\right)\frac{\partial}{\partial\phi}\right], \tag{4.85}$$

$$W_3 = \frac{1}{2i}\left[2\cos\theta\,\frac{\partial}{\partial\alpha} - \frac{\sin\theta}{\tan(\alpha/2)}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\right], \tag{4.86}$$

$$W^2 = -\left[\frac{\partial^2}{\partial\alpha^2} + \frac{1}{\tan(\alpha/2)}\frac{\partial}{\partial\alpha} + \frac{1}{4\sin^2(\alpha/2)}\left\{\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\right], \tag{4.87}$$

$$W_+ = W_1 + iW_2 = \frac{e^{i\phi}}{2i}\left[2\sin\theta\,\frac{\partial}{\partial\alpha} + \frac{\cos\theta}{\tan(\alpha/2)}\,\frac{\partial}{\partial\theta} + i\,\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\,\frac{\partial}{\partial\phi} + \frac{i}{\tan((\alpha/2))\sin\theta}\,\frac{\partial}{\partial\phi}\right], \tag{4.88}$$

$$W_- = W_1 - iW_2 = \frac{e^{-i\phi}}{2i}\left[2\sin\theta\,\frac{\partial}{\partial\alpha} + \frac{\cos\theta}{\tan(\alpha/2)}\,\frac{\partial}{\partial\theta} - i\,\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\,\frac{\partial}{\partial\phi} - \frac{i}{\tan(\alpha/2)\sin\theta}\,\frac{\partial}{\partial\phi}\right], \tag{4.89}$$

and the passive T_i generators take the form

$$T_1 = \frac{-i}{2}\left[2\sin\theta\,\cos\phi\,\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta\,\cos\phi}{\tan(\alpha/2)} + \sin\phi\right)\frac{\partial}{\partial\theta} - \left(\frac{\sin\phi}{\tan(\alpha/2)\sin\theta} - \frac{\cos\theta\,\cos\phi}{\sin\theta}\right)\frac{\partial}{\partial\phi}\right], \tag{4.90}$$

$$T_2 = \frac{-i}{2}\left[2\sin\theta\,\sin\phi\,\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta\,\sin\phi}{\tan(\alpha/2)} - \cos\phi\right)\frac{\partial}{\partial\theta} - \left(-\frac{\cos\theta\,\sin\phi}{\sin\theta} - \frac{\cos\phi}{\tan(\alpha/2)\sin\theta}\right)\frac{\partial}{\partial\phi}\right], \tag{4.91}$$

$$T_3 = \frac{-i}{2}\left[2\cos\theta\,\frac{\partial}{\partial\alpha} - \frac{\sin\theta}{\tan(\alpha/2)}\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\phi}\right]. \tag{4.92}$$

T_i are related to W_i by changing α into $-\alpha$.

With respect to the part $U(1)_Q$, of generator Q, which commutes with the remaining generators, the action of an infinitesimal element $\delta\chi$ of this este group acting on the kinematical variables is:

$$\delta t = 0, \quad \delta\beta = \delta\chi, \quad \delta r_i = 0, \quad \delta u_i = 0, \quad \delta\rho_i = 0,$$

so that the quantum representation of this generator is

$$Q = -i\frac{\partial}{\partial\beta}$$

Because it commutes with the others, we can separate variables in the wave function in terms of this new variable

$$Qf(\beta) = qf(\beta), \quad f(\beta) \sim e^{iq\beta},$$

and the general form of the wave function can be written as

$$\psi(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha}, \beta) = e^{iq\beta}\Psi(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha}),$$

and the new variable β apears as a general phase.

The normalized eigenvectors of $W^2 = T^2$ and W_3 and T_3 for $w = t = 1/2$, written in the form $|w_3, t_3>$, (which are also eigenvectors of Z^2 with $z = 0$) are written as $|0; s_3, t_3>$

$$\Phi_1 = |1/2, -1/2> = i\sqrt{2}\sin(\alpha/2)\sin\theta e^{i\phi}, \tag{4.93}$$

$$\Phi_2 = |-1/2, -1/2> = \sqrt{2}\left(\cos(\alpha/2) - i\cos\theta\sin(\alpha/2)\right) \tag{4.94}$$

$$\Phi_3 = |1/2, 1/2> = -\sqrt{2}\left(\cos(\alpha/2) + i\cos\theta\sin(\alpha/2)\right), \tag{4.95}$$

$$\Phi_4 = |-1/2, 1/2> = -i\sqrt{2}\sin(\alpha/2)\sin\theta e^{-i\phi}. \tag{4.96}$$

The rising and lowering operators $W_\pm$ and the corresponding $T_\pm$ transform them among each other. $\{\Phi_1, \Phi_2\}$ are related by $W_\pm$, and similarly the $\{\Phi_3, \Phi_4\}$ while the sets $\{\Phi_1, \Phi_3\}$ and $\{\Phi_2, \Phi_4\}$ are separately related by $T_\pm$. For instance

$$W_-\Phi_1 = \Phi_2, \quad W_-\Phi_2 = 0, \quad W_-\Phi_3 = \Phi_4,$$

$$T_-\Phi_1 = \Phi_3, \quad T_-\Phi_3 = 0, \quad T_-\Phi_2 = \Phi_4.$$

They form an orthonormal set with respect to the normalized invariant measure defined on $SU(2)$

$$dg(\alpha, \theta, \phi) = \frac{1}{4\pi^2}\sin^2(\alpha/2)\sin\theta\, d\alpha\, d\theta\, d\phi,$$

$$\alpha \in [0, 2\pi], \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi].$$

$$\int_{SU(2)} dg(\alpha, \theta, \phi) = 1.$$

The wavefunction ψ can be separated in the form

$$\psi(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha}) = \sum_{i=1}^{i=4} \phi_i(t, \boldsymbol{r})\chi_i(\boldsymbol{u}, \boldsymbol{\alpha})$$

where the four χ_i can be classified according to the eigenvalues $|s_3, t_3>$. The functions $\phi_i(t, \boldsymbol{r})$ can be chosen as eigenfunctions of the Klein-Gordon operator

$$p_\mu p^\mu \phi_i(t, \boldsymbol{r}) = m_i^2 \phi_i(t, \boldsymbol{r}).$$

The functions $\chi(\boldsymbol{u}, \boldsymbol{\alpha})$ can also be separated because the total spin $\boldsymbol{S}$ with $s = 1/2$, is the sum of the two parts $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$, with $[\boldsymbol{Z}, \boldsymbol{W}] = 0$, so that since the $\boldsymbol{W}$ part contributes with $w = 1/2$ then the $\boldsymbol{Z}$ part contributes with $z = 0$ or $z = 1$. The $z = 0$ contribution corresponds to the functions $\chi_i(\boldsymbol{\alpha})$ independent of the velocity variables and the orthonormal set are the above Φ_i, $i = 1, 2, 3, 4$, which can also be written in the form $|z; s_3, t_3>$, with $z = 0$.

Because $\boldsymbol{Z} = -i\boldsymbol{u} \times \nabla_u$, for the $z = 1$ part the eigenvectors of Z^2 and Z_3 are the spherical harmonics $Y_1^i(\beta, \lambda)$, $i = -1, 0, 1$. The variables β and λ represent the orientation of the velocity vector $\boldsymbol{u}$. Because $[Z_i, W_j] = 0$, we can again separate the variables in the functions $\chi(\boldsymbol{u}, \boldsymbol{\alpha})$.

In this case the $\chi(\boldsymbol{u}, \boldsymbol{\alpha}) = \sum \phi_i(\beta, \lambda)\lambda_i(\alpha, \theta, \phi)$. The four orthonormal vectors, eigenvectors of S_3, Z^2 with $z = 1$ and T_3, $|1; s_3, t_3>$, are now

$$\Psi_1 = |1; 1/2, 1/2 >= \frac{1}{\sqrt{3}}\left(Y_1^0(\beta,\lambda)\Phi_1 - \sqrt{2}Y_1^1(\beta,\lambda)\Phi_2\right), \tag{4.97}$$

$$\Psi_2 = |1; -1/2, 1/2 >= \frac{1}{\sqrt{3}}\left(-Y_1^0(\beta,\lambda)\Phi_2 + \sqrt{2}Y_1^{-1}(\beta,\lambda)\Phi_1\right), \tag{4.98}$$

$$\Psi_3 = |1; 1/2, -1/2 >= \frac{1}{\sqrt{3}}\left(Y_1^0(\beta,\lambda)\Phi_3 - \sqrt{2}Y_1^1(\beta,\lambda)\Phi_4\right), \tag{4.99}$$

$$\Psi_4 = |1; -1/2, -1/2 >= \frac{1}{\sqrt{3}}\left(-Y_1^0(\beta,\lambda)\Phi_4 + \sqrt{2}Y_1^{-1}(\beta,\lambda)\Phi_3\right). \tag{4.100}$$

where Φ_i are the same as those in (4.93-4.96) and the spherical harmonics $Y_1^i(\beta, \lambda)$ are

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin(\beta)e^{i\widetilde{\lambda}}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos(\beta), \quad Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\sin(\beta)e^{-i\widetilde{\lambda}}. \tag{4.101}$$

The Z_i operators are given by

$$Z_1 = i\sin\lambda\frac{\partial}{\partial\beta} + i\frac{\cos\beta}{\sin\beta}\cos\lambda\frac{\partial}{\partial\lambda}, \quad Z_2 = -i\cos\lambda\frac{\partial}{\partial\beta} + i\frac{\cos\beta}{\sin\beta}\sin\lambda\frac{\partial}{\partial\lambda},$$

$$Z_3 = -i\frac{\partial}{\partial\lambda}.$$

The rising and lowering operators $Z_\pm$ are

$$Z_\pm = e^{\pm i\lambda}\left(\pm\frac{\partial}{\partial\beta} + i\frac{\cos\beta}{\sin\beta}\frac{\partial}{\partial\lambda}\right),$$

so that

$$Z_- Y_1^1 = \sqrt{2}\, Y_1^0, \quad Z_- Y_1^0 = \sqrt{2}\, Y_1^{-1}.$$

The four spinors Ψ_i are orthonormal with respect to the invariant measure

$$dg(\beta, \lambda; \alpha, \theta, \phi) = \frac{1}{4\pi^2}\sin^2(\alpha/2)\sin\theta\sin\beta\, d\alpha\, d\theta\, d\phi\, d\beta d\lambda$$

$$\alpha \in [0, 2\pi], \quad \beta, \theta \in [0, \pi], \quad \lambda, \phi \in [0, 2\pi].$$

Similarly as before, the rising and lowering operators $S_\pm = Z_\pm + W_\pm$ and the corresponding $T_\pm$ transform the Ψ_i among each other. In particular $\{\Psi_1, \Psi_2\}$ are related by $S_\pm$, and similarly $\{\Psi_3, \Psi_4\}$ while the sets $\{\Psi_1, \Psi_3\}$ and $\{\Psi_2, \Psi_4\}$ are separately related by $T_\pm$. This is the reason why the general spinor in this representation is a four-component object.

In the $z = 0$ basis Φ_i (4.93-4.96), the spin operators and the basis vectors of the body frame take the form

$$\boldsymbol{S} = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix} = \boldsymbol{W},$$

$$T_1 = \frac{1}{2}\begin{pmatrix}0 & \mathbb{I}\\ \mathbb{I} & 0\end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix}0 & -i\mathbb{I}\\ i\mathbb{I} & 0\end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{pmatrix},$$

$$\boldsymbol{e}_1 = \frac{-1}{3}\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}, \quad \boldsymbol{e}_2 = \frac{-1}{3}\begin{pmatrix}0 & -i\boldsymbol{\sigma}\\ i\boldsymbol{\sigma} & 0\end{pmatrix}, \quad \boldsymbol{e}_3 = \frac{-1}{3}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & -\boldsymbol{\sigma}\end{pmatrix},$$

in terms of the Pauli $\boldsymbol{\sigma}$ matrices and the 2×2 unit matrix $\mathbb{I}$.

In the $z = 1$ basis Ψ_i (4.97-4.100), the operators S_i and T_i take the same matrix form as above, while the $\boldsymbol{e}_i$ are

$$\boldsymbol{e}_1 = \frac{1}{9}\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \frac{1}{9}\begin{pmatrix} 0 & -i\boldsymbol{\sigma} \\ i\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{e}_3 = \frac{1}{9}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}.$$

In all cases, the 6 Hermitian traceless matrices S_i, T_j, the nine Hermitian traceless matrices e_{ij} and the 4×4 unit matrix are linearly independent and they completely define a Hermitian basis for Dirac's algebra, so that any other translation invariant observable of the particle will be expressed as a real linear combination of the above 16 Hermitian matrices. We have used this fact previously to explicitely obtain Dirac's equation for this model.

Both representations are orthogonal to each other, $< \Phi_i | \Psi_j >= 0$, and they produce two different irreducible representations of the group, so that they describe two different kinds of particles of the same spin $1/2$.

The matrix representation of the Z_i and W_i operators in the basis Ψ_i are given by

$$\boldsymbol{Z} = \frac{2}{3}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \boldsymbol{W} = \frac{-1}{6}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix},$$

although the spinors Ψ_i are not eigenvectors of Z_3 and W_3. In the basis Φ_i, the representation of the Z_i are vanishing matrices because in this case $\boldsymbol{S} = \boldsymbol{W}$. We see that $\boldsymbol{Z}$ is a positive multiple, greater than $\boldsymbol{S}$ and therefore it has the same direction, while $\boldsymbol{W}$ has the opposite direction, as it corresponds to the picture of the front page.

In both cases $z = 0$ and $z = 1$, we have that

$$T_1\boldsymbol{e}_1 + T_2\boldsymbol{e}_2 + T_3\boldsymbol{e}_3 = \boldsymbol{e}_1 T_1 + \boldsymbol{e}_2 T_2 + \boldsymbol{e}_3 T_3 = \boldsymbol{W},$$

which justifies that the operators T_i are the components of the rotative angular momentum $\boldsymbol{W}$, projected into the corresponding axis of the local frame $\boldsymbol{e}_i$.

Chapter 5

Electromagnetic structure of the electron

5.1 Structure of the spinning electron

Let us consider that the classical electron is described by the model whose charge is moving in circles at the speed of light in the center of mass frame.

One of the immediate questions concerning the classical structure of the electron is, what is the associated electromagnetic field of the particle? We see that the charge is accelerated and according to the classical electromagnetic theory, the particle must necessarily radiate continuously. However, from the mechanical point of view we have produced a classical free system, such that properties like the mechanical energy and mechanical linear and angular momentum are conserved in time. The Lagrangian that describes the system is Poincaré invariant, and if we think about a free system, the corresponding field structure cannot produce loss of energy and linear momentum. The free particle has to have associated an electromagnetic field without radiation. Radiation has to be produced whenever the center of mass of the particle is accelerated, *i.e.*, when the particle is no longer free.

There must exist radiationless solutions of Maxwell's equations, associated to point charges moving in circles at the speed of light. One possibility is to consider solutions derived from the Liénard-Wiechert potentials $(A^{\mu}_{ret}+A^{\mu}_{adv})/2$, where A^{μ}_{ret} and A^{μ}_{adv} are the corresponding retarded and advanced potentials. But, even if we take as the probable electric field $(\boldsymbol{E}_{ret}+\boldsymbol{E}_{adv})/2$, it is neither static nor Coulomb-like, and therefore it does not look like the estimated electric field of a point electron. Nevertheless, the point particle model is an approximation of an elementary particle considered as a spinless particle, and in nature there seems to exist no spinless particles.

We are going to compute the instantaneous electromagnetic field of the model of Dirac particle and consider next a static solution: the time average retarded field during a complete turn of the charge. We are going to see the Coulomb behaviour of this field but not of the corresponding advanced time average field.

5.1.1 Covariant formulation

We can give a covariant formulation of the Lienard-Wiechert potentials and also of the electric and magnetic fields which are independent of the evolution parameter τ. Let us call $z^{\mu}(\tau) \equiv (ct_p(\tau), \boldsymbol{r}_p(\tau))$, to the kinematical variables which describe the evolution of the center of charge of an elementary particle. Let $x^{\mu} \equiv (ct, \boldsymbol{x})$ be the instant and location of the observation point and call $R^{\mu} = x^{\mu} - z^{\mu}(\tau)$. The covariant expression of the Lienard-Wiechert potentials

generated by the charge at that point is written as,

$$A^\mu(x) = \frac{e}{c}\frac{\dot{z}^\mu(\tau)}{(x - z(\tau))^\nu \dot{z}_\nu(\tau)} = \frac{e}{c}\frac{\dot{z}^\mu}{R^\nu \dot{z}_\nu}, \quad e = \frac{q}{4\pi\epsilon_0}, \quad A^\mu \equiv \frac{e}{c}\frac{\{1, \boldsymbol{\beta}\}}{R(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})},$$

where τ is the value of the arbitrary evolution parameter at the retarded instant, $\boldsymbol{\beta} = \boldsymbol{u}_p(\tau)/c$, $R = |\boldsymbol{R}|$ and $\boldsymbol{n} = \boldsymbol{R}/R$. The potentials at that point are not independent because $A_\mu R^\mu = e/c$ and it implies that $R^\nu R_\nu = 0$ at the retarded instant. If we take the derivative of this expression with respect to x^μ, taking into account that $\tau(x)$ is a function of the observation point x, and using the most compact notation $(a \cdot b) \equiv a^\mu b_\mu$ we get

$$\partial_\mu(R \cdot R) = 0 = R_\mu - (R \cdot \dot{z})\frac{\partial \tau}{\partial x^\mu}, \quad \tau_{,\mu} \equiv \frac{\partial \tau}{\partial x^\mu} = \frac{R_\mu}{(R \cdot \dot{z})} \equiv \frac{\{1, -\boldsymbol{n}\}}{(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})c\dot{t}_p}.$$

In this way, by supressing the global factor e/c,

$$A_{\mu,\nu} = \frac{\ddot{z}_\mu \tau_{,\nu}}{(R \cdot \dot{z})} - \frac{\dot{z}_\mu}{(R \cdot \dot{z})^2}\left(\dot{z}_\nu + (R \cdot \ddot{z})\tau_{,\nu} - (\dot{z} \cdot \dot{z})\tau_{,\nu}\right)$$

and after the substitution of $\tau_{,\nu}$ it leads to

$$A_{\mu,\nu} = \frac{\ddot{z}_\mu R_\nu}{(R \cdot \dot{z})^2} - \frac{\dot{z}_\mu \dot{z}_\nu}{(R \cdot \dot{z})^2} - \frac{\dot{z}_\mu (R \cdot \ddot{z}) R_\nu}{(R \cdot \dot{z})^3} + \frac{\dot{z}_\mu (\dot{z} \cdot \dot{z}) R_\nu}{(R \cdot \dot{z})^3}$$

If we rise the index ν

$$A_\mu{}^{,\nu} = \frac{\ddot{z}_\mu R^\nu}{(R \cdot \dot{z})^2} - \frac{\dot{z}_\mu \dot{z}^\nu}{(R \cdot \dot{z})^2} - \frac{\dot{z}_\mu (R \cdot \ddot{z}) R^\nu}{(R \cdot \dot{z})^3} + \frac{\dot{z}_\mu (\dot{z} \cdot \dot{z}) R^\nu}{(R \cdot \dot{z})^3}.$$

After contraction of μ with ν, is easy to see that the first term cancels out with the third and also the second with the fourth, and thus $A_\mu{}^{,\mu} = 0$ and the potentials satisfy Lorenz condition.

The field, without the global factor e/c, looks like

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = \frac{\dot{z}_\nu R_\mu - \dot{z}_\mu R_\nu}{(R \cdot \dot{z})^3}\left((\dot{z} \cdot \dot{z}) - (R \cdot \ddot{z})\right) + \frac{\ddot{z}_\nu R_\mu - \ddot{z}_\mu R_\nu}{(R \cdot \dot{z})^2}. \tag{5.1}$$

This expression of the electromagnetic field can also be written as

$$F_{\mu\nu} = \frac{1}{(R \cdot \dot{z})}\frac{d}{d\tau}\left(\frac{\dot{z}_\nu R_\mu - \dot{z}_\mu R_\nu}{(R \cdot \dot{z})}\right). \tag{5.2}$$

The potentials and the fields are homogenoeus functions of zero degree of the derivatives with respect to τ of the kinematical variables of the particle which creates the field, and therfore the result is independent of the evolution parameter used to describe the evolution of the sources of the field.

Jacobi's identities

The field of a point charge (5.1) satisfy Jacobi's identities

$$R_\alpha F_{\mu\nu} + R_\mu F_{\nu\alpha} + R_\nu F_{\alpha\mu} = 0, \quad \forall \alpha, \mu, \nu.$$

With $\alpha = 0$, $\mu = i$ And $\nu = j$, we arrive to

$$F_{ij} = \frac{R_i F_{j0} - R_j F_{i0}}{R_0}, \quad \boldsymbol{B} = \frac{1}{c}\boldsymbol{n} \times \boldsymbol{E},$$

because $F_{12} = -B_z$, $F_{10} = -E_x/c$.

Time of the particle

If we use as evolution parameter the time of the particle, $\tau = t_p$, at the retarded instant, since $R^\mu \equiv (R, \boldsymbol{R})$, we get

$$\dot{z}^\mu \equiv (c, \boldsymbol{u}(t_p)), \quad \ddot{z}^\mu \equiv (0, \boldsymbol{a}(t_p)), \quad (R\cdot\dot{z}) = cR(1-\boldsymbol{n}\cdot\boldsymbol{\beta}), \quad (\dot{z}\cdot\dot{z}) = c^2(1-\beta^2), \quad (R\cdot\ddot{z}) = -\boldsymbol{R}\cdot\boldsymbol{a}.$$

Because $E_i/c = F_{0i}$, And $R_0 = R$, the first term of (5.1) gives rise to the velocity field, which looks

$$\frac{1}{c}E_i = \frac{q}{4\pi\epsilon_0 c}\frac{-u_i R + cR_i}{c^3R^3(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}c^2(1-\beta^2), \quad \boldsymbol{E}_\beta = \frac{q}{4\pi\epsilon_0}\frac{1-\beta^2}{R^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}(\boldsymbol{n}-\boldsymbol{\beta}).$$

The acceleration terms of (5.1) give rise, respectively, to

$$\frac{-\boldsymbol{u}R + c\boldsymbol{R}}{c^3R^3(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}(\boldsymbol{R}\cdot\boldsymbol{a}), \quad -\frac{\boldsymbol{a}}{c^2R(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2}$$

and finally, in the International System of Units the acceleration field is

$$\boldsymbol{E}_a = \frac{q}{4\pi\epsilon_0}\left[\frac{(\boldsymbol{n}-\boldsymbol{\beta})(\boldsymbol{n}\cdot\boldsymbol{a})}{c^2R(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3} - \frac{\boldsymbol{a}}{c^2R(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2}\right] = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{n}\times((\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a})}{c^2R(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3} \tag{5.3}$$

Observer time

If we use as evolution parameter the time of the observer $\tau = t$ at the retarded instant t_r, then

$$\dot{z}^\mu \equiv \left(c\frac{dt_p}{dt}, \frac{d\boldsymbol{r}_p}{dt}\right), \quad \ddot{z}^\mu \equiv \left(c\frac{d^2t_p}{dt^2}, \frac{d^2\boldsymbol{r}_p}{dt^2}\right),$$

But from $R = c(t - t_p)$, taking the derivative of this expression with respect to t, it gives

$$\frac{dt_p}{dt} = 1 - \frac{1}{c}\frac{dR}{dt}, \quad \frac{d^2t_p}{dt^2} = -\frac{1}{c}\frac{d^2R}{dt^2}$$

If we also take the derivative with respect to t of $(R\cdot R) = R^2 - \boldsymbol{R}^2 = 0$, we get:

$$2R\frac{dR}{dt} - 2\boldsymbol{R}\cdot\frac{d\boldsymbol{R}}{dt} = 0, \quad R\frac{dR}{dt} - \boldsymbol{R}\cdot\frac{d\boldsymbol{R}}{dt} = 0,$$

and from $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_p$, taking the derivative with respect to t,

$$\frac{d\boldsymbol{R}}{dt} = -\frac{d\boldsymbol{r}_p}{dt}, \quad \frac{d^2\boldsymbol{r}_p}{dt^2} = -\frac{d^2\boldsymbol{R}}{dt^2}.$$

and thus

$$R^\mu = (R, \boldsymbol{R}), \quad \dot{z}^\mu = \left(c - \frac{dR}{dt}, -\frac{d\boldsymbol{R}}{dt}\right), \quad \ddot{z}^\mu = \left(-\frac{d^2R}{dt^2}, -\frac{d^2\boldsymbol{R}}{dt^2}\right).$$

The part

$$(R\cdot\dot{z}) = Rc - \left(R\frac{dR}{dt} - \boldsymbol{R}\cdot\frac{d\boldsymbol{R}}{dt}\right) = Rc,$$

because the term in brackets vanishes.

$$(\dot{z}\cdot\dot{z}) = c^2\left(1 - \frac{1}{c}\frac{dR}{dt}\right)^2 - \left(\frac{d\boldsymbol{R}}{dt}\right)^2 = c^2\left(1 - \frac{2}{c}\frac{dR}{dt}\right) + \left(\frac{dR}{dt}\right)^2 - \left(\frac{d\boldsymbol{R}}{dt}\right)^2$$

$$(R\cdot\ddot{z}) = -R\frac{d^2R}{dt^2} + \boldsymbol{R}\cdot\frac{d^2\boldsymbol{R}}{dt^2}, \quad (\dot{z}\cdot\dot{z}) - (R\cdot\ddot{z}) = c^2\left(1 - \frac{2}{c}\frac{dR}{dt}\right) + \frac{1}{2}\frac{d^2}{dt^2}\left(R^2 - \boldsymbol{R}^2\right) = c^2\left(1 - \frac{2}{c}\frac{dR}{dt}\right).$$

because the last term $R^2 - \boldsymbol{R}^2 = 0$. Since

$$\dot{z}_i R_0 - \dot{z}_0 R_i = -\frac{d\boldsymbol{r}_{pi}}{dt} R + c\frac{dt_p}{dt} R_i = R\frac{dR_i}{dt} + \left(c - \frac{dR}{dt}\right) R_i,$$

$$\ddot{z}_i R_0 - \ddot{z}_0 R_i = -\frac{d^2\boldsymbol{r}_{pi}}{dt^2} R + c\frac{d^2 t_p}{dt^2} R_i = R\frac{d^2 R_i}{dt^2} - \frac{d^2 R}{dt^2} R_i,$$

the field becomes:

$$\frac{1}{c}\boldsymbol{E} = \frac{e}{c}\frac{1}{c^3 R^3}\left(R\frac{d\boldsymbol{R}}{dt} + \left(c - \frac{dR}{dt}\right)\boldsymbol{R}\right) c^2 \left(1 - \frac{2}{c}\frac{dR}{dt}\right) + \frac{e}{c}\frac{1}{c^2 R^2}\left(R\frac{d^2\boldsymbol{R}}{dt^2} - \frac{d^2 R}{dt^2}\boldsymbol{R}\right),$$

i.e.,

$$\boldsymbol{E} = \frac{e\boldsymbol{R}}{R^3} + \frac{eR}{c}\frac{d}{dt}\left(\frac{\boldsymbol{R}}{R^3}\right) + \frac{e}{c^2}\frac{d^2}{dt^2}\left(\frac{\boldsymbol{R}}{R}\right) = \frac{q}{4\pi\epsilon_0}\left[\frac{\boldsymbol{n}}{R^2} + \frac{R}{c}\frac{d}{dt}\left(\frac{\boldsymbol{n}}{R^2}\right) + \frac{1}{c^2}\frac{d^2\boldsymbol{n}}{dt^2}\right], \tag{5.4}$$

the expression of the field of Heaviside-Feynman.

For the magnetic field we get

$$\boldsymbol{B} = \frac{1}{c}\boldsymbol{n} \times \boldsymbol{E} = \frac{\mu_0 q}{4\pi}\left[\frac{1}{R}\boldsymbol{n} \times \frac{d\boldsymbol{n}}{dt} + \frac{1}{c}\boldsymbol{n} \times \frac{d^2\boldsymbol{n}}{dt^2}\right] \tag{5.5}$$

Gauss theorem

All components of the field are relevant. Let us compute the outgoing flux of the field around a sphere of radius R centered at the charge, and at any time. All points of this surface have, at the instant t, the same retarded point for the position of the charge. We are going to take the retarded velocity in the direction of OZ and the unit vector $\boldsymbol{n}$ from the retarded charge to the observation point, at an angle θ with respect to the velocity. The part of the acceleration term of the electric field (5.3) is perpendicular to the vector $\boldsymbol{n}$, and therefore its flux vanishes. The other part has two terms, one in the direction of $\boldsymbol{n}$ and another in the direction of $\boldsymbol{\beta}$, such that when projected into the direction orthogonal to the sphere, which is the same as the direction of the vector $\boldsymbol{n}$ from the retarded point, if gives

$$\boldsymbol{E}(t, \boldsymbol{r}) = \frac{q(1-\beta^2)}{4\pi\epsilon_0 R^2 (1 - \boldsymbol{n}\cdot\boldsymbol{\beta})^3}(\boldsymbol{n} - \boldsymbol{\beta}), \quad \boldsymbol{E}\cdot\boldsymbol{n} = \frac{q(1-\beta^2)}{4\pi\epsilon_0 R^2 (1 - \boldsymbol{n}\cdot\boldsymbol{\beta})^2}, \quad dS = R^2 \sin\theta d\phi d\theta$$

$$\oint \boldsymbol{E}\cdot\boldsymbol{n} dS = \frac{q(1-\beta^2)}{2\epsilon_0}\int_0^\pi \frac{\sin\theta d\theta}{(1 - \beta\cos\theta)^2} = \frac{q}{\epsilon_0}$$

because after the change $\cos\theta = u$, $\sin\theta d\theta = -du$ and the

$$\int_1^{-1} \frac{-du}{(1-\beta u)^2} = \frac{-1}{\beta}\frac{1}{(1-\beta u)}\bigg|_1^{-1} = \frac{2}{1-\beta^2}.$$

This result is independent of the value β, and it is also valid in our case with $\beta = 1$. Therefore, in the case of the field of a Dirac particle we cannot eliminate this part of the field, because although the numerator vanishes, it also vanish $\boldsymbol{n} - \boldsymbol{\beta}$ and the denominator $(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})$ in the direction where $\boldsymbol{n} = \boldsymbol{\beta}$. In the limit, the field does not vanish, it is singular. To compute the field of spinning particles, outside the zitterbewegung plane, only the acceleration field contributes but we cannot eliminate the velocity field on the zitterbewegung plane. If we do not include this velocity, the Maxwell equation $\nabla\cdot\boldsymbol{E} = \rho/\epsilon_0$, will not be satisfied, i.e., Gauss theorem will not hold.

5.1.2 Radiation field according to Dirac

Dirac[1] considers that the electron(the positron) has a point like property, the electric charge e, localized at a point, $z^\mu(\tau)$, which correspond to the localized kinematical variables of our spinning particle. He assumes that the field created by this point is the retarded field $F^{\mu\nu}_{\rm ret}$. This field comes from the potentials A^μ which satisfy the Lorenz condition $\partial_\mu A^\mu = 0$ and the equation related to the sources $\Box A^\mu = \mu_0 j^\mu$, in the International Sistema of Units. Of all possible solutions of these equations the retarded solution is selected. If the particle is going to radiate it is necessary to be under the presence of some external field $A^\mu(x)_{\rm in}$ which satisfy the sourceless equations $\partial_\mu A^\mu(x)_{\rm in} = 0$ and $\Box A^\mu{}_{\rm in} = 0$, in the region where the particle is located, and thus the total field in the surrounding of the charge is

$$F^{\mu\nu}_{\rm act} = F^{\mu\nu}_{\rm ret} + F^{\mu\nu}_{\rm in}.$$

It is the superposition of the external field $F^{\mu\nu}_{\rm in}$ and the retarded field created by the charge. Dirac also states that by symmetry the real field can also be written as the superposition of the advanced solution $F^{\mu\nu}_{\rm adv}$, and another outgoing field $F^{\mu\nu}_{\rm out}$,

$$F^{\mu\nu}_{\rm act} = F^{\mu\nu}_{\rm adv} + F^{\mu\nu}_{\rm out},$$

which would represent the outgoing radiation from the region where the charge is located, including the radiation produced by the charge. The difference

$$F^{\mu\nu}_{\rm rad} = F^{\mu\nu}_{\rm out} - F^{\mu\nu}_{\rm in} = F^{\mu\nu}_{\rm ret} - F^{\mu\nu}_{\rm adv},$$

would be a sourceless field and would represent the electromagnetic field radiated by the electron. If the electron would be under no external field, the radiation would necessarily be zero. When defining $A^\mu_{\rm rad} = A^\mu_{\rm ret} - A^\mu_{\rm adv}$, this implies that the field radiated by the particle satisfies Lorenz condition $\partial_\mu A^\mu_{\rm rad} = 0$ and the sourceless equation $\Box A^\mu{}_{\rm rad} = 0$, because it is a field which is not linked to the particle and has abandoned it.

As another alternative to Dirac method, let us calculate this field in the surroundings of the charge, as was proposed by Barut[2]. The point is very close to the charge but in the trajectory $x = z(\tau)$, at the retarded and advanced instants $\tau_B = \tau - \sigma$, and $\tau_A = \tau + \sigma$, respectively. We shall use the expression of the fields (5.1) and we represent it as

$$F_{\mu\nu}(x) = \frac{1}{(R\cdot\dot z)^3}\left(\dot z_\nu R_\mu\left((\dot z\cdot\dot z) - (R\cdot\ddot z)\right) + \ddot z_\nu R_\mu (R\cdot\dot z)\right) - \{\nu \longleftrightarrow \mu\}$$

where $\{\nu \longleftrightarrow \mu\}$ represents the same term as the first but with ν and μ interchanged. We have to determine

$$F^{\rm rad}_{\mu\nu}(z(\tau)) = \lim_{\sigma\to 0}\left(F^{\rm ret}_{\mu\nu}(\tau_B) - F^{\rm adv}_{\mu\nu}(\tau_A)\right).$$

To do that, let us make the expansion to lowest orders of the different terms. For the retarded field we have

$$R(\tau_B) = z(\tau) - z(\tau_B) = \dot z(\tau_B)\sigma + \ddot z(\tau_B)\sigma^2/2 + \dddot z(\tau_B)\sigma^3/6 + \cdots$$

$$(R\cdot\dot z)_{\tau_B} = (\dot z\cdot\dot z)_{\tau_B}\sigma + (\dot z\cdot\ddot z)_{\tau_B}\sigma^2/2 + (\dot z\cdot\dddot z)_{\tau_B}\sigma^3/6 + \cdots = (\dot z\cdot\dot z)_{\tau_B}\sigma - (\ddot z\cdot\ddot z)_{\tau_B}\sigma^3/6 + \cdots,$$

$$(R\cdot\ddot z)_{\tau_B} = (\ddot z\cdot\dot z)_{\tau_B}\sigma + (\ddot z\cdot\ddot z)_{\tau_B}\sigma^2/2 + (\ddot z\cdot\dddot z)_{\tau_B}\sigma^3/6 + \cdots = (\ddot z\cdot\ddot z)_{\tau_B}\sigma^2/2 + (\ddot z\cdot\dddot z)_{\tau_B}\sigma^3/6 + \cdots,$$

since for the point particle $(\dot z\cdot\dot z) = c^2$, $(\dot z\cdot\ddot z) = 0$, $(\ddot z\cdot\ddot z) + (\dot z\cdot\dddot z) = 0$ and so on.

[1] P.A.M. Dirac, *Classical theory of radiating electrons*, Proc. Roy. Soc. **A 167**, 148 (1938)
[2] A.O. Barut, *Electrodynamics and classical theory of fields and particles*, Dover NY (1980).

For the advanced field we have

$$R(\tau_A) = z(\tau) - z(\tau_A) = -\dot{z}(\tau_A)\sigma + \ddot{z}(\tau_A)\sigma^2/2 - \dddot{z}(\tau_A)\sigma^3/6 + \cdots$$

$$(R\cdot\dot{z})_{\tau_A} = -(\dot{z}\cdot\dot{z})_{\tau_A}\sigma + (\dot{z}\cdot\ddot{z})_{\tau_A}\sigma^2/2 - (\dot{z}\cdot\dddot{z})_{\tau_A}\sigma^3/6 + \cdots = -(\dot{z}\cdot\dot{z})_{\tau_A}\sigma + (\ddot{z}\cdot\ddot{z})_{\tau_A}\sigma^3/6 + \cdots,$$

$$(R\cdot\ddot{z})_{\tau_A} = (\ddot{z}\cdot\dot{z})_{\tau_A}\sigma + (\ddot{z}\cdot\ddot{z})_{\tau_A}\sigma^2/2 + (\ddot{z}\cdot\dddot{z})_{\tau_A}\sigma^3/6 + \cdots = (\ddot{z}\cdot\ddot{z})_{\tau_A}\sigma^2/2 - (\ddot{z}\cdot\dddot{z})_{\tau_A}\sigma^3/6 + \cdots,$$

To lowest order, the retarded field at some intermediate point $(z(\tau))$, is

$$F^{\rm ret}_{\mu\nu}(z(\tau)) = -\frac{(\dot{z}_\nu\ddot{z}_\mu - \dot{z}_\mu\ddot{z}_\nu)_{\tau_B} + O(\sigma)}{2\sigma(\dot{z}\cdot\dot{z})^2(1-(\ddot{z}\cdot\ddot{z})/(\dot{z}\cdot\dot{z})\,\sigma^2/6)^3}.$$

For the advanced field to lowest order we also get

$$F^{\rm adv}_{\mu\nu}(z(\tau)) = -\frac{(\dot{z}_\nu\ddot{z}_\mu - \dot{z}_\mu\ddot{z}_\nu)_{\tau_A} + O(\sigma)}{2\sigma(\dot{z}\cdot\dot{z})^2(1-(\ddot{z}\cdot\ddot{z})/(\dot{z}\cdot\dot{z})\,\sigma^2/6)^3}.$$

The radiation field on the particle is

$$F^{\rm rad}_{\mu\nu} = \lim_{\sigma\to 0}(F^{\rm ret}_{\mu\nu} - F^{\rm adv}_{\mu\nu}) = \frac{1}{(\dot{z}\cdot\dot{z})^2}\lim_{\sigma\to 0}\frac{(\dot{z}_\nu\ddot{z}_\mu - \dot{z}_\mu\ddot{z}_\nu)_{\tau_A} - (\dot{z}_\nu\ddot{z}_\mu - \dot{z}_\mu\ddot{z}_\nu)_{\tau_B}}{2\sigma},$$

i.e.,

$$F^{\rm rad}_{\mu\nu} = \frac{1}{(\dot{z}\cdot\dot{z})^2}\frac{d}{d\tau}(\dot{z}_\nu\ddot{z}_\mu - \dot{z}_\mu\ddot{z}_\nu) = \frac{\dot{z}_\nu\dddot{z}_\mu - \dot{z}_\mu\dddot{z}_\nu}{(\dot{z}\cdot\dot{z})^2} \tag{5.6}$$

because $\tau_A - \tau_B = 2\sigma$, and we also see that it is a homogeneous function of zero degree of the derivatives of the kinematical variables. The result obtained by Dirac is the same as this but with a coefficint of 2/3.

If this radiation field corresponds to a photon emitted at that instant, it would produce a reaction force on the particle which would modify the linear momentum in a value

$$\frac{dp^\mu}{d\tau} = eF^{\mu\nu}_{\rm rad}\dot{z}_\nu = \frac{e^2}{4\pi\epsilon_0 c}\left(\frac{1}{(\dot{z}\cdot\dot{z})}\dddot{z}^\mu + \frac{(\ddot{z}\cdot\ddot{z})}{(\dot{z}\cdot\dot{z})^2}\dot{z}^\mu\right),$$

wher we have written $(\ddot{z}\cdot\ddot{z}) = -(\dot{z}\cdot\dddot{z})$ and including the constant coefficient in the International Sistem of units $e/4\pi\epsilon_0 c$. This force has opposite direction to the velocity of the point because the coefficient $(\ddot{z}\cdot\ddot{z}) < 0$, and the third derivative $\dddot{z}$, has opposite direction to $\dot{z}$ suggesting that the emmited photon has the direction of the velocity of the particle. For the spinning particle we have the problem that the four-vector $\dot{z}$, satisfies $(\dot{z}\cdot\dot{z}) = 0$. If we had that $(\dot{z}\cdot\dot{z}) \neq 0$, for the free Dirac particle when analyzed in the CM frame, the reaction term vanishes because the third derivative has opposite direction to the velocity. It is necessary that the motion of the particle would no longer be free to produce radiation.

In our spinning model the charge is moving at the speed c and describes a curly trajectory such that the field created expands at the speed c and never reaches the position of the charge at later times, and thus, therefore there would not exist the self-force described by Dirac.

5.1.3 The instantaneous electromagnetic field of a Dirac particle

The retarded(or advanced) electric field of a point charge at the observation point $\boldsymbol{x}$ at the instant t, is given by

$$\boldsymbol{E} = \boldsymbol{E}_\beta + \boldsymbol{E}_a, \quad e \equiv \frac{q}{4\pi\epsilon_0},$$

where

$$\boldsymbol{E}_\beta(t,\boldsymbol{x}) = \frac{e(1-\beta^2)}{R^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}(\boldsymbol{n}-\boldsymbol{\beta}), \tag{5.7}$$

$$\boldsymbol{E}_a(t,\boldsymbol{x}) = \frac{e}{Rc^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}\,\boldsymbol{n}\times((\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a})\,, \tag{5.8}$$

are the velocity and acceleration fields, respectively. The observables $\boldsymbol{r}$, $\boldsymbol{u} = d\boldsymbol{r}/dt$ and $\boldsymbol{a} = d\boldsymbol{u}/dt$, are respectively, the position, velocity and acceleration of the charge, evaluated at the retarded(or advanced) instant $t_r = t - R/c$, (or $t_a = t + R/c$). The vector $\boldsymbol{\beta} = \boldsymbol{u}/c$, and

$$\boldsymbol{n} = \frac{\boldsymbol{x}-\boldsymbol{r}}{|\boldsymbol{x}-\boldsymbol{r}|}, \quad R = |\boldsymbol{x}-\boldsymbol{r}|.$$

The magnetic field is $\boldsymbol{B} = \boldsymbol{n}\times\boldsymbol{E}/c$. Because the center of charge of a Dirac particle is moving at the speed c, $\beta = 1$, and the velocity field $\boldsymbol{E}_\beta$ vanishes, except at those points on the XOY plane where $\boldsymbol{n} = \boldsymbol{\beta}$, where it is singular. From here we deduce that outside the plane XOY the only contribution to the field is (5.8) which is always orthogonal to the unit vector $\boldsymbol{n}$ from the retarded position and which decreases at large distances like $1/R$. On the plane XOY the electric field is also perpendicular to the vector $\boldsymbol{n}$, except on those points where the retarded point satisfies $\boldsymbol{n} = \boldsymbol{\beta}$ where the field becomes infinite.

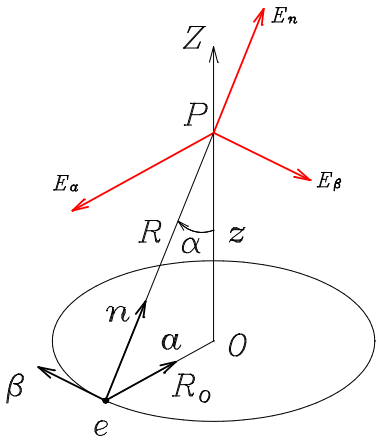


Figure 5.1: **Components $\boldsymbol{E}_n$, $\boldsymbol{E}_\beta$ and $\boldsymbol{E}_a$ of the electric field at point P. $\boldsymbol{E}_n$ and $\boldsymbol{E}_\beta$ are of the same intensity and that part of $\boldsymbol{E}_a$ projected along the vector $\boldsymbol{n}$ cancels the part $\boldsymbol{E}_n$, and thus the resultant field $\boldsymbol{E}$ is always orthogonal to the retarded unit vector $\boldsymbol{n}$ and forms in the normal plane an angle α with the vector $\boldsymbol{E}_{a_\perp}$.**

The field (5.8) has three parts, and because the acceleration $a = c^2/R_0$, is written as

$$\boldsymbol{E} = \frac{e\boldsymbol{n}(\boldsymbol{n}\cdot\widehat{\boldsymbol{a}})}{RR_0(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3} - \frac{e\boldsymbol{\beta}(\boldsymbol{n}\cdot\widehat{\boldsymbol{a}})}{RR_0(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3} - \frac{e\widehat{\boldsymbol{a}}}{RR_0(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2} = \boldsymbol{E}_n + \boldsymbol{E}_\beta + \boldsymbol{E}_a.$$

where $\widehat{\boldsymbol{a}}$ is a unit vector along the direction of the acceleration. At the point P of the OZ axis and in those units where $e = 1$ and $R_0 = 1$, because $\boldsymbol{n}\cdot\boldsymbol{\beta} = 0$, these fields take the values $E_n = \sin\alpha/R = E_\beta$ and $E_a = 1/R$ and are represented at the figure **5.1**. The component of $\boldsymbol{E}_a$ along the vector $\boldsymbol{n}$ cancels the component $\boldsymbol{E}_n$, and the resultant of the field is contained on the plane perpendicular to the vector $\boldsymbol{n}$, of intensity $1/R$ in these units and at an angle α with the projection on this plane of the component $\boldsymbol{E}_a$.

Efectively, the component of $\boldsymbol{E}_a$ orthogonal to $\boldsymbol{n}$, denoted by $\boldsymbol{E}_{a_\perp}$ is of intensity $\cos\alpha/R$, and when added to $\boldsymbol{E}_\beta$ of intensity $\sin\alpha/R$ on the perpendicular plane to $\boldsymbol{n}$ produces a final vector $\boldsymbol{E}$ of intensity $1/R$ and which forms the same angle α with the proyection $\boldsymbol{E}_{a_\perp}$.

If we represent at the point P all resultant fields during a complete turn of the charge, we obtain a cone of revolution, of angle $\pi/2-\alpha$ and the generatrix has a size $1/R$ in these units. In the figure **5.2** we represent the values of the field at every point of coordinate $z=1,2,3$ and 4 during a complete turn of the charge.

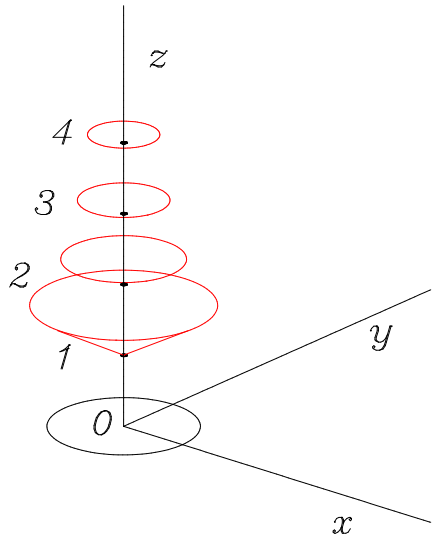


Figure 5.2: **Revolution cones which represent respectively, the instantaneous electric fields during a complete turn of the charge, at the points of the axis OZ of coordinates $z=1,2,3,4$. At the point 1 the semiangle of the cone is $\alpha=\pi/4$, and it is the same angle between the field and the vector $-\boldsymbol{a}_\perp$, at the retarded instant.**

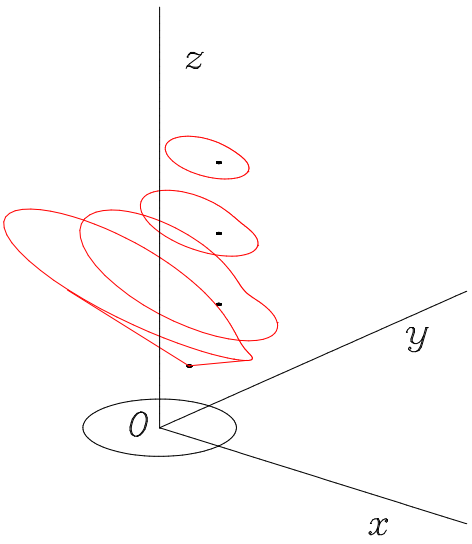


Figure 5.3: **Instantaneous electric fields during a complete turn of the charge of values from bottom to top** $(0.5,0,1)$, $(1,0,2)$, $(1,0,3)$ **and** $(1,0,4)$, **respectively.**

In the figures **5.3**, **5.4** And **5.5** we represent the resultant field at the points shown. The local intensity is increasing but as we shall see, its time average value is radial and decreases with the distance like $1/r^2$, with respect to the origin.

In cartesian components, if we call ϕ the phase of the charge at the retarded position, the

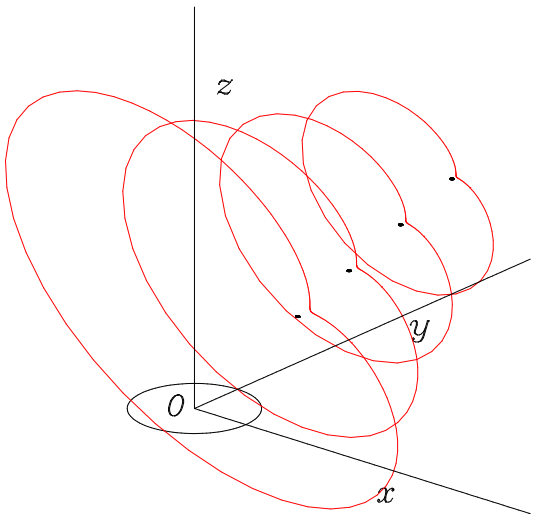


Figure 5.4: **Instantaneous electric fields during a complete turn of the charge and where the corresponding vertex is located at those black dots of coordinates** (x, y, z)**, along the straight line** $y = 0$, $z = x$**, and from bottom to top** $(2, 0, 2)$, $(3, 0, 3)$, $(4, 0, 4)$ **and** $(5, 0, 5)$, **respectively.**

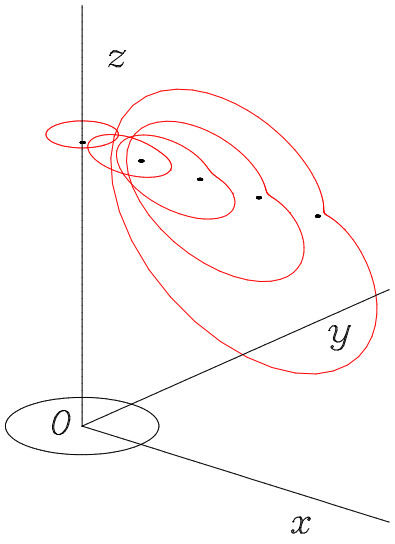


Figure 5.5: **Instantaneous electric fields during a complete turn of the charge and where the corresponding vertex is located at those black dots of coordinates** (x, y, z)**, along the straight line** $y = 0$, $z = 4$**, and from left to right at** $(0, 0, 4)$ $(1, 0, 4)$, $(2, 0, 4)$, $(3, 0, 4)$ **and** $(4, 0, 4)$, **respectively.**

components of the electric field, up to a global factor e/R_0^2, are

$$E_x = \frac{(1 - x\cos\phi - y\sin\phi)(x - \cos\phi + R\sin\phi)}{(R + x\sin\phi - y\cos\phi)^3} + \frac{R\cos\phi}{(R + x\sin\phi - y\cos\phi)^2}, \tag{5.9}$$

$$E_y = \frac{(1 - x\cos\phi - y\sin\phi)(y - \sin\phi - R\cos\phi)}{(R + x\sin\phi - y\cos\phi)^3} + \frac{R\sin\phi}{(R + x\sin\phi - y\cos\phi)^2}, \tag{5.10}$$

$$E_x = \frac{z(1 - x\cos\phi - y\sin\phi)}{(R + x\sin\phi - y\cos\phi)^3}, \quad R = \sqrt{1 + x^2 + y^2 + z^2 - 2x\cos\phi - 2y\sin\phi}. \tag{5.11}$$

In all these figures we have to imagine the corresponding cone, with origin at the observation point, which subtends the circular motion of the charge. Then, for every retarded position, we have to depict at the observation point a perpendicular straight line to the cone generatrix up to reach the corresponding point on the red line. The vector from the observation point to the red line, which is orthogonal to the corresponding generatrix of the cone, represents the instantaneous electric field.

5.1.4 The time average electric and magnetic field of a Dirac particle

Let us assume that we have a test charge in the neighborhood of the electron. The frequency of the zitterbewegung is very high, of order $\sim 10^{21}$ s^{-1}. If our test particle is moving slowly, then presumably the detected electric field will be some time average field during a complete turn of the charge.

The complete analytical expression of a time average field at any arbitrary point has not yet been obtained. However, to obtain an estimate, let us compute the average field on some particular point. Let us consider that the electron is at rest, with the center of mass at the origin of a reference frame. The constant spin is pointing along the OZ axis. We shall try to calculate this average field at a point P of coordinate z in this OZ axis. In Figure 5.6, we represent the different magnitudes at the retarded time $\tilde{t}$, needed to apply equation (5.8).

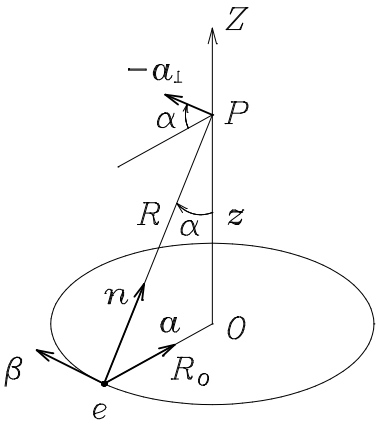


Figure 5.6: **Instantaneous electric field of the electron at point P has a component along $-a_\perp$ and $-\boldsymbol{\beta}$, and we can check that it is orthogonal to $\boldsymbol{n}$.**

In that particular point shown in the figure, $\boldsymbol{n} \cdot \boldsymbol{\beta} = 0$, and thus

$$\boldsymbol{E} = \frac{e}{c^2 R}\left(\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{a}) - \boldsymbol{n} \times (\boldsymbol{\beta} \times \boldsymbol{a})\right) = \frac{e}{c^2 R}\left(-\boldsymbol{a}_\perp - \boldsymbol{\beta}(\boldsymbol{n} \cdot \boldsymbol{a})\right),$$

where the vector $\boldsymbol{a}_\perp = \boldsymbol{a} - \boldsymbol{n}(\boldsymbol{a}\cdot\boldsymbol{n})$, is the component of the acceleration orthogonal to the unit vector $\boldsymbol{n}$. For the observation point P, the expression $\boldsymbol{n}\cdot\boldsymbol{a}$ is constant at any retarded point, and the time average of $\boldsymbol{\beta}$ during a complete turn is zero, and for the vector $\boldsymbol{a}_\perp$ it reduces to its z-component $a_\perp \sin\alpha$. Since the acceleration in this frame is $a = c^2/R_0$, $a_\perp = a\cos\alpha$ and $\sin\alpha = R_0/R$ and $\cos\alpha = z/R$, the time average electric field at point P is

$$\boldsymbol{E}(z) = \frac{ez}{(R_0^2 + z^2)^{3/2}}\hat{\boldsymbol{z}}, \tag{5.12}$$

where $\hat{\boldsymbol{z}}$ is a unit vector along the OZ axis. The advanced field has exactly the same expression. For the antiparticle we have to change e for $-e$, and the velocity $\boldsymbol{\beta}$ by $-\boldsymbol{\beta}$, and the average makes no difference of (5.12).

If we had use the Heaviside-Feynman expresion (5.4), the average field would be the instantaneous Coulomb field at the point P, and when taking the average value during a complete turn of the charge, its projection along the direction OZ will be

$$\boldsymbol{E}(z) = \frac{e}{R^2}\cos\alpha\hat{\boldsymbol{z}},$$

which gives again (5.12), because the other two contributions $d(\boldsymbol{n}/R^2)/dt$ and $d^2\boldsymbol{n}/dt^2$ are orthogonal to OZ and its average value is zero.

This is a radial static field from the origin of the reference frame with a Coulomb-like behaviour $1/z^2$, but it does not diverge at the origin. We depict this field in Figure 5.7, and compared with the Coulomb field of a point charge at the origin, where we take as a unit of length the radius R_0 of the internal motion.

We can clearly see the fitting of the average field and the Coulomb field for large z, around $|z| \geq 5R_0$. The maximum of the average field takes place at $z = R_0/\sqrt{2}$. If we consider that the static field of a pointlike electron is this time average field, then the electrostatic energy does not diverge and the energy will be renormalized. The instantaneous field diverges at the charge position like $1/R$, the energy behaves like $1/R^2$ and the volume element goes like $4\pi R^2 dR$, and therefore there is no divergence of the energy in the surrounding of the charge. Nevertheless there are other points of the zitterbewegung plane, in which the instantaneous field also diverges, and the computation of the energy is still to be done.

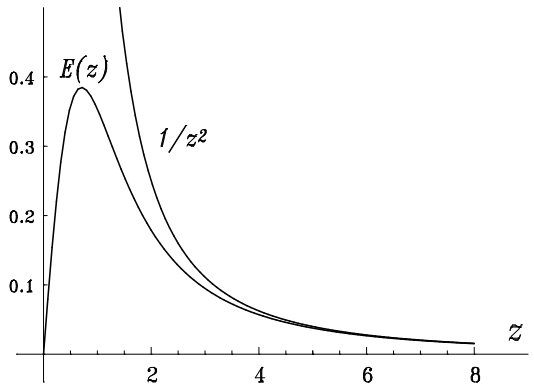


Figure 5.7: **Average retarded (or advanced) electric field (5.12) and Coulomb field along the OZ axis.**

However, if we are involved in high energy processes, our test particle is moving sufficiently fast relative to the electron, then the field it feels is the instantaneous $1/R$ field, which is greater than the average field, and becomes important for points closer to the electron. This means that

the average energy density of the local instantaneous field is greater than the average Coulomb-like energy density, and we can naively interpret this difference, from the classical point of view, as the energy associated to the cloud of virtual photons in the surroundings of the particle. Is this the corresponding infinite energy which is usually cancelled out in the renormalization of quantum electrodynamics?

To compute numerically the average field at an arbitrary position, let us consider the different magnitudes depicted in Figure 5.8.

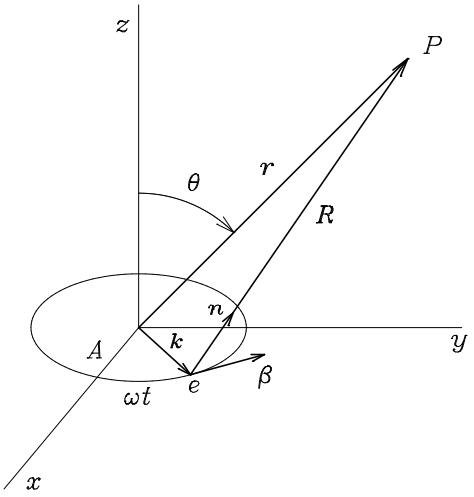


Figure 5.8: **Charge motion and observation point P.**

If at time $t = 0$ the charge is located at point A on the OX axis, then at time t the different observables shown in the figure are described in Cartesian coordinates and in the laboratory frame by

$$\boldsymbol{k} = R_0[\cos\omega t, \sin\omega t, 0] \equiv R_0\widetilde{\boldsymbol{k}}, \quad \boldsymbol{\beta} = \frac{\boldsymbol{u}}{c} = [-\sin\omega t, \cos\omega t, 0],$$

$$\boldsymbol{r} = [x, y, z], \quad \boldsymbol{a} = \frac{d\boldsymbol{u}}{dt} = \frac{c^2}{R_0}[-\cos\omega t, -\sin\omega t, 0] = \frac{c^2}{R_0}\widehat{\boldsymbol{a}},$$

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{k} = R_0(\widetilde{\boldsymbol{r}} - \widetilde{\boldsymbol{k}}), \quad \boldsymbol{n} = \frac{\boldsymbol{R}}{R}, \quad R = |\boldsymbol{R}| = R_0\widetilde{R}.$$

With these definitions, field (5.8) can be written as

$$\boldsymbol{E}(t, \boldsymbol{r}) = \left(\frac{e}{R_0^2}\right) \frac{\boldsymbol{n} \times ((\boldsymbol{n} - \boldsymbol{\beta}) \times \widehat{\boldsymbol{a}})}{(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})^3\widetilde{R}}.$$

We want to compare the time average value of this field with the static Coulomb field of a point charge e at the center of mass

$$\boldsymbol{E}_0(\boldsymbol{r}) = \left(\frac{e}{R_0^2}\right) \frac{\widehat{\boldsymbol{r}}}{r^2},$$

where $\widehat{\boldsymbol{r}}$ is a unit vector in the radial direction. The constant factor in brackets in front of these formulae will be dropped out from now on. In this way the unit of length is the zitterbewegung radius R_0.

When the charge is at the point indicated in Figure 5.8, the retarded field it produces at point P is evaluated at the observation time $t_o = t + R/c$. Thus $dt_o = dt + d\widetilde{R}/\omega$, because

$R_0/c = 1/\omega$. If we express $d\widetilde{R}$ in terms of dt, we get $dt_o = (N(t)/\widetilde{R}(t))dt$, where N and $\widetilde{R}$ are explicitly given by

$$\widetilde{R}(t) = \sqrt{(\widetilde{x} - \cos\omega t)^2 + (\widetilde{y} - \sin\omega t)^2 + \widetilde{z}^2},$$

$$N(t) = \widetilde{R}(t) + \widetilde{x}\sin\omega t - \widetilde{y}\cos\omega t.$$

We are going to average the field at P with respect to the observation time at that point during a complete period of the motion of the charge T. If we define a dimensionless evolution time $\tau = \omega t$, then $\omega T = 2\pi$ and thus

$$\frac{1}{T}\int_0^T E(t_o)\,dt_o = \frac{1}{T}\int_0^T E(t)\frac{N(t)}{\widetilde{R}(t)}dt = \frac{1}{2\pi}\int_0^{2\pi} E(\tau)\frac{N(\tau)}{\widetilde{R}(\tau)}d\tau. \tag{5.13}$$

In terms of the τ evolution the different expressions are

$$\boldsymbol{n}\times(\boldsymbol{n}\times\widehat{\boldsymbol{a}}) = \boldsymbol{n}(\boldsymbol{n}\cdot\widehat{\boldsymbol{a}}) - \widehat{\boldsymbol{a}},$$

and

$$\boldsymbol{n}(\boldsymbol{n}\cdot\widehat{\boldsymbol{a}}) = \frac{1 - \widetilde{x}\cos\tau - \widetilde{y}\sin\tau}{\widetilde{R}^2}[\widetilde{x} - \cos\tau, \widetilde{y} - \sin\tau, \widetilde{z}],$$

$$\widehat{\boldsymbol{a}} = [-\cos\tau, -\sin\tau, 0],$$

while

$$\boldsymbol{n}\times(\boldsymbol{\beta}\times\widehat{\boldsymbol{a}}) = \frac{1}{\widetilde{R}}[\widetilde{y} - \sin\tau, -\widetilde{x} + \cos\tau, 0],$$

and

$$1 - \boldsymbol{n}\cdot\boldsymbol{\beta} = \frac{1}{\widetilde{R}}\left(\widetilde{R} + \widetilde{x}\sin\tau - \widetilde{y}\cos\tau\right).$$

We are interested in the radial and transversal part of the field $E_r = \boldsymbol{E}\cdot\widehat{\boldsymbol{r}}$, $E_\theta = \boldsymbol{E}\cdot\widehat{\boldsymbol{\theta}}$, and $E_\phi = \boldsymbol{E}\cdot\widehat{\boldsymbol{\phi}}$, respectively. Here $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\phi}}$ are respectively the usual unit vectors in polar spherical coordinates. If we consider that the observation point P is on the plane XOZ, then we have to take $\widetilde{x} = r\sin\theta$, $\widetilde{y} = 0$ and $\widetilde{z} = r\cos\theta$, where r is the radial separation from the origin in units of R_0.

The final expressions for the field components are

$$E_r(r,\theta,\tau) = \frac{(\widetilde{R}^2 - r^2 - 1)\sin\theta\cos\tau + \widetilde{R}\sin\theta\sin\tau + r(1 + \sin^2\theta\cos^2\tau)}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3},$$

$$E_\theta(r,\theta,\tau) = \frac{\left[(\widetilde{R}^2 - 1)\cos\tau + \widetilde{R}\sin\tau + r\sin\theta\cos^2\tau\right]\cos\theta}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3},$$

$$E_\phi(r,\theta,\tau) = \frac{(\widetilde{R}^2 - 1)\sin\tau + \widetilde{R}(r\sin\theta - \cos\tau) + r\sin\theta\sin\tau\cos\tau}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3},$$

with

$$\widetilde{R} = \sqrt{r^2 - 2r\sin\theta\cos\tau + 1}.$$

To take the time average value of the above fields we have to perform the integration (5.13) so that the above expressions of E_r, E_θ and E_ϕ have to be multiplied by $N(\tau)/\widetilde{R}(\tau)$, where now

$$N(\tau) = \widetilde{R} + r\sin\theta\sin\tau.$$

The average retarded radial electric field for $\theta = 0$ is already depicted in Figure 5.7 but we also include it in the next Figure 5.9. We see the Coulomb behavior of the radial component for the directions $\theta = 0, \pi/3, \pi/4, \pi/6$. Similarly, in Figure 5.10 is displayed the transversal component of the average retarded electric field $< E_\theta(r,\theta) >$ for the same directions, that goes to zero very quickly. For $\theta = \pi/2$, we see that $< E_\theta(r,\pi/2) >= 0$. The average $< E_\phi(r,\theta) >$ vanishes everywhere for any $\theta \neq \pi/2$. On the plane $\theta = \pi/2$ the numerical routine fails.

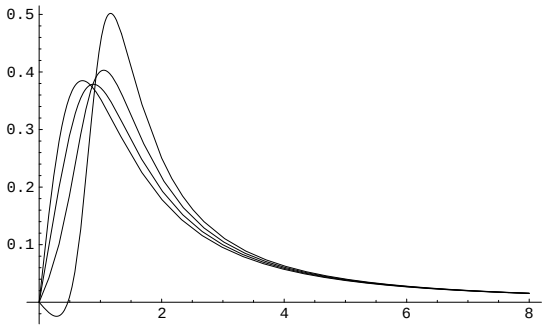


Figure 5.9: **Time average $< E_r(r) >$ of the radial component of the retarded electric field in the directions $\theta = 0, \pi/3, \pi/4$ and $\pi/6$.**

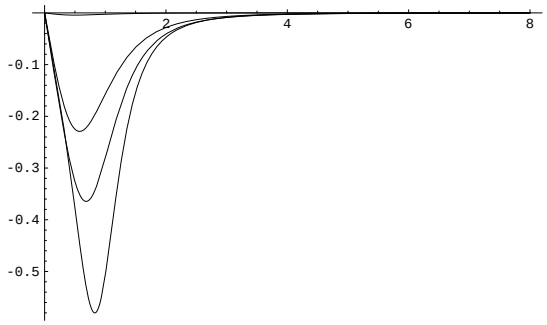


Figure 5.10: **Time average of the component $< E_\theta(r) >$ of the retarded electric field in the directions $\theta = 0, \pi/3, \pi/4$ and $\pi/6$. It goes to zero very quickly. For $\theta = \pi/2$ it vanishes everywhere.**

The average magnetic field can be computed in the same way. Here we shall consider only the retarded solution and we will compare it with the magnetic field produced by an intrinsic magnetic moment $\boldsymbol{\mu}$ placed at the center of mass. This magnetic field is [3]

$$\boldsymbol{B}_0(\boldsymbol{r}) = \frac{3\widehat{\boldsymbol{r}}(\widehat{\boldsymbol{r}} \cdot \boldsymbol{\mu}) - \boldsymbol{\mu}}{c^2 r^3}.$$

For our system the magnetic moment produced by the moving charge is of value $ecR_0/2$ in the direction of OZ, so that in units of R_0 it can be written as

$$\boldsymbol{B}_0(\boldsymbol{r}) = \left(\frac{e}{2cR_0^2}\right) \frac{3\widehat{\boldsymbol{r}}(\widehat{\boldsymbol{r}} \cdot \widehat{\boldsymbol{z}}) - \widehat{\boldsymbol{z}}}{\widetilde{r}^3}.$$

The nonvanishing components are

$$B_{0r}(r,\theta) = \left(\frac{e}{cR_0^2}\right) \frac{\cos\theta}{\widetilde{r}^3}, \quad B_{0\theta}(r,\theta) = \left(\frac{e}{cR_0^2}\right) \frac{\sin\theta}{2\widetilde{r}^3}. \tag{5.14}$$

[3] J.D. Jackson, *Classical Electrodynamics*, John Wiley and Sons, NY 3rd. ed. (1998).

In our model, the instantaneous magnetic field is $\boldsymbol{B} = \boldsymbol{n} \times \boldsymbol{E}/c$. Their components can be written, after deleting a constant factor e/cR_0^2, as:

$$B_r(r,\theta,\tau) = \frac{(1 - r\sin\theta\cos\tau)\cos\theta}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3},$$

$$B_\theta(r,\theta,\tau) = \frac{r\cos\tau(1+\sin^2\theta) - (1+r^2)\sin\theta - \widetilde{R}r\sin\tau}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3},$$

$$B_\phi(r,\theta,\tau) = \frac{(\widetilde{R}\cos\tau + \sin\tau)r\cos\theta}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3}.$$

To proceed with the retarded time average integral we have to multiply the above fields by $N(t)/\widetilde{R}(t)$, as before. The numerical integration is compared with the analytical expression of the magnetic field of a dipole (5.14) for different directions.

The magnetic dipole field (5.14) goes to infinity when $r \to 0$. In Figures 5.11-5.13 we show the matching of the $B_{0r}(r)$ components of the dipole and the computed time average value $< B_r(r,\theta) >$, for $r > R_0$ and in the directions given by $\theta = \pi/6, \pi/4$ and $\pi/3$. Similarly, in Figures 5.14-5.16, for the corresponding $B_{0\theta}(r,\theta)$ and $< B_\theta(r,\theta) >$ components.

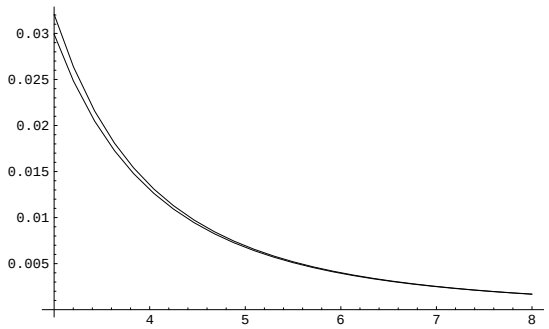


Figure 5.11: **Radial components of the dipole field $B_{0r}(r)$ and the time average retarded magnetic field $< B_r(r) >$, along the direction $\theta = \pi/6$.**

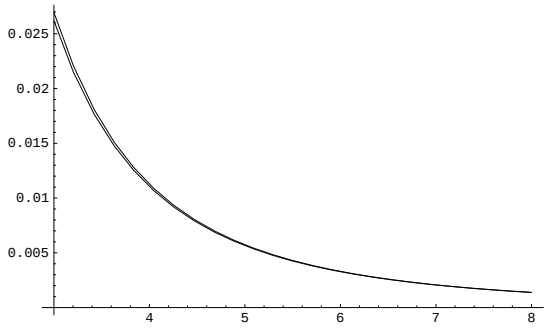


Figure 5.12: **Radial components of the dipole field $B_{0r}(r)$ and the time average retarded magnetic field $< B_r(r) >$, along the direction $\theta = \pi/4$.**

The computed time averages $< B_r(r) >$ and $< B_\theta(r) >$ do not diverge at the origin but have the behavior depicted in 5.17 and 5.18, respectively, when represented along the directions $\theta = 0, \pi/3, \pi/4$ and $\pi/6$, and they take the values $\cos\theta$ and $-\sin\theta$ respectively, at point $r = 0$.

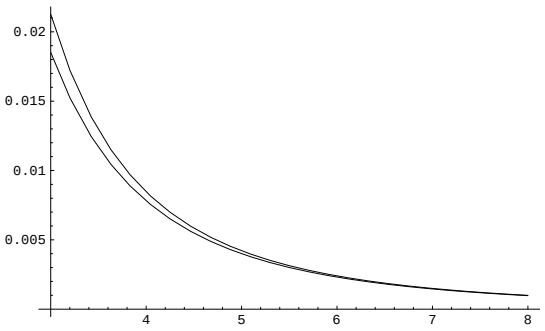


Figure 5.13: **Radial components of the dipole field $B_{0r}(r)$ and the time average retarded magnetic field $< B_r(r) >$, along the direction $\theta = \pi/3$.**

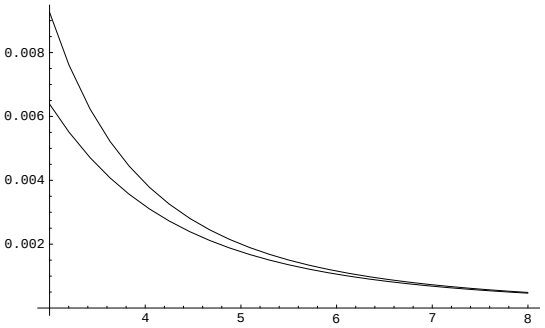


Figure 5.14: **Time average retarded magnetic field $< B_\theta(r) >$ and the dipole field $B_{0\theta}(r)$, along the direction $\theta = \pi/6$.**

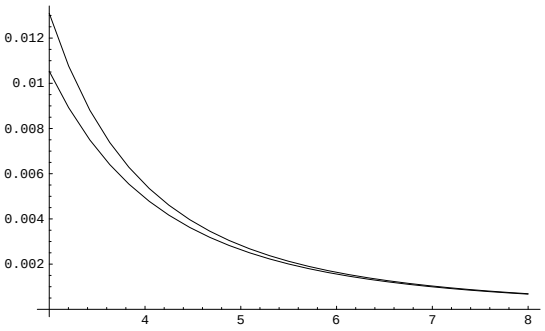


Figure 5.15: **Time average retarded magnetic field $< B_\theta(r) >$ and the dipole field $B_{0\theta}(r)$, along the direction $\theta = \pi/4$.**

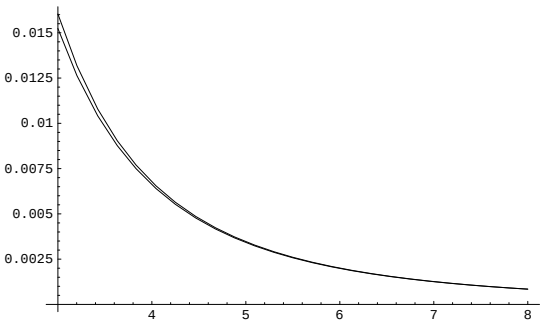


Figure 5.16: **Time average retarded magnetic field $< B_\theta(r) >$ and the dipole field $B_{0\theta}(r)$, along the direction $\theta = \pi/3$.**

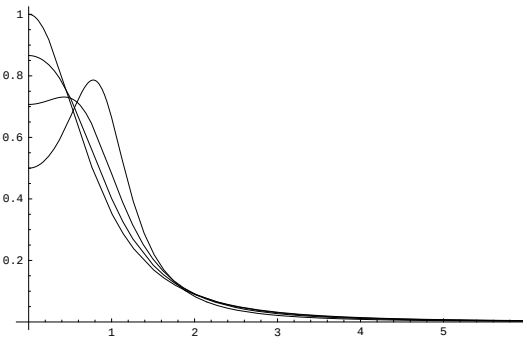


Figure 5.17: **Time average retarded magnetic field** $< B_r(r) >$ **along the directions** $\theta = 0, \pi/3, \pi/4$ **and** $\pi/6$ **and its behavior at** $r = 0$**. For** $\theta = \pi/2$ **it vanishes everywhere.**

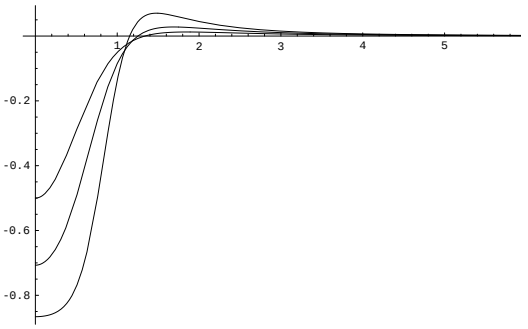


Figure 5.18: **Time average retarded magnetic field** $< B_\theta(r) >$ **along the directions** $\theta = 0, \pi/3, \pi/4$ **and** $\pi/6$ **and its behavior at** $r = 0$**.**

The time average value of the transversal component $< B_\phi(r, \theta) >$ vanishes everywhere for all directions.

To end this section we can think about the possibility of computing the average fields using the advanced solutions in spite of the retarded ones.

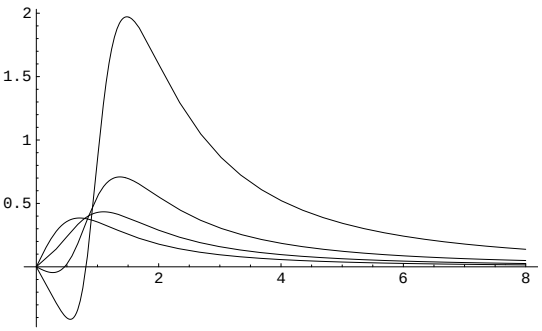


Figure 5.19: **Time average radial component** $< E_r(r) >$ **of the advanced electric field in the directions** $\theta = 0, \pi/3, \pi/4$ **and** $\pi/6$**.**

In that case the observation time will be related with the laboratory time by $t_o = t - R/c$, and therefore $dt_o = (M(t)/\widetilde{R}(t))dt$, where $\widetilde{R}(t)$ is the same as before, but

$$M(t) = \widetilde{R}(t) - \widetilde{x}\sin\omega t + \widetilde{y}\cos\omega t.$$

Then, if we depict, for instance, the advanced average radial electric field in Figure 5.19, for the same directions as in Figure 5.9, we see the different behavior in these radial directions and, although the field decreases for large distances, it nevertheless does not fit with a Coulomb field.

The numerical routine fails to compute the corresponding integrals for $\theta = \pi/2$ where we have some indefiniteness of the integrands for observation points lying on the XOY plane. There are no singularities for points inside the circle of radius R_0. We have a divergence of order $1/r$ for points on this circle, but this divergence can be removed by taking a principal value of the time integral. Finally, the quotient term $1 - \boldsymbol{n} \cdot \boldsymbol{\beta}$ can vanish for some observation points on the XOY plane outside the circle of radius R_0, whenever the retarded $\boldsymbol{n}$ and $\boldsymbol{\beta}$ become parallel vectors. But this can happen only for a single point of the retarded charge position in the average integral and perhaps some kind of principal value should be taken to properly obtain a finite average value. The difficulties of obtaining an analytical estimate for these integrals make this analysis incomplete. Nevertheless, the nice fitting of the average electric field with a Coulomb field and the average magnetic field with the field of a magnetic dipole, for distances of a few Compton wave lengths away, except on the $\theta = \pi/2$ plane where we have not been able to obtain an estimate, suggests that we devote some effort to renormalize and improve the model at a classical level.

5.1.5 Electromagnetic energy and angular momentum

If we compute the electromagnetic energy associated to the instantaneous field (5.8), since $\boldsymbol{B} = \boldsymbol{n} \times \boldsymbol{E}/c$, it implies that the energy density is

$$h(t_o) = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 = \epsilon_0 E^2 - \frac{1}{2}\epsilon_0 (\boldsymbol{n} \cdot \boldsymbol{E})^2,$$

and therefore the energy at any instant of observation time t_o is

$$\mathcal{E}(t_o) = \int_{\mathbb{R}^3} h(t_o) dV.$$

The value of $E^2(t_o)$ of the field, has to be evaluated from the location of the charge in the corresponding retarded time t, with $t_o = t + R/c$.

Since $\boldsymbol{E} = \boldsymbol{E}_\beta + \boldsymbol{E}_a$, as we have analyzed Gauss theorem in section **5.1.1**, it is necessary to consider both fields, although $\boldsymbol{E}_\beta$ vanishes escept at those points on the plane XOY when $\beta \to 1$.

$$\boldsymbol{E}^2 = E_\beta^2 + E_a^2 + 2\boldsymbol{E}_\beta \cdot \boldsymbol{E}_a.$$

Because

$$\boldsymbol{E}_\beta = \frac{e(1-\beta^2)}{R^2(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} (\boldsymbol{n} - \boldsymbol{\beta}),$$

$$\boldsymbol{E}_a = \frac{e}{Rc^2(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} \boldsymbol{n} \times ((\boldsymbol{n} - \boldsymbol{\beta}) \times \boldsymbol{a}).$$

The following vectors, in terms of an arbitary value of β are

$$\boldsymbol{n} = \frac{1}{\widetilde{R}}[\widetilde{x} - \cos\omega t, \widetilde{y} - \sin\omega t, \widetilde{z}],$$

$$\boldsymbol{\beta} = \beta[-\sin\omega t, \cos\omega t, 0],$$

$$\boldsymbol{a} = \frac{\beta^2 c^2}{R_0}[-\cos\omega t, -\sin\omega t, 0].$$

The different scalar products are

$$\boldsymbol{a} \cdot \boldsymbol{n} = \frac{\beta^2 c^2}{R_0 \widetilde{R}} (1 - \widetilde{x}\cos\omega t - \widetilde{y}\sin\omega t)$$

$$\boldsymbol{n}\cdot\boldsymbol{\beta}=\frac{\beta}{\widetilde{R}}\left(-\widetilde{x}\sin\omega t+\widetilde{y}\cos\omega t\right),$$
$$1-\boldsymbol{n}\cdot\boldsymbol{\beta}=\frac{1}{\widetilde{R}}\left(\widetilde{R}+\beta\widetilde{x}\sin\omega t-\beta\widetilde{y}\cos\omega t\right)$$

For the part E_β^2, $(\boldsymbol{n}-\boldsymbol{\beta})^2=(1+\beta^2-2\boldsymbol{n}\cdot\boldsymbol{\beta})$, and therefore

$$E_\beta^2=\frac{e^2}{R_0^4}\frac{(1-\beta^2)^2(1+\beta^2-2\beta\left(-\widetilde{x}\sin\omega t+\widetilde{y}\cos\omega t\right)/\widetilde{R})}{\widetilde{R}^4(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^6}. \tag{5.15}$$

For the part of $\boldsymbol{E}_a^2$,

$$[\boldsymbol{n}\times((\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a})]^2=a^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2-(1-\beta^2)(\boldsymbol{a}\cdot\boldsymbol{n})^2,$$

and thus $\boldsymbol{E}_a^2$ is written as

$$E_a^2=\frac{e^2(a^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2-(1-\beta^2)(\boldsymbol{a}\cdot\boldsymbol{n})^2)}{R^2c^4(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^6},$$

$$E_a^2=\frac{e^2}{R_0^4}\frac{\beta^4\left[\widetilde{R}^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2-(1-\beta^2)\left(1-\widetilde{x}\cos\omega t-\widetilde{y}\sin\omega t\right)^2\right]}{\widetilde{R}^4(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^6} \tag{5.16}$$

in terms of the dimensionless distance $\widetilde{R}=R/R_0$. Finally, the part $2\boldsymbol{E}_\beta\cdot\boldsymbol{E}_a$ contains the term

$$(\boldsymbol{n}-\boldsymbol{\beta})\cdot[\boldsymbol{n}\times((\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a})]=(\beta^2-\boldsymbol{n}\cdot\boldsymbol{\beta})(\boldsymbol{n}\cdot\boldsymbol{a}),$$

because the other term $\boldsymbol{\beta}\cdot\boldsymbol{a}=0$. We get

$$2\boldsymbol{E}_\beta\cdot\boldsymbol{E}_a=\frac{e^2}{R_0^4}\frac{2\beta^2(1-\beta^2)(\beta^2-\boldsymbol{n}\cdot\boldsymbol{\beta})\left(1-\widetilde{x}\cos\omega t-\widetilde{y}\sin\omega t\right)}{\widetilde{R}^4(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^6}. \tag{5.17}$$

For the part $(\boldsymbol{n}\cdot\boldsymbol{E})^2$ only $\boldsymbol{E}_\beta$ contributes,

$$-\frac{1}{2}(\boldsymbol{n}\cdot\boldsymbol{E})^2=-\frac{e^2}{R_0^4}\frac{(1-\beta^2)^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2}{2\widetilde{R}^4(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^6}, \tag{5.18}$$

We are going to suppress the dimensionless factor ϵ_0e^2/R_0^4, and to write it in the International System of Units, we have to replace $e\to e/4\pi\epsilon_0$.

The energy density $h(x,y,z,t_o)$ in dimensionless variables is the sum of the four terms (5.15), (5.16), (5.17) and (5.18), and the energy becomes:

$$\mathcal{E}(t_o)=\frac{\epsilon_0e^2R_0^3}{(4\pi\epsilon_0)^2R_0^4}\int_{\mathbb{R}^3}h(x,y,z,t_o)d\widetilde{V}=\frac{e^2 2mc}{4\pi\epsilon_0 4\pi\hbar}\int_{\mathbb{R}^3}h(x,y,z,t_o)d\widetilde{V}=\frac{e^2 2mc}{4\pi\epsilon_0 4\pi\hbar}M,$$

where the volume element $dV=R_0^3d\widetilde{V}$ is written in terms of dimensionless variables and therefore the result of the integration is the strict dimensionles numerical value M. The integral depends on the value of β and once the integral is performed we have to take the limit $\beta\to1$. Since the center of mass of the electron is at rest, $R_0=\hbar/2mc$ and if we assume that the whole energy of the electron is purely electromagnetic, we can identify this energy with the rest energy, and thus

$$mc^2=\frac{e^2 2mc}{4\pi\epsilon_0 4\pi\hbar}M,\quad 1=\frac{e^2}{4\pi\epsilon_0 hc}M=\frac{\alpha}{2\pi}M,\quad \alpha\simeq\frac{1}{137}, \tag{5.19}$$

wher α is the fine structure constant. The value of e^2 will be uniquely determined(up to a sign) once the above integral is performed. This integral has difficulties, and if our conjecture is correct it will produce the value $M = 2\pi/\alpha \approx 861.009$.

$$e^2 = \frac{4\pi\epsilon_0 hc}{M}.$$

An elementary particle with a pure electromagnetic structure, will have a unique value (up to a sign) for the electric charge. and therefore electrons, muons and tau particles, and their corresponding antiparticles, which only interact in the electro-weak way, will all have the same charge, independently of their masses. In the case of quarks, not all internal structure is pure electromagnetic, because it also interacts strongly, and therefore in the equation (5.19) the computed electromagnetic energy will be a fraction$k$ of its total rest energ mc^2, and therefore con lo que

$$e^2 = k\frac{4\pi\epsilon_0 hc}{M}, \quad k < 1.$$

Quarks will have an electric charge smaller than the charge of the particles which only interact electroweakly. The theory should give the value $k = 1/9$ or $4/9$ for the up and down quarks, respectively.

The value of the energy must be independent of the observation time t_o, since the motion of the center of charge is stationary, and must be the same as the corresponding time average value,

$$\mathcal{E} = \frac{1}{T}\int_0^T \mathcal{E}(t_o)dt_o = \frac{\epsilon_0}{T}\int_{\mathbb{R}^3} dV \int_0^T E^2 dt_o,$$

for any value T, in particular for the period of the internal motion.

Electromagnetic angular momentum

Let us compute the electromagnetic angular momentum of the system with respect to the origin. The projection of this angular momentum along the direction OZ is identified with the mechanical angular momentum of the particle $-\hbar/2$, and we get another equation

$$-\frac{\hbar}{2} = \int_{\mathbb{R}^3} (\boldsymbol{r}\times\boldsymbol{g})_z dV,$$

$$\boldsymbol{g} = \epsilon_0 \boldsymbol{E}\times\boldsymbol{B} = \epsilon_0 \boldsymbol{E}\times(\boldsymbol{n}\times\boldsymbol{E}/c) = \frac{\epsilon_0}{c}\left(E^2\,\boldsymbol{n} - \boldsymbol{E}(\boldsymbol{n}\cdot\boldsymbol{E}_\beta)\right),$$

because $\boldsymbol{E}_a$ is perpendicular to $\boldsymbol{n}$. The part

$$\frac{\epsilon_0}{c}E^2(\boldsymbol{r}\times\boldsymbol{n})_z = \frac{R_0}{\widetilde{R}}(\widetilde{y}\cos\omega t - \widetilde{x}\sin\omega t), \tag{5.20}$$

contains E^2, which is the sum of the three terms (5.15), (5.16) and (5.17). The part $\boldsymbol{r}\times\boldsymbol{E} = \boldsymbol{r}\times\boldsymbol{E}_\beta + \boldsymbol{r}\times\boldsymbol{E}_a$, and we have the two terms

$$(\boldsymbol{r}\times(\boldsymbol{n}-\boldsymbol{\beta}))_z = \frac{R_0}{\widetilde{R}}(\widetilde{y}\cos\omega t - \widetilde{x}\sin\omega t) - \beta x\cos\omega t - \beta y\sin\omega t,$$

$$(\boldsymbol{r}\times\boldsymbol{a})_z = \beta^2 c^2(\widetilde{y}\cos\omega t - \widetilde{x}\sin\omega t),$$

which are multiplied by

$$\boldsymbol{n}\cdot\boldsymbol{E}_\beta = \frac{e(1-\beta^2)}{R^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2}.$$

Because $(\boldsymbol{r}\times\boldsymbol{E})_z = xE_y - yE_x$,

$$(\boldsymbol{r}\times\boldsymbol{E}_\beta)_z = \frac{e(1-\beta^2)}{R^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}\left(\frac{1}{\widetilde{R}}(y\cos\omega t - x\sin\omega t) - \beta x\cos\omega t - \beta y\sin\omega t\right),$$

$$(\boldsymbol{r}\times\boldsymbol{E}_a)_z = \frac{e}{Rc^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}\left[(\boldsymbol{a}\cdot\boldsymbol{n})(\boldsymbol{r}\times(\boldsymbol{n}-\boldsymbol{\beta}))_z - (1-\boldsymbol{n}\cdot\boldsymbol{\beta})(\boldsymbol{r}\times\boldsymbol{a})_z\right],$$

and thus

$$-\frac{\epsilon_0}{c}(\boldsymbol{n}\cdot\boldsymbol{E}_\beta)(\boldsymbol{r}\times\boldsymbol{E}_\beta)_z = -\frac{\epsilon_0 e^2(1-\beta^2)^2 R_0}{cR^4(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^5}\left(\frac{1}{\widetilde{R}}(\widetilde{y}\cos\omega t - \widetilde{x}\sin\omega t) - \beta\widetilde{x}\cos\omega t - \beta\widetilde{y}\sin\omega t\right). \tag{5.21}$$

Similarly

$$-\frac{\epsilon_0}{c}(\boldsymbol{n}\cdot\boldsymbol{E}_\beta)(\boldsymbol{r}\times\boldsymbol{E}_a)_z = -\frac{\epsilon_0 e^2(1-\beta^2)\beta^2}{cR^3\widetilde{R}(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^5}\times$$

$$\left[(1-\widetilde{x}\cos\omega t - \widetilde{y}\sin\omega t)\left(\frac{1}{\widetilde{R}}(\widetilde{y}\cos\omega t - \widetilde{x}\sin\omega t) - \beta\widetilde{x}\cos\omega t - \beta\widetilde{y}\sin\omega t\right) - \right.$$

$$\left. - \left(\widetilde{R} + \beta\widetilde{x}\sin\omega t - \beta\widetilde{y}\cos\omega t\right)(\widetilde{y}\cos\omega t - \widetilde{x}\sin\omega t)\right] \tag{5.22}$$

$$-\frac{\hbar}{2} = \int_{\mathbb{R}}^{3} S_z dV,$$

where S_z is the sum of the equations (5.20), (5.21) and (5.22).

Let us assume that the integral is performed at the observation instant $t_0 = 0$, when the center of charge is at the point $A \equiv (1,0,0)$. For this point, the retarded distance up to the charge is $\widetilde{R} = 0$. For another arbitrary point $P \equiv (x,y,z)$ the corresponding retarded point is $B \equiv (\cos\theta, \sin\theta, 0)$, and because the radius of the circle is 1, $\theta = -\widetilde{R}$, as is shown in the figure.

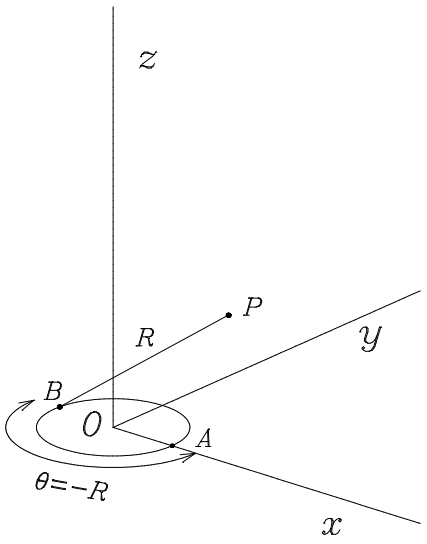


Figure 5.20: **The dimensionless distance R form the observation point P to the retarded point B is the same as the arc length AB.**

With the following change of variables

$$x = R\sin\lambda\cos\phi + \cos R, \quad y = R\sin\lambda\sin\phi - \sin R, \quad z = R\cos\lambda, \quad R\in[0,\infty],\ \lambda\in[0,\pi]\ \phi\in[0,2\pi].$$

The Jacobian of the transformation is

$$J = R^2\sin\lambda(1-\sin\lambda\sin(R+\phi)),$$

and $R = \widetilde{R}$.

As a summary, the integral for the electromagnetic energy, wich gives the constant M, in the limit $\beta \to 1$, is

$$M = \int_0^\infty dR \int_0^\pi d\lambda \int_0^{2\pi} d\phi \frac{8 \sin\lambda}{(2 - \cos(R - \lambda + \phi) + \cos(R + \lambda + \phi))^3},$$

For the integral of the angular momentum we get

$$N = \int_0^\infty dR \int_0^\pi d\lambda \int_0^{2\pi} d\phi \frac{-\sin^2\lambda \sin(R + \phi)}{R(1 - \sin\lambda \sin(R + \phi))^3}.$$

Both integrals when performed with Mathematica give 0, but if the calculation is done numericaly it produces very high oscillating values even for a finite limit of the range of the integration variable R.

Chapter 6

Some spin features

6.1 Gyromagnetic ratio

If we have a charged point particle of mass m and charge e moving in space, and let us compute its angular momentum $\boldsymbol{J}$ and magnetic moment $\boldsymbol{\mu}$ with respect to some point, these properties satisfy

$$\boldsymbol{\mu} = \frac{e}{2m}\boldsymbol{J}$$

In the case of the electron, the relationship between the spin $\boldsymbol{S}$ and its magnetic moment $\boldsymbol{\mu}$ with respect to the center of mass, is

$$\boldsymbol{\mu} = g\frac{e}{2m}\boldsymbol{S}, \quad g = 2.$$

The dimensionless magnitude g is called the gyromagnetic ratio, because determines the relationship between the magnetic property of generating a magnetic field with the mechanical property associated to the rotation.

The $g = 2$ gyromagnetic ratio of the electron was considered for years a success of Dirac's electron theory. [1] Later, Levy-Leblond [2] obtained similarly $g = 2$ but from a $s = 1/2$ non-relativistic wave equation. Proca [3] found $g = 1$ for spin 1 particles and this led Belinfante [4] to conjecture that the gyromagnetic ratio for elementary particles is $g = 1/s$, irrespective of the value s of its spin. He showed this to be true for quantum systems of spin 3/2, and a few years later the conjecture was analyzed and checked by Moldauer and Case [5] to be right for any half-integer spin, and by Tumanov [6] for the value $s = 2$. In all these cases a minimal electromagnetic coupling was assumed.

Weinberg [7] made the prediction $g = 2$ for the intermediate bosons of the weak interactions when analyzing the interaction of W bosons with the electromagnetic field by requiring a good high-energy behavior of the scattering amplitude. The discovery of the charged $W^{\pm}$ spin 1 bosons with $g = 2$, contradictory to Belinfante's conjecture, corroborated Weinberg's prediction and raised the question as to whether $g = 2$ for any elementary particle of arbitrary spin.

Jackiw [8] has given another dynamical argument confirming that the gyromagnetic ratio of spin-1 fields is $g = 2$, provided a nonelectromagnetic gauge invariance is accepted. He also gives

[1] P.A.M. Dirac, *Proc. Roy. Soc. London* **A117**, 610 (1928).

[2] J.M. Levy-Leblond, *Comm. Math. Phys.* **6**, 286 (1967).

[3] A. Proca, *Compt. Rend.* **202**, 1420 (1936); *Journ. Phys. Radium*, **49**, 245 (1988).

[4] F.J. Belinfante, *Phys. Rev.* **92**, 997 (1953).

[5] P.A. Moldauer and K.M. Case, *Phys. Rev.* **102**, 279 (1956).

[6] V.S. Tumanov, *Sov. Phys. JETP*, **19**, 1182 (1964).

[7] S. Weinberg, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru and H. Pendleton, MIT press, Cambridge, MA (1970), p. 283.

[8] R. Jackiw, *Phys. Rev.* D **57**, 2635 (1998).

some *ad hoc* argument for $s = 2$ fields, consistent with the $g = 2$ prescription.

Ferrara *et al.* [9] in a Lagrangian approach for massive bosonic and fermionic strings, by the requirement of a smooth fixed-charge $M \to 0$ limit, get $g = 2$ as the most natural value for particles of arbitrary spin. However the only known particles which fulfill this condition are leptons and charged $W^{\pm}$ bosons, *i.e.*, charged fermions and bosons of the lowest admissible values of spin. No other higher spin charged elementary particles have been found.

The aim of this section, instead of using dynamical arguments as in the previous attempts, is to give a kinematical description of the gyromagnetic ratio of elementary particles [10] which is based upon the double content of their spin operator structure.

The general structure of the quantum mechanical angular momentum operator with respect to the origin of the observer frame, in either relativistic or nonrelativistic approach, is

$$\boldsymbol{J} = \boldsymbol{r} \times \frac{\hbar}{i}\nabla + \boldsymbol{S} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{S}, \tag{6.1}$$

where the spin operator is

$$\boldsymbol{S} = \boldsymbol{u} \times \frac{\hbar}{i}\nabla_u + \boldsymbol{W}, \tag{6.2}$$

and ∇_u is the gradient operator with respect to the velocity variables and $\boldsymbol{W}$ is a linear differential operator that operates only on the orientation variables $\boldsymbol{\alpha}$ and therefore commutes with the other. For instance, in the $\boldsymbol{\rho} = \boldsymbol{n}\tan(\alpha/2)$ parameterization $\boldsymbol{W}$ is written as

$$\boldsymbol{W} = \frac{\hbar}{2i}\left[\nabla_\rho + \boldsymbol{\rho} \times \nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \nabla_\rho)\right]. \tag{6.3}$$

The first part in (6.2), related to the zitterbewegung spin, has integer eigenvalues because it has the form of an orbital angular momentum in terms of the $\boldsymbol{u}$ variables. Half-integer eigenvalues come only from the operator (6.3). This operator $\boldsymbol{W}$ takes into account the change of orientation, *i.e.*, the rotation of the particle.

We have seen in either relativistic or non-relativistic examples that if the only spin content of the particle $\boldsymbol{S}$ is related to the zitterbewegung part $\boldsymbol{Z} = \boldsymbol{u} \times \boldsymbol{U}$, then the relationship between the magnetic moment and zitterbewegung spin is given by

$$\boldsymbol{\mu} = \frac{e}{2}\,\boldsymbol{k} \times \frac{d\boldsymbol{k}}{dt} = -\frac{e}{2m}\boldsymbol{Z}, \tag{6.4}$$

i.e., with a normal up to a sign gyromagnetic ratio $g = 1$. If the electron has a gyromagnetic ratio $g = 2$, this implies necessarily that another part of the spin is coming from the angular velocity of the body, but producing no contribution to the magnetic moment.

Therefore for the electron, both parts $\boldsymbol{W}$ and $\boldsymbol{Z}$ contribute to the total spin. But the $\boldsymbol{W}$ part is related to the angular variables that describe orientation and does not contribute to the separation $\boldsymbol{k}$ between the center of charge and the center of mass. It turns out that the magnetic moment of a general particle is still related to the motion of the charge by the expression (6.4), *i.e.*, in terms of the $\boldsymbol{Z}$ part but not to the total spin $\boldsymbol{S}$. It is precisely when we try to express the magnetic moment in terms of the total spin that the concept of gyromagnetic ratio arises.

Now, let us assume that both $\boldsymbol{Z}$ and $\boldsymbol{W}$ terms contribute to the total spin $\boldsymbol{S}$ with their lowest admissible nonvanishing values, and must have unique values. This is the idea contained in the atomic principle in the sense that an elementary particle has no excited states they must have fixed values for its physical properties.

For a Dirac particle we have found that the total spin is $s = 1/2$ and that both parts have opposite orientations while the total spin $\boldsymbol{S}$ has the same orientation as the part $\boldsymbol{Z}$. The orbital

[9] S. Ferrara, M. Porrati and V.L. Telegdi, *Phys. Rev.* D **46**, 3529 (1992).
[10] M. Rivas, J.M.Aguirregabiria and A. Hernández, *Phys. Lett.* **A 257**, 21 (1999).

part $\boldsymbol{Z}$ cannot have a vanishing value because the motion of the center of charge does not go through the center of mass and the lowest admisible value from the quantum point of view is $z = 1$ and therefore $w = 1/2$ in the opposite direction for the rotative part. When we express the magnetic moment in terms of the total spin, because $\boldsymbol{Z} = 2\boldsymbol{S}$, is why the gyromagnetic ratio takes the value $g = 2$.

The experimental value of g for leptons is

$$g_e = 2.00231930436182, \quad g_\mu = 2.0023318418, \quad g_\tau = 2.00235442,$$

up to the maximum number of verified digits. We see that is a little bigger than the predicted value $g = 2$, of Dirac equation. The reason could be that one feature is the prediction and the other the measurement. When we measure a property we have to interact with the lepton and this interaction modifies the kinematics. When we introduce the particle in a magnetic field the motion of the center of charge is modified and basically the magnetic moment is the intensity of this current times the area enclosed by this trajectory. If the trajectory is modified the area too and therefore the measurement of the magnetic moment is not the predicted. The deviation would depend on the intensity of the magnetic field during the experiment.

There is a quantum radiative correction which is independent of the external magnetic field, which was determined by Schwinger (1948), which takes the value $\alpha/\pi = 0.0023228$, in terms of the fine structure constant such that the experimental result is:

$$g = 2 + \frac{\alpha}{\pi} + g(B),$$

where the part $g(B)$ depends on the external magnetic field.

In the case of protons and neutrons, the relationship between their magnetic moment and spin is:

$$\boldsymbol{\mu}_p = g_p \frac{e}{2m_p}\boldsymbol{S}, \quad g_p = 5.585694, \quad \boldsymbol{\mu}_n = g_n \frac{e}{2m_p}\boldsymbol{S}, \quad g_n = -3.826085,$$

where m_p is the proton mass. These are two particles of spin 1/2 which do not satisfy Dirac equation, because this equation predicts that g has to be $g = 2$. Protons and neutrons are not elementary particles, and it is the internal motion of the charged quarks which produce the measured magnetic moment. This is very clear in the case of the neutron, a particle of total charge 0, but with a nonvanishing magnetic moment.

6.2 The electron clock

In the De Broglie thesis [11] it is postulated that: Every piece of isolated matter has an internal periodic motion, of an unknown nature, whose frequency is $\nu = mc^2/h$. Nevertheless, Dirac finds that the frequency associated to the motion of the point $\boldsymbol{r}$ of the electron, is twice than the frequency postulated by De Broglie. We have shown that this internal frequency corresponds to the motion at the speed of light, of the center of charge around the center of mass, and which describes an elementary particle of spin 1/2. This model satisifes Dirac equation when quantized. This model is depicted is the front page. The motion is a circle of radius $R_0 = \hbar/2mc$ and frequency $\nu_0 = 2mc^2/h$, in the center of mass frame.

[11] L. de Broglie, Thèse de doctorat (1924). Sommaire: *...nous admettons dans le présent travail l'existence d'un phénomène periodique d'une nature encore à préciser qui serait lié à tout morceau isolé d'énergie et qui dépendrait de sa masse propre par l'équation de Planck-Einstein.*

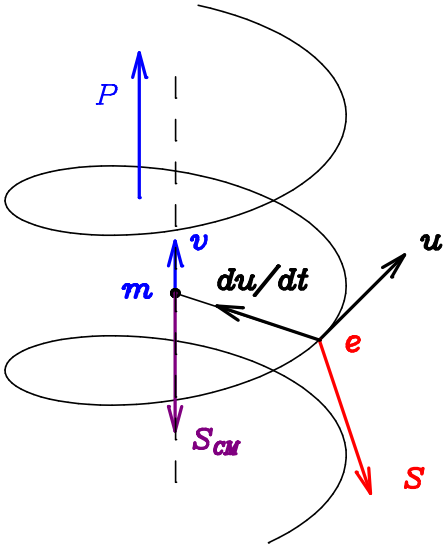


Figure 6.1: **Motion of the center of mass of the electron with velocity v and hellical motion, at the speed of light, of the center of charge of the free electron. The two angular momenta S and S_{CM} are depicted. They show that S_{CM} is conserved, while S preces saround the linear momentum. The transversal velocity of the center of charge is $u_\perp = \sqrt{c^2 - v^2}$, and therefore it takes more time for a moving electron to complete a turn. The electron clock of a moving electron is going slower by a factor $\gamma(v)$ than the clock of the electron at rest.**

6.2.1 Measuring the electron clock

If the electron has the internal periodic motion described in our model, when the center of mass moves with constant velocity, the trajectory of the center of charge has also a spatial periodicity. We can talk of its *wavelength* as its spatial period, or equivalently the length run by the center of mass during a complete turn of the center of charge. The frequency depends also an the motion of the center of mass.

Let us assume the the center of mass is moving at the speed $\boldsymbol{v}$ as is depicted in the figure 6.1. The center of charge follows a hellical trajectory at the speed of light, then its transversal velocity is $u_\perp = \sqrt{c^2 - v^2}$, and therefore a moving electron takes more time to complete a turn, and the electron clock is slower than for the electron at rest. If we call $T_0 = 2\pi R_0/c$ to the period of this internal motion for the center of mass observer, then for the observer who sees the electron moving at the speed $\boldsymbol{v}$ it takes more time $T = 2\pi R_0/u_\perp = \gamma(v)T_0$, where $\gamma(v) = (1 - v^2/c^2)^{-1/2}$.

If we sent an electron beam with a velocity $\boldsymbol{v}$ through a crystal, for instance a silicon crystal, and the velocity is such that the spatial periodicity of the lattice $d = 3.84$Å for Si, and the spatial periodicity of the beam $\lambda = vT$, are commensurables, i.e., either $d = k\lambda$, or $\lambda = nd$, with k and n integer numbers, then a resonant scattering of the beam with the atoms of the lattice can happen. If every electron gets a transversal linear momentum Δp when interacts with an atom, and a longitudinal linear momentum negligible when compared with p, when the electron has crossed a region with N atoms, the transversal linear momentum will be $N\Delta p$, because the interaction with each atom will be basically the same. This electron will be deflected by an angle of order $N\Delta p/p$. Gouanére et al.[12], propose to measure the intensity of the electron beam which crosses the crystal in the forward direction. Then there will exist some resonant linear momenta for which the intensity of the beam will decrease because of this resonant transversal scattering.

[12] M Gouanère, M. Spighel, N. Cue, MJ. Gaillard, R. Genre, R. Kirsch,JC. Poizat, J. Remillieux, P. Catillon and L. Roussel *Found. Phys.* **38**, 659, (2008).

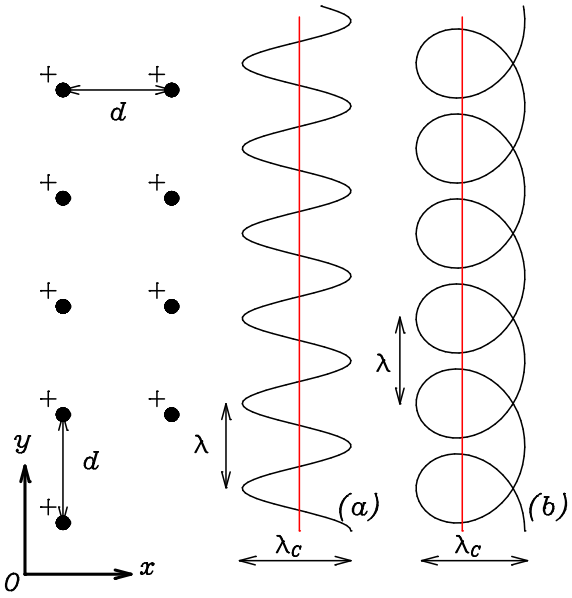


Figure 6.2: **On the left part of the figura it is represented the distribution of silicon atoms. The nucleai are separated by a distance $d = 3.84$Å. On the right we have two possible motions. Motion (a) is the projection on the XOY plane of the motion of the center of charge of an electron polarized in the forward direction. In (b) the spin of the electron is transversal to the motion of the center of mass. It is also depicted in red, the corresponding trajectory of the center od mass. This motion and the crystal lattice are not depicted at the same scale. $\lambda_C = 2R \sim 10^{-13}$m is the amplitude of the transversal oscillation of the center of charge. and $\lambda = vT$ is the distance the center of mass runs during a complete turn of the center of charge.**

In the figure 6.2 it is represented the motion of two polarized electrons, one longitudinally (a) and the other (b) transversally to the motion of the center of mass. We compare the spatial periodicity of the motion of the center of charge of each electron with the periodicity of the silicon lattice.

When $d = k\lambda$, the electron interacts with every atom of the lattice in the same way, and in the case $\lambda = nd$, the interaction is every n atoms. In each interaction the transfer of transversal linear momentum will be basically the same. A greater λ implies also a greater linear momentum, and therefore the dispersion angle will be smaller. Since

$$d = k\lambda = kvT = k\gamma(v)vT_0 = \frac{k\gamma(v)v}{\nu_0} = \frac{km\gamma(v)v}{m\nu_0} = \frac{kp}{m\nu_0}$$

there will exist some resonant linear momenta

$$p_{Dk} = \frac{m\nu_0 d}{k} = \frac{161.748}{k}\text{MeV}/c, \quad k = 1, 2, \ldots \text{(Dirac frequency)}$$

for which the detector will measure a decrease in the intensity of the outgoing beam.

In the mentioned experiment Gouanére et al., used a detector located at 3 m from the silicon crystal with a window of 3×3 mm, so that electrons scattered by an angle greater than 0.001 rad will not be detected. They try to measure De Broglie frequency, which is half Dirac's frequency. In this case the resonant momenta satisfy

$$p_{Bk} = \frac{m\nu_0 d}{k} = \frac{80.874}{k}\text{MeV}/c, \quad k = 1, 2, \ldots \text{(De Broglie frequency)}$$

and they establish in their experiment a range for the linear momentum between 54 and 110 MeV/c, to obtain, at least, the first resonant frequency corresponding to $k = 1$. What they

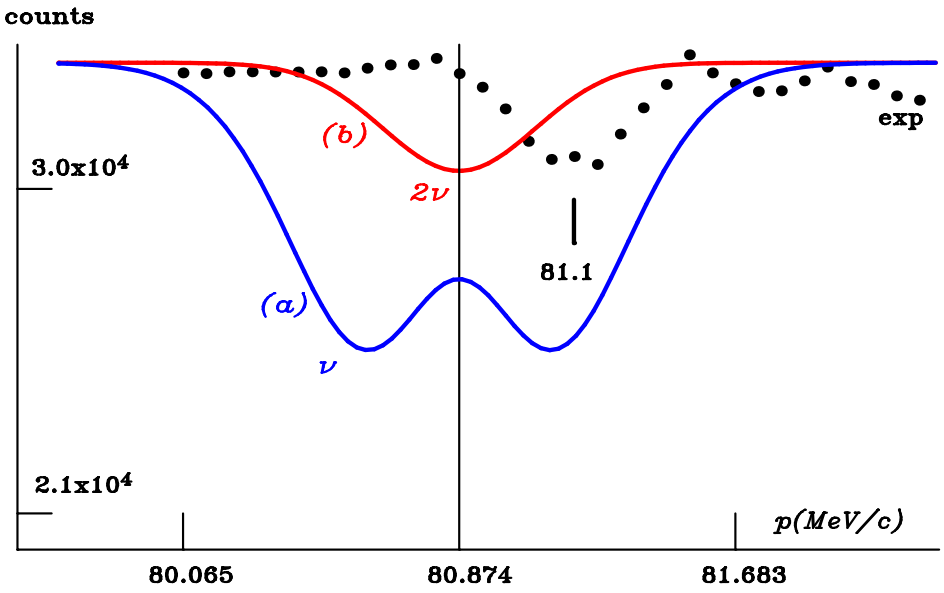


Figure 6.3: **Transcription of figure 4 of the Gouanere et al reference of 2008, which shows the experimental outcome of the detected number of electrons versus the linear momentum p of the electron beam in MeV/c (dotted line). Curve (a) (in blue) represents their Monte Carlo calculation for de Broglie's frequency ν. Curve (b) (in red) represents their Monte Carlo calculation by considering that the electron internal frequency is twice de Broglie's frequency 2ν. It matches with the experimental result except for a shift from 80.874 MeV/c to 81.1 MeV/c. This shift could be related to the temperature of the sample and therefore with a change of the parameter d of the lattice.**

obtained, see figure 6.3, was the resonant peak for the value $p = 81.1$ MeV/c instead of the expected p =80.874 MeV/c, which corresponds to $k = 2$ in the case when the internal frequency is that of our model or Dirac's frequency.

If the electron clock had De Broglie frequency $\nu_0 = mc^2/h$, then the resonant peaks will be those of the figure 6.4, while if the frequency is that of Dirac, twice De Broglie frequency, the resultant peaks will be those of the figure 6.5. All De Broglie peaks can also be obtained if the frequency is that of Dirac but not conversely. The presence of one kind of peaks or the other, will show as a first glance, how to discriminate between these alternative frequencies. The accurate measurement of the peaks represents an accurate measurement of the internal frequency of the electron ν_0. It would be desirable to enlarge the energy range of the experiment of Gouanére et al. to detect those peaks below 80.874 MeV/c. This will show the existence of this internal periodic motion and will allow us to determine the frequency of a high precission clock, the clock of the electron.

The accurate measurement of this frequency will be used to define a natural unit of time, associated to physical phenomena related the electrons and their interactions. With the natural unit of velocity c, this allows us to obtain a natural unit of length, and therefore in our classical kinematical description of the electron, all kinematical variables can be taken dimensionless in this sytem of units, thus justifying the additional scale invariance of the model.

6.3 Instantaneous electric dipole

The internal motion of the charge of the electron in the center of mass frame is a circle at the speed of light. The position of the charge in this frame is related to the total spin by eq.

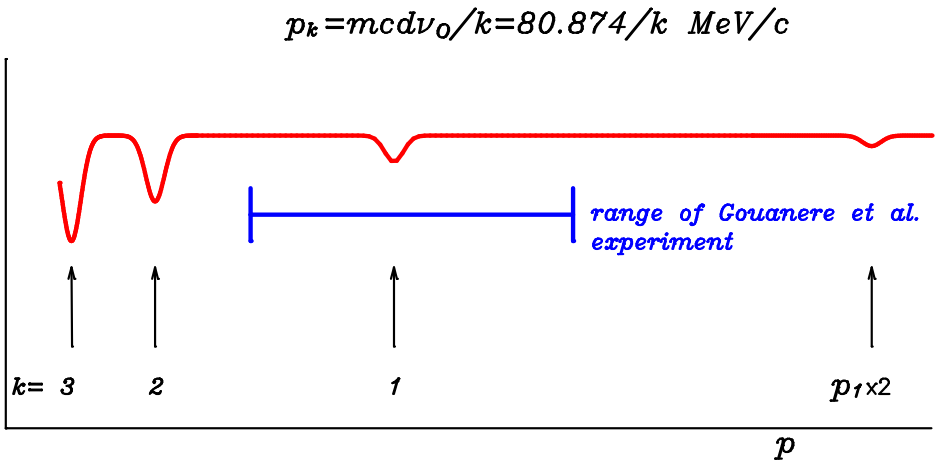


Figure 6.4: **Different resonant peaks of the interaction of the electron beam with the silicon lattice, if assumed that the internal electron frequency is De Broglie's frequency $\nu_0 = mc^2/h$. Some of the previsible peaks of the following figure do not appear in this ansatz.**

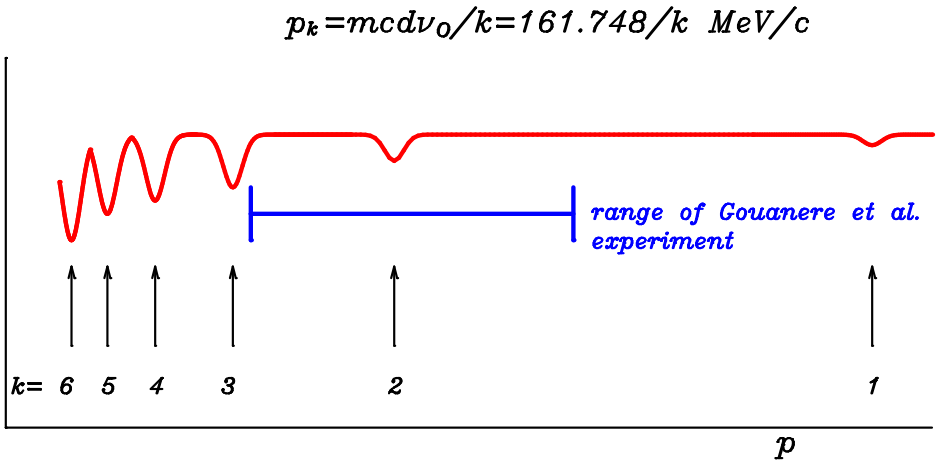


Figure 6.5: **Different resonant peaks of the interaction of the electron beam with the silicon lattice, if assumed that the internal electron frequency is twice De Broglie's frequency $\nu_0 = 2mc^2/h$. Only the peaks corresponding to p_k, $k = 1, \ldots, 6$, are depicted. The picks corresponding to $P_3, p_5, \ldots$ do not appear in the previous figure.**

(2.167), *i.e.*,

$$\boldsymbol{k} = \frac{1}{mc^2}\, \boldsymbol{S} \times \boldsymbol{u}, \tag{6.5}$$

where $\boldsymbol{S}$ is the total constant spin and $\boldsymbol{u} = d\boldsymbol{k}/dt$, with $u = c$ is the velocity of the charge. In addition to this motion there is a rotation of a local frame linked to the particle that gives rise to some angular velocity, but this rotation has no effect on the electric dipole structure. (See Fig. 6.6 where the angular velocity and the local frame are not depicted).

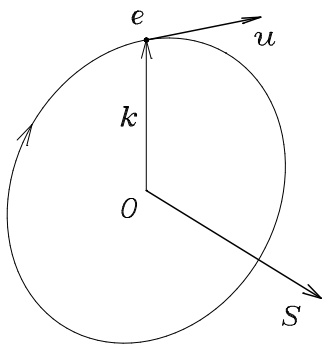


Figure 6.6: **Electron charge motion in the C.M. frame.**

Now, from the point of view of the center of mass observer, the particle behaves as though it has a magnetic moment related to the particle current by the usual classical expression

$$\boldsymbol{\mu} = \frac{1}{2}\int \boldsymbol{k} \times \boldsymbol{j}\; d^3\boldsymbol{r} = \frac{e}{2}\, \boldsymbol{k} \times \frac{d\boldsymbol{k}}{dt},$$

where e is the charge and $\boldsymbol{j}(\boldsymbol{r} - \boldsymbol{k}) = e\, d\boldsymbol{k}/dt\; \delta^3(\boldsymbol{r} - \boldsymbol{k})$ is the particle current density. The orbital term $\boldsymbol{k} \times d\boldsymbol{k}/dt$ is related to the zitterbewegung part of spin that quantizes with integer values and which for spin 1/2 and spin 1 charged particles is twice the total spin $\boldsymbol{S}$, giving rise to a pure kinematical interpretation of the gyromagnetic ratio $g = 2$ for this model as seen in the previous section.

But also in the center of mass frame the particle has an oscillating instantaneous electric dipole moment $\boldsymbol{d} = e\boldsymbol{k}$, that is thus related to the total spin by

$$\boldsymbol{d} = \frac{e}{mc^2}\, \boldsymbol{S} \times \boldsymbol{u}. \tag{6.6}$$

This instantaneous electric dipole, which fulfills the usual definition of the momentum of the point charge e with respect to the origin of the reference frame, is translation invariant because it is expressed in terms of a relative position vector $\boldsymbol{k}$. It can never be interpreted as some kind of fluctuation of a spherical symmetry of a charge distribution. Even in this kind of model, it is not necessary to talk about charge distributions, because all particle attributes are defined at a single point $\boldsymbol{r}$.

In his original 1928 article, [13] Dirac obtains that the Hamiltonian for the electron has, in addition to the Hamiltonian of a free point particle of mass m, two new terms that in the

[13] P.A.M. Dirac, *Proc. Roy. Soc. London*, **A117**, 610 (1928).

presence of an external electromagnetic field are

$$\frac{e\hbar}{2m}\boldsymbol{\Sigma}\cdot\boldsymbol{B}+\frac{ie\hbar}{2mc}\boldsymbol{\alpha}\cdot\boldsymbol{E}=-\boldsymbol{\mu}\cdot\boldsymbol{B}-\boldsymbol{d}\cdot\boldsymbol{E}, \tag{6.7}$$

where

$$\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}, \quad \text{and} \quad \boldsymbol{\alpha}=\gamma_0\boldsymbol{\gamma},$$

i.e., $\boldsymbol{\Sigma}$ is expressed in terms of $\boldsymbol{\sigma}$ Pauli-matrices and $\boldsymbol{\alpha}$ is Dirac's velocity operator when written in terms of Dirac's gamma matrices.

We shall show that the quantum counterpart of expression (6.6) is in fact the electric dipole term of Dirac's Hamiltonian (6.7). The difficulty to relate the classical expression of the dipole moment with its quantum version lies how to interpret the 'cross' product in (6.6), introduced by J. Willard Gibbs in 1884, in terms Dirac matrices or in terms of the matrix (or geometric) product of the elements of Dirac's algebra that represent the quantum version of the above observables, so that a short explanation to properly interpret these observables as elements of a Clifford algebra is given in what follows.

Both, velocity operator $\boldsymbol{u}=c\boldsymbol{\alpha}$ and spin operator $\boldsymbol{S}$ are bivectors in Dirac's algebra, considered as elements of the Geometric or Clifford algebra of space-time in the sense of Hestenes. [14]

In fact, Dirac's alpha matrices are written as a product of two gamma matrices $\alpha_i=\gamma_0\gamma_i$ and also the spin components $S_j=(i\hbar/2)\,\gamma_k\gamma_l$, j,k,l cyclic $1,2,3$, and where the four gamma matrices, γ_μ, $\mu=0,1,2,3$ are interpreted as the four basic vectors of Minkowski's space-time that generate Dirac's Clifford algebra. They satisfy $\gamma_\mu\cdot\gamma_\nu=\eta_{\mu\nu}$, *i.e.*, $\gamma_0^2=1$ and $\gamma_i^2=-1$, where the dot means the inner product in Dirac's Clifford algebra. We thus see that velocity and spin belong to the even subalgebra of Dirac's algebra and therefore they also belong to Pauli algebra or geometric algebra of three-dimensional space. Under spatial inversions $\gamma_0\rightarrow\gamma_0$ and $\gamma_i\rightarrow-\gamma_i$, the velocity operator changes its sign and it is thus a spatial vector, while the spin is invariant under this transformation as it corresponds to a spatial bivector or pseudovector.

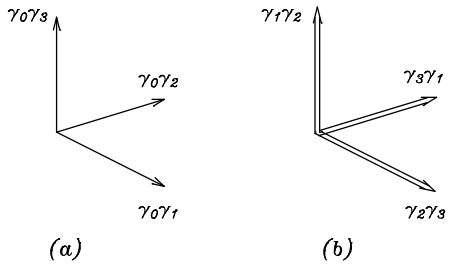


Figure 6.7: **A basis for vectors** (a) **and bivectors (pseudovectors)** (b) **of Pauli algebra.**

The relationship between the cross product and the outer and inner product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in Pauli algebra is,

$$\boldsymbol{a}\times\boldsymbol{b}=-i\boldsymbol{a}\wedge\boldsymbol{b}=\boldsymbol{b}\cdot(i\boldsymbol{a}), \tag{6.8}$$

where $\wedge$ represents the symbol for the outer product in the geometric algebra, the imaginary unit i represents the unit pseudoscalar three-vector and $i\boldsymbol{a}$ is the dual bivector of vector $\boldsymbol{a}$.

The inner product of a vector $\boldsymbol{b}$ and a bivector A is expressed in terms of the geometric product in the form

$$\boldsymbol{b}\cdot A=\frac{1}{2}(\boldsymbol{b}A-A\boldsymbol{b}) \tag{6.9}$$

[14] D. Hestenes, *Space-Time algebra*, Gordon and Breach, NY (1966); D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, D. Reidel Pub. Co. Dordrecht, (1984).

where in Dirac's or Pauli algebra the geometric product $\boldsymbol{b}A$ is just the ordinary multiplication of matrices.

If we choose a basis of vectors and pseudovectors as in Fig. 6.7, where the double-lined objects of part (b) represent the dual vectors of the corresponding spatial bivectors, and express in these bases the observables of Fig. 6.6, then the spatial velocity vector $\boldsymbol{u} = c\gamma_0\gamma_2$ and the pseudovector $\boldsymbol{S} = (\hbar/2)\gamma_2\gamma_3$ and therefore, using (6.8) and (6.9) we get

$$\boldsymbol{S} \times \boldsymbol{u} = \boldsymbol{u} \cdot (i\boldsymbol{S}) = \frac{ic\hbar}{2}\left(\frac{1}{2}\left(\gamma_0\gamma_2\gamma_2\gamma_3 - \gamma_2\gamma_3\gamma_0\gamma_2\right)\right) = \frac{-ic\hbar}{2}\gamma_0\gamma_3.$$

Now vector $\boldsymbol{k} = R\gamma_0\gamma_3$ with $R = \hbar/2mc$, and by substitution in (6.6) we get the desired result,

$$d_3 = -\frac{ie\hbar}{2mc}\alpha_3.$$

From this relation, if we use as a unit of length the radius R_0, this implies that the separation, from the quantum mechanical point of view, between the CC and the CM, in dimensionless units is

$$\widetilde{\boldsymbol{r}} - \widetilde{\boldsymbol{q}} = -i\boldsymbol{\alpha}.$$

From the algebraic poitn of view, Dirac velocity operator in dimensionless units is $\widetilde{\boldsymbol{u}} = \boldsymbol{\alpha}$. As we have seen in section **4.2.3**, a rotation of value $\pi/2$ in the phase of the internal motion, $e^{i\pi/2} = i$ is equivalent to rotate $\pi/2$ the velocity vector, and this defines a dimensionles vector from CC to CM, and thus the opposite vector is $\widetilde{\boldsymbol{r}} - \widetilde{\boldsymbol{q}} = -i\boldsymbol{\alpha}$.

6.3.1 Darwin term of Dirac's Hamiltonian

When analyzing Dirac's equation in the presence of an external electric field $\boldsymbol{E}$, Darwin [15] found in the expansion of the Hamiltonian an energy term, that bears his name, of the form

$$-\frac{\hbar^2 e}{8m^2c^2}\,\nabla\cdot\boldsymbol{E} \equiv \frac{\hbar^2 e}{8m^2c^2}\nabla^2 V. \tag{6.10}$$

The usual interpretation of this term [16] corresponds to the idea of the zitterbewegung and therefore to the fluctuation of the position of the electron $\boldsymbol{r}$ around the center of mass $\boldsymbol{q}$. In our model this is very well understood because for the spinning electron, there is a separation S/mc between the center of mass and center of charge. Thus, by expanding the interaction potential around the center of mass $\boldsymbol{q}$ we get

$$V(\boldsymbol{q} + \delta\boldsymbol{r}) = V(\boldsymbol{q}) + \delta\boldsymbol{r}\cdot\nabla V + \frac{1}{2}\delta r_i \delta r_j \frac{\partial^2 V}{\partial r_i \partial r_j} + \cdots.$$

The fluctuation of the relative coordinates of the center of charge position vanish and thus $< \delta r_i >= 0$. Similarly $< \delta r_i \delta r_j >= 0$ for $i \neq j$ and the lowest order non-vanishing terms come from the fluctuations of

$$< (\delta r_1)^2 >=< (\delta r_2)^2 >=< (\delta r_3)^2 >= \frac{1}{3} < |\delta\boldsymbol{r}|^2 >= \frac{1}{3}\frac{S^2}{m^2c^2}.$$

We thus get

$$V(\boldsymbol{q} + \delta\boldsymbol{r}) = V(\boldsymbol{q}) + \frac{S^2}{6m^2c^2}\nabla^2 V,$$

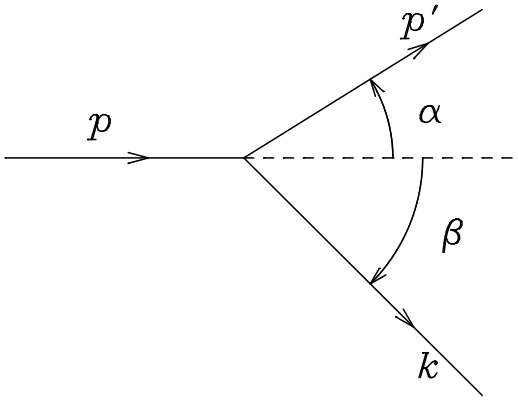
[15] C.G. Darwin, *Proc. Roy. Soc. (London)*, **A118**, 654 (1928).
[16] J.J. Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley, Reading Mass. (1967), p.119.

but for a spin 1/2 particle $S^2 = 3\hbar^2/4$ and by multiplying the above expression by the electric charge e we get the electrostatic potential energy of a charge at point $\boldsymbol{q}$, $eV(\boldsymbol{q})$, and the additional Darwin term (6.10). One important feature is that the Darwin term, related to the separation between the center of mass and center of charge, can also be used within a non-relativistic context, as shown by Fushchich et al. [17]

6.4 Compton Effect

Circularly polarized light corresponds to a beam of photons where all the photons have their spins oriented in the same direction, forward or backward [18]. In the interaction of a plane electromagnetic wave with a pointelectron at rest the scattered electron always moves forward, at a certain angle with the direction of the wave motion. The wave transfers a linear momentum which has a component in the direction of the propagation of the wave. If we interpret this direction with the direction of the motion of the photons in the beam, we can make a relativistic analysis of the interaction between the electron and photon, considered as point particles. This analysis was performed by Arthur H. Compton in 1923.

In fact, Let us assume that we have an electron at rest and an incoming photon along the axis OZ of linear momentum p and energy $H = pc = hf$. After the collision, the photon comes out with a linear momentum p' at an angle α with the initial direction and the electron with a linear momentum k at the direction β, as depicted in the figure. Conservation of linear



momentum and energy leave us to:

$$p'\cos\alpha + k\cos\beta = p, \quad p'\sin\alpha = k\sin\beta, \quad cp + mc^2 = cp' + \sqrt{m^2c^4 + c^2k^2}. \tag{6.11}$$

It is a system of three equations with four unknowns, α, β, p' and k, which has no unique solution. But if we fix one of these variables, for instance the outgoing direction of the photon α, we get a unique value for the remaining ones p', k and β. Since $\boldsymbol{k} = \boldsymbol{p} - \boldsymbol{p}'$,

$$k^2 = p^2 + p'^2 - 2\boldsymbol{p}\cdot\boldsymbol{p}' = \frac{1}{c^2}\left(H^2 + H'^2 - 2HH'\cos\alpha\right)$$

[17] V.I. Fushchich, A. Nikitin and V.A. Sagolub, *Rep. Math. Phys.* **13**, 175 (1978).

[18] R. A. Beth, *Mechanical Detection and Measurement of the Angular Momentum of Light*, Phys. Rev. **50**, 115 (1936).

and by substituting this in

$$c(p-p')+mc^2=\sqrt{m^2c^4+c^2k^2}$$

taking the square, we arrive to

$$mc^3(p-p')=mc^2(H-H')=HH'(1-\cos\alpha),\quad \frac{1}{H'}-\frac{1}{H}=\frac{1-\cos\alpha}{mc^2}$$

and therefore the relation between the frequencies of the photons involved in this reaction is

$$\frac{1}{f'}-\frac{1}{f}=\frac{h}{mc^2}(1-\cos\alpha). \tag{6.12}$$

and for the wavelengths

$$\lambda'-\lambda=\lambda_C(1-\cos\alpha),\quad \lambda_C=\frac{h}{mc}=2.426\cdot 10^{-12}\text{m},$$

where λ_C represents the Compton wavelength of the electron. In the models of spinning electrons the separation between the center of charge and the center of mass is $R_0=\hbar/2mc=1.93{\cdot}10^{-13}$m.

The variation of the energy of the scattered photon in terms of the dispersion angle is given by

$$hf'=\frac{hf}{1+\dfrac{hf}{mc^2}(1-\cos\alpha)}=hfA(\alpha),\quad A(\alpha)=\frac{1}{1+q(1-\cos\alpha)},$$

where $q=p/mc$ and it is represented in the figure:

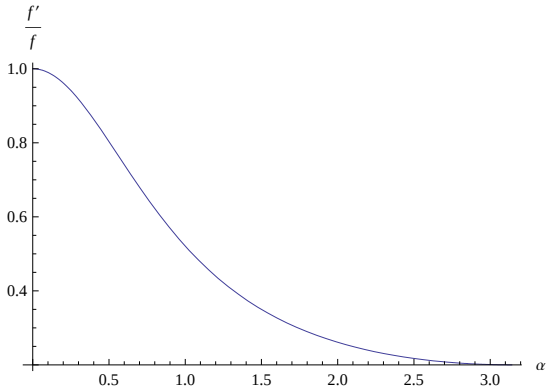


Figure 6.8: **Representation of the function $A(\alpha)=f'/f$, for an incoming photon of $hf=2mc^2$.**

In this figure, when $\alpha=\pi$, the photon goes backward, $f'/f=0.2$, and this corresponds to an experiment with photons of energy $hf=2mc^2\approx 1$ MeV, twice as much as the rest energy of the electron.

The linear momenta of the outgoing photon and electron are, respectively:

$$p'=\frac{pmc}{mc+p(1-\cos\alpha)}=pA(\alpha),\quad k=p\sqrt{1+A(\alpha)^2-2A(\alpha)\cos\alpha}, \tag{6.13}$$

and the dispersion angle of the electron

$$\sin\beta=\frac{A(\alpha)\sin\alpha}{\sqrt{1+A(\alpha)^2-2A(\alpha)\cos\alpha}},\quad \cos\beta=\frac{1-A(\alpha)\cos\alpha}{\sqrt{1+A(\alpha)^2-2A(\alpha)\cos\alpha}}. \tag{6.14}$$

We see in the figures **6.10** and **6.11** that for low dispersion angles of the photon, $\alpha\approx 0$, $\sin\beta\approx 1$, and the electron goes out basically at 90° of the incoming photon. It only goes forward $\sin\beta=0$, when the scattered photon is coming backwards $\alpha\approx\pi$. For $\alpha\approx\pi/2$, we get in both cases $\sin\beta\approx 0.3$ and 0.62, respectively.

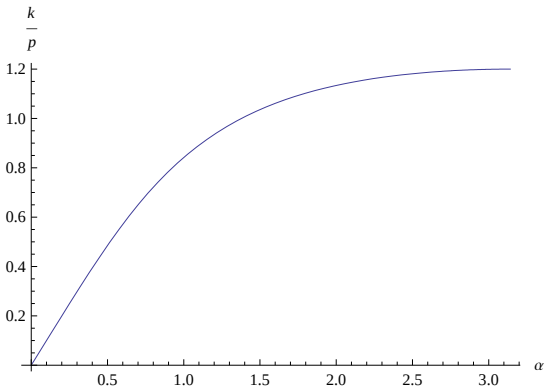


Figure 6.9: **Variation with α of the linear momentum k of the otgoing electron.**

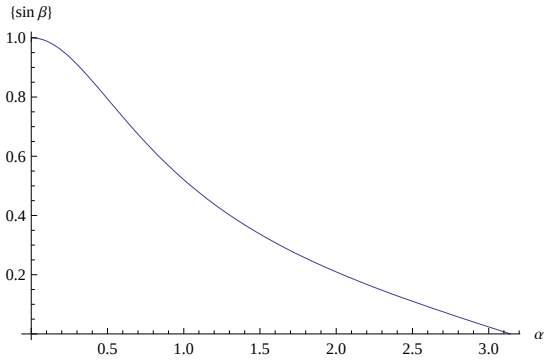


Figure 6.10: **Variation of the dispersion angle β of the outgoing electron in terms of the dispersion angle of the photon, in the case $H \approx 1\text{MeV} = 2mc^2$.**

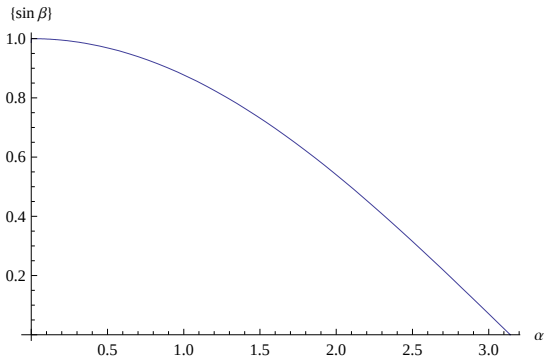


Figure 6.11: **Variation of the dispersion angle of the outgoing electron β in terms of the photon dispersion angle, for an incoming photon of energy 5 eV, $H \approx 10^{-5}mc^2$.**

6.4.1 Contribution of the spin

If we take into account the spins of the photon Z and of the electron S, the interaction also conserves the total angular momentum of this system.

Let us take the OZ axis along the direction of the motion of the incoming photon and the plane XOZ as the plane containing the linear momenta of the outgoing particles.Let θ and ϕ be the initial orientation of the spin of the electron, and θ' And ϕ' its final orientation. Because the spin of the photon has the direction of its linear momentum, up to a sign, the conservation of the three components of the total angular momentum leads to:

$$S\sin\theta\cos\phi = Z_2\sin\alpha + S\sin\theta'\cos\phi', \quad S\sin\theta\sin\phi = S\sin\theta'\sin\phi',$$

$$Z_1 + S\cos\theta = Z_2\cos\alpha + S\cos\theta',$$

i.e.,

$$\sin\theta\cos\phi = (Z_2/S)\sin\alpha + \sin\theta'\cos\phi', \quad \sin\theta\sin\phi = \sin\theta'\sin\phi',$$

$$(Z_1/S) + \cos\theta = (Z_2/S)\cos\alpha + \cos\theta',$$

where Z_1 and Z_2 are the components of the spin of the photon in the direction of its motion, and we know that takes the value $Z_1 = Z_2 = \pm\hbar$. It is a system of three equations with three unknowns θ', ϕ' and α, which will give us finally the spin orientation of the electron θ', ϕ' and the outgoing direction of the photon α. Together the conservation equations of the previous section, we see that the direction of the outgoing photon α depends on the initial orientation of the spin of the electron, which can be controlled by means of an external magnetic field.

From the first two equations we arrive to

$$\sin^2\theta' = (Z_2/S)^2\sin^2\alpha - 2(Z_2/S)\sin\alpha\sin\theta\cos\phi + \sin^2\theta.$$

If in the last one we separate the term of θ' and take the squared of it,

$$\cos^2\theta' = \left(Z_1/S + \cos\theta - (Z_2/S)\cos\alpha\right)^2$$

adding together, we obtain an equation with a single unknown, α, the outgoing angle of the photon:

$$\sin\alpha\sin\theta\cos\phi = (Z_1/Z_2 - \cos\alpha)(Z_1/S + \cos\theta).$$

If Z_1 and Z_2 have the same orientation with respect to the corresponding linear momentum of the photon, $Z_1/Z_2 = +1$, and in terms of the half angle we get

$$\cot(\alpha/2) = \frac{Z_1/S + \cos\theta}{\sin\theta\cos\phi}, \quad Z_1/Z_2 = +1,$$

but if Z_1 and Z_2 have the opposite orientation $Z_1/Z_2 = -1$,

$$\tan(\alpha/2) = -\frac{Z_1/S + \cos\theta}{\sin\theta\cos\phi}, \quad Z_1/Z_2 = -1.$$

Because $\cot(z/2) = \sqrt{(1+\cos z)/(1-\cos z)} = 1/\tan(z/2)$, this implies that

$$\cos\alpha = \frac{(Z_1/S + \cos\theta)^2 - \sin^2\theta\cos^2\phi}{(Z_1/S + \cos\theta)^2 + \sin^2\theta\cos^2\phi}, \quad Z_1/Z_2 = 1. \tag{6.15}$$

$$\cos\alpha = \frac{\sin^2\theta\cos^2\phi - (Z_1/S + \cos\theta)^2}{(Z_1/S + \cos\theta)^2 + \sin^2\theta\cos^2\phi}, \quad Z_1/Z_2 = -1. \tag{6.16}$$

which is the opposite to the previous one. For the angle ϕ' one obtains:

$$\tan\phi' = \frac{\tan\phi}{1 - \dfrac{Z_2 \sin\alpha}{S \sin\theta\cos\phi}} \tag{6.17}$$

while θ' is obtained from the last equation

$$\cos\theta' = \cos\theta + Z_1/S - Z_2/S\cos\alpha. \tag{6.18}$$

If we take the absolute value of the spin of the electron as $S = \sqrt{(1/2+1)/2}\hbar$, $Z_1/S = \pm 2/\sqrt{3}$. If we represent the value of $\cos\alpha$, we obtain the figures 6.12. To know where the $\cos\alpha$ reach

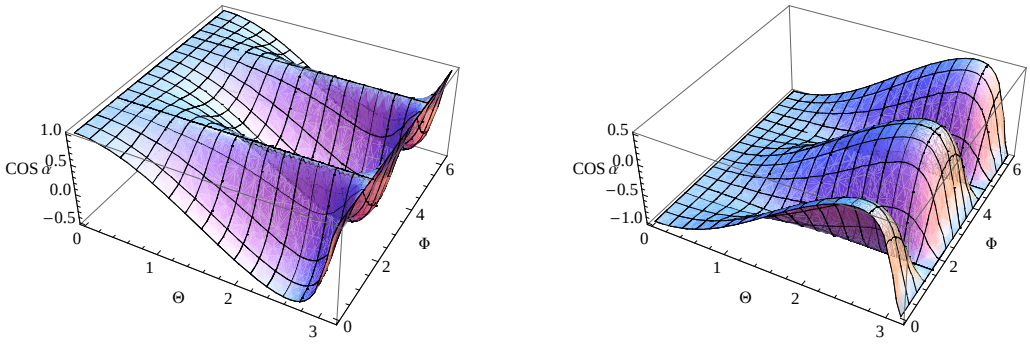
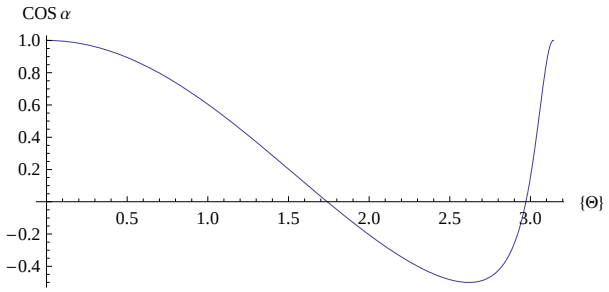


Figure 6.12: **Variation of $\cos\alpha$ in the case$Z_1/Z_2 = 1$, $Z_1/S = 2/\sqrt{3}$ and $Z_1/Z_2 = -1$, respectively.**

a maximum or a minimum, is sufficient to take the derivative with respect to θ and ϕ in the expression (6.15), equate both derivatives to zero, and we obtain the system of equations

$$\sin\theta\cos^2\phi(Z_1/S + \cos\theta)(1 + Z_1/S\cos\theta) = 0, \quad \sin^2\theta\sin 2\phi(Z_1/S + \cos\theta) = 0,$$

such that the minimum is reached for $\cos\theta = -S/Z_1 = -\sqrt{3}/2$, $\theta = 5\pi/6$, $\phi = 0, \pi$, and takes the value $\cos\alpha = -1/2$. For this value, the scattered photon leaves at an angle $\alpha = 2\pi/3$, or $120°$, which is the maximum backward direction of any photon. A greater backward direction would imply that the electron would be moving towards the photon. For $\phi = \pi$, the variation of $\cos\alpha$ with θ of (6.15) is that of the figure



If Z_1 is oriented backwards, we obtain this maximum outgoing direction, but the spin of the electron has to have the orientation $\cos\theta = \sqrt{3}/2$, $\theta = \pi/6$, $\phi = 0, \pi$.

In the first case, if $\theta = 0$, or aproximately zero, whatever ϕ be, results $\alpha = 0$ and the photon is not deviated. the same happens in the second case if $\theta = \pi$. Polarized electrons in the direction of the motion of the photons seem to be invisible.

We have taken as the plane ZOX the plane which contains both photons, the initial and the final dispersed photon, and in this plane the angle of dispersion α is determined by the

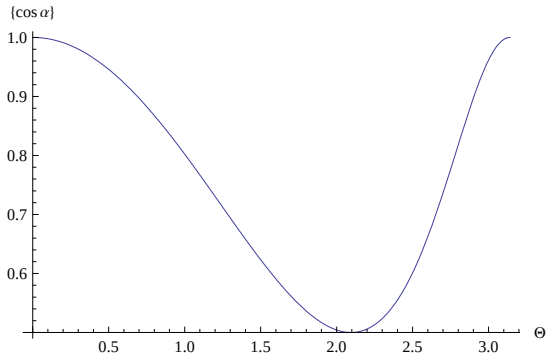


Figure 6.13: **If $Z_1/S = 2$, the variation of $\cos\alpha$ with the spin orientation leaves to $\phi = \pi$ and the outgoing photon can only exit within the angle $\pm 60^\circ$ with respect to the incoming direction.**

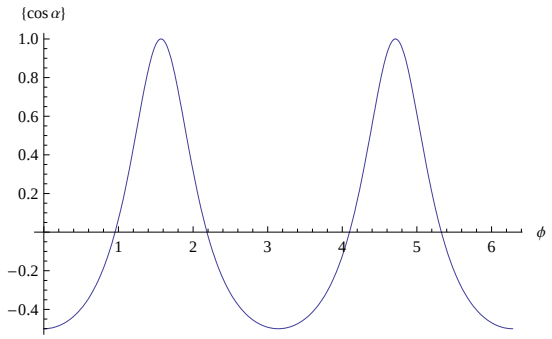


Figure 6.14: **Variation of $\cos\alpha$ versus ϕ, for the value $\theta = 5\pi/6$ in the first case, or $\theta = \pi/6$ in the second.**

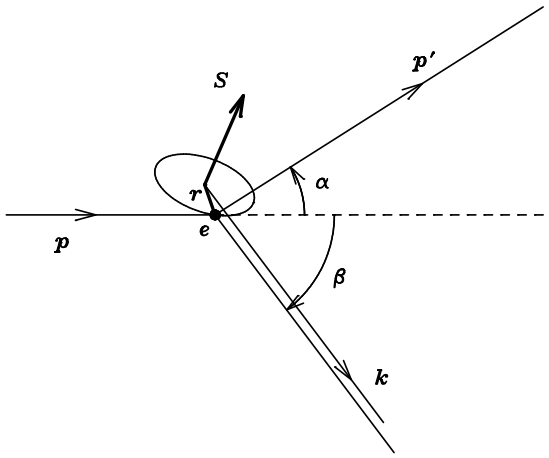
orientation of the spin of the electron (θ, ϕ). Conversely, if what we control is the orientation of the spin of the electron, then the dispersion plane is undefined by a rotation of arbitrary value $-\phi$, where the outgoing photon makes an angle α with the initial direction. In another form, if what we control in the laboratory is θ, the orientation of the outgoing photon is $(\alpha, -\phi)$ with respect to the plane subtended by the direction of the incoming photon and the direction of the spin of the electron.

If the experiment is not performed with free electrons, but rather with the electrons bound to atoms, the interaction can excited the atoms to upper states and the problem is different. Because the electron is an elementary particle, it is not possible to modify the absolute value of its spin, which is what we have considerd in this analysis.

Experimentaly outgoing photons with a dispersion angle greater than 150° with respect to the incoming direction are found, but according to this analysis the greater angle is of 120°. In all this calculation we have assumed a strict point-like electron. This experimental fact invalidates this calculation, and we have to take into account the feature that the center of mass and center of charge of a Dirac particle, are different points. From the electromagnetic point of view, the photon collides with the center of charge of the electron. We have to recalculate all these items because we have to consider also the angular momentum of the moving electron with respect to its center of charge.

6.4.2 Model of spinning electron

We are going to consider that the pointlike photon collides with the center of charge of the electron. We take this center of charge as the origin of the laboratory reference frame as depicted in the figure: The incoming photon goes along the OZ axis, the outgoing photon is



contained in the XOZ plane with an angle α with the direction OZ and the spin of the electron at rest $\boldsymbol{S}$ has the usual orientation θ And ϕ. The scattered electron has a linear momentum $\boldsymbol{k}$ also contained in the plane XOZ with an angle β with respect to OZ.

We show in the picture the structure of the electron at rest as a Dirac particle with the center of mass and center of charge two different points. The vector $\boldsymbol{r}$ represents the relative position of the center of charge with respect to the center of mass. We call ψ the phase of the internal motion of the center of charge and $R_0 = S/mc$, the radius of this internal motion at the speed of light.

If the spin of the electron were oriented along OZ axis, the vector $\boldsymbol{r}$ should be given by

$$\boldsymbol{r}_0 = R_0 \begin{pmatrix} \cos\psi \\ \sin\psi \\ 0 \end{pmatrix},$$

but because it has an orientation θ and ϕ, we have:

$$\boldsymbol{r} = \mathcal{R}_z(\phi)\mathcal{R}_y(\theta)\boldsymbol{r}_0 = R_0 \begin{pmatrix} \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi \\ \cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi \\ -\sin\theta\cos\psi \end{pmatrix}$$

Before the collision, the angular momentum of the system electron-photon, with respect to the origin, is just the sum of the spins of both particles, because the electron is at rest, $\boldsymbol{J}_1 = \boldsymbol{Z}_1 + \boldsymbol{S}$ and expressed by its cartesian components we have:

$$\boldsymbol{J}_1 = \begin{pmatrix} S\sin\theta\cos\phi \\ S\sin\theta\sin\phi \\ Z_1 + S\cos\theta \end{pmatrix}$$

After the collision the photon has a spin $\boldsymbol{Z}_2$ and the electron spin is oriented with angles θ' and ϕ', and also a linear momentum $\boldsymbol{k}$. The angular momentun with respect to the origin after the collision is

$$\boldsymbol{J}_2 = \boldsymbol{Z}_2 + \boldsymbol{S}' - \boldsymbol{r}\times\boldsymbol{k}, \quad \boldsymbol{k} = k\begin{pmatrix} -\sin\beta \\ 0 \\ \cos\beta \end{pmatrix},$$

and expressed by components:

$$\boldsymbol{r}\times\boldsymbol{k} = R_0 k \begin{pmatrix} (\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\cos\beta \\ \sin\theta\cos\psi\sin\beta - (\cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi)\cos\beta \\ (\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\sin\beta \end{pmatrix}$$

$$\boldsymbol{Z}_2 = Z_2\begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}, \qquad \boldsymbol{S}' = S\begin{pmatrix} \sin\theta'\cos\phi' \\ \sin\theta'\sin\phi' \\ \cos\theta' \end{pmatrix}.$$

From energy and linear momentum conservation the relationship between β and α of the previous section (6.14) still holds. If the total angular momentum is conserved, and if we express the primed angles in terms of the others, we get a system of equations:

$$\boldsymbol{S}' = \boldsymbol{J}_1 - \boldsymbol{Z}_2 + \boldsymbol{r}\times\boldsymbol{k}$$

$$S\sin\theta'\cos\phi' = S\sin\theta\cos\phi - Z_2\sin\alpha + R_0 k\,(\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\cos\beta,$$

$$S\sin\theta'\sin\phi' = S\sin\theta\sin\phi + R_0 k\,(\sin\theta\cos\psi\sin\beta - (\cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi)\cos\beta),$$

$$S\cos\theta' = Z_1 + S\cos\theta - Z_2\cos\alpha + R_0 k\,(\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\sin\beta.$$

If we define the magnitudes $a = Z_1/S$, $b = Z_2/S$ and $R = R_0 k/S$, these equations look:

$$\begin{aligned} \sin\theta'\cos\phi' &= \sin\theta\cos\phi - b\sin\alpha + R\,(\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\cos\beta \\ \sin\theta'\sin\phi' &= \sin\theta\sin\phi + R\,(\sin\theta\cos\psi\sin\beta - (\cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi)\cos\beta) \\ \cos\theta' &= a + \cos\theta - b\cos\alpha + R\,(\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\sin\beta. \end{aligned}$$

Here we have to substitute $\sin\beta$ and $\cos\beta$ in terms of $A(\alpha)$ and since $R_0 = S/mc$, the coefficient

$$R = \frac{R_0 k}{S} = q\sqrt{1 + A(\alpha)^2 - 2A(\alpha)\cos\alpha},$$

with $q = p/mc$, and thus the terms

$$R\cos\beta = q(1 - A(\alpha)\cos\alpha), \quad R\sin\beta = qA(\alpha)\sin\alpha.$$

The final equations to obtain the orientation of the spin of the electron are

$$\sin\theta'\cos\phi' = \sin\theta\cos\phi - b\sin\alpha + q(\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)(1 - A(\alpha)\cos\alpha),$$

$$\sin\theta'\sin\phi' = \sin\theta\sin\phi + q\left(A(\alpha)\sin\theta\cos\psi\sin\alpha - (\cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi)(1 - A(\alpha)\cos\alpha)\right),$$

$$\cos\theta' = a + \cos\theta - b\cos\alpha + qA(\alpha)(\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\sin\alpha.$$

If we take the phase $\psi = 0$, at the instant when the photon collides with the center of charge of the electron, the above equations simplify to:

$$\sin\theta'\cos\phi' = \sin\theta\cos\phi - b\sin\alpha + q\cos\theta\sin\phi(1 - A(\alpha)\cos\alpha), \tag{6.19}$$

$$\sin\theta'\sin\phi' = \sin\theta\sin\phi + q\left(A(\alpha)\sin\theta\sin\alpha - \cos\theta\cos\phi(1 - A(\alpha)\cos\alpha)\right), \tag{6.20}$$

$$\cos\theta' = a + \cos\theta - b\cos\alpha + qA(\alpha)\cos\theta\sin\phi\sin\alpha. \tag{6.21}$$

They are three extra equations that together the other three (6.11) will supply a system of six equations to determine $\alpha, \beta, p', k, \theta'$ and ϕ', if the initial orientation of the spin of the electron θ, ϕ is known.

By controlling the initial orientation of the spin of the electron, i.e., the values θ and ϕ, there would be determined the direction α and the energy p' of the outgoing photons and this device could be used as a fine tunning device for producing photons of very accurate frequency on a cone of semiangle α around the dispersion center, with the only requirement of acting on the free electrons (for instance in a Penning trap) with some external magnetic field.

6.5 Classical Tunneling

As a consequence of the zitterbewegung and therefore of the separation between the center of mass and center of charge, we shall see that spinning particles can have a non-vanishing crossing of potential barriers.

Let us consider a spinning particle with spin of (anti)orbital type, as described in Section 2.2, under the influence of a potential barrier. The Langrangian of this system is given by:

$$L = \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}} - \frac{m}{2\omega^2}\frac{\dot{\boldsymbol{u}}^2}{\dot{t}} - eV(\boldsymbol{r})\dot{t}. \tag{6.22}$$

Sharp walls correspond classically to infinite forces so that we shall consider potentials that give rise to finite forces like those of the shape depicted in Fig. 6.15, where V_0 represents the top of the potential.

Then the external force $F(x)$, is constant and directed leftwards in the region $x \in (-a, 0)$ and rightwards for $x \in (0, b)$, vanishing outside these regions.

Potentials of this kind can be found for instance in the simple experiment depicted in Figure 6.16 in which an electron beam, accelerated with some acceleration potential V_a, is sent into the uniform field region of potential V_0 contained between the grids or plates A, C and B.

In Figure 6.16 from a strict classical viewpoint a spinless electron stops at the dotted line and is rejected backwards. But a classical spinning electron can cross the barrier provided its kinetic energy is above some minimum value, although below the top of the potential. This minimum value depends on the separation between plates.

Let us assume for simplicity that the spin is pointing up or down in the z direction such that the point charge motion takes place in the XOY plane. Let q_x, q_y and $q_z = 0$, be the coordinates of the center of mass and x, y and $z = 0$, the position of the charge.

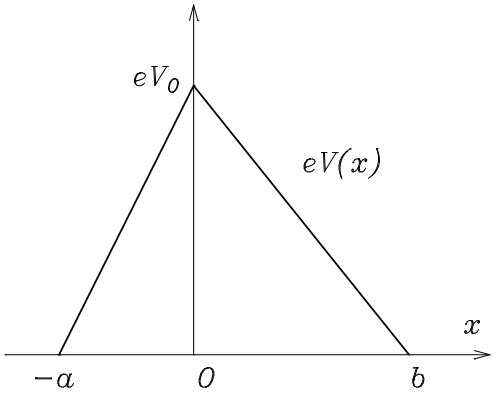


Figure 6.15: **Triangular potential barrier.**

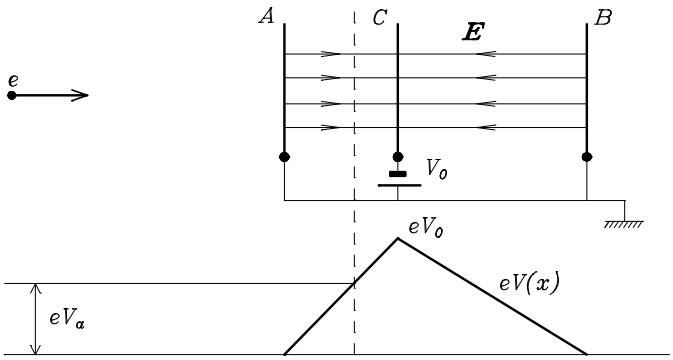


Figure 6.16: **Electron beam into a potential barrier. A classical spinless electron never crosses the dotted line. It is stopped there and rejected backwards. A spinning particle of the same kinetic energy might cross the barrier.**

The dynamical equations are

$$\frac{d^2q_x}{dt^2} = \frac{1}{m}F(x), \quad \frac{d^2q_y}{dt^2} = 0, \tag{6.23}$$

$$\frac{d^2x}{dt^2} + \omega^2(x - q_x) = 0, \quad \frac{d^2y}{dt^2} + \omega^2(y - q_y) = 0, \tag{6.24}$$

where

$$F(x) = \begin{cases} -eV_0/a, & \text{for } x \in (-a, 0), \\ eV_0/b, & \text{for } x \in (0, b), \\ 0, & \text{otherwise.} \end{cases}$$

Equations (6.23) are nonlinear and we have not been able to obtain an analytical solution in closed form. We shall try to find a numerical solution. To make the corresponding numerical analysis we shall define different dimensionless variables. Let R be the average separation between the center of charge and center of mass. In the case of circular internal motion, it is just the radius R_0 of the zitterbewegung. Then we define the new dimensionless position variables:

$$\hat{q}_x = q_x/R, \quad \hat{q}_y = q_y/R, \quad \hat{x} = x/R, \quad \hat{y} = y/R, \quad \hat{a} = a/R, \quad \hat{b} = b/R.$$

The new dimensionless time variable $\alpha = \omega t$ is just the phase of the internal motion, such that the dynamical equations become

$$\frac{d^2\hat{q}_x}{d\alpha^2} = A(\hat{x}), \quad \frac{d^2\hat{q}_y}{d\alpha^2} = 0,$$

$$\frac{d^2\hat{x}}{d\alpha^2} + \hat{x} - \hat{q}_x = 0, \quad \frac{d^2\hat{y}}{d\alpha^2} + \hat{y} - \hat{q}_y = 0,$$

where $A(\hat{x})$ is given by

$$A(\hat{x}) = \begin{cases} -eV_0/\hat{a}m\omega^2R^2, & \text{for } \hat{x} \in (-\hat{a}, 0), \\ eV_0/\hat{b}m\omega^2R^2, & \text{for } \hat{x} \in (0, \hat{b}), \\ 0, & \text{otherwise.} \end{cases}$$

In the case of the relativistic electron, the internal velocity of the charge is $\omega R = c$, so that the parameter $e/mc^2 = 1.9569 \times 10^{-6}\text{V}^{-1}$, and for potentials of order of 1 volt we can take the dimensionless parameter $eV_0/m\omega^2R^2 = 1.9569 \times 10^{-6}$.

If we choose as initial conditions for the center of mass motion

$$\hat{q}_y(0) = 0, \quad d\hat{q}_y(0)/d\alpha = 0,$$

then the center of mass is moving along the OX axis. The above system reduces to the analysis of the one-dimensional motion where the only variables are $\hat{q}_x$ and $\hat{x}$. Let us call from now on these variables q and x respectively and remove all hats from the dimensionless variables. Then the dynamical equations to be solved numerically are just

$$\frac{d^2q}{d\alpha^2} = A(x), \quad \frac{d^2x}{d\alpha^2} + x - q = 0, \tag{6.25}$$

where $A(x)$ is given by

$$A(x) = \begin{cases} -1.9569 \times 10^{-6}\, a^{-1}V_0, & \text{for } x \in (-a, 0), \\ 1.9569 \times 10^{-6}\, b^{-1}V_0, & \text{for } x \in (0, b), \\ 0, & \text{otherwise.} \end{cases} \tag{6.26}$$

Numerical integration has been performed by means of the computer package *Dynamics Solver*. [19] The quality of the numerical results is tested by using the different integration schemes this program allows, ranging from the very stable embedded Runge-Kutta code of eight order due to Dormand and Prince to very fast extrapolation routines. All codes have adaptive step size control and we check that smaller tolerances do not change the results.

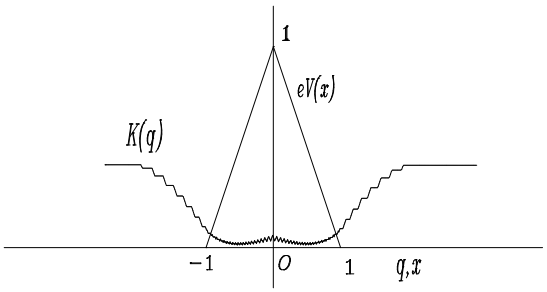


Figure 6.17: **Kinetic Energy during the crossing for the values** $a = b = 1$**.**

With $a = b = 1$, and in energy units such that the top of the barrier is 1, if we take an initial kinetic energy K below this threshold, $K = m\dot{q}(0)^2/2eV_0 = 0.41$ we obtain for the center of mass motion the graphic depicted in Fig. 6.17, where is shown the variation of the kinetic energy of the particle $K(q)$, with the center of mass position during the crossing of the barrier. There is always crossing with a kinetic energy above this value. In Fig. 6.18, the same graphical evolution with $a = 1$ and $b = 10$ and $K = 0.9055$ for a potential of 10^3 Volts in which the different stages in the evolution are evident. Below the initial values for the kinetic energy of 0.4 and 0.9 respectively, the particle does not cross these potential barriers and it is rejected backwards.

If in both examples the parameter a is ranged from 1 to 0.05, thus making the left slope sharper, there is no appreciable change in the crossing energy, so that with $a = 1$ held fixed we can compute the minimum crossing kinetic energies for different b values, $K_c(b)$.

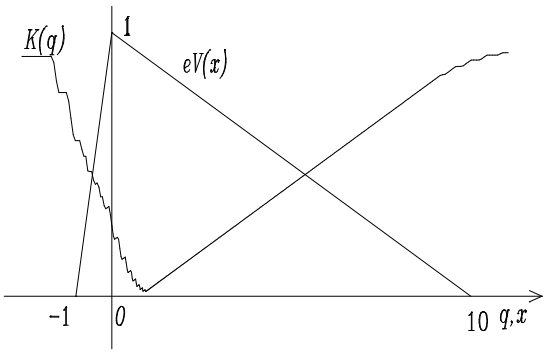


Figure 6.18: **Kinetic Energy during the crossing for the values** $a = 1$**,** $b = 10$**.**

To compare this model with the quantum tunnel effect, let us quantize the system. In the quantization of generalized Lagrangians developed in the Chapter 3, the wave function for this system is a squared-integrable function $\psi(t, \boldsymbol{r}, \boldsymbol{u})$, of the seven kinematical variables and the generators of the Galilei group have the form:

$$H = i\hbar\frac{\partial}{\partial t},\ \boldsymbol{P} = -i\hbar\nabla,\ \boldsymbol{K} = m\boldsymbol{r} - t\boldsymbol{P} + i\hbar\nabla_u,\ \boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{Z}, \tag{6.27}$$

[19] J.M. Aguirregabiria, *Dynamics Solver*, computer program for solving different kinds of dynamical systems, which is available from his author through the web site <http://tp.lc.ehu.es/jma.html> at the server of the Theoretical Physics dept. of The University of the Basque Country, Bilbao (Spain).

where ∇_u is the gradient operator with respect to the $\boldsymbol{u}$ variables. These generators satisfy the commutation relations of the extended Galilei group, [20] and the spin operator is given by $\boldsymbol{Z} = -i\hbar\boldsymbol{u} \times \nabla_u$.

One Casimir operator of this extended Galilei group is the Galilei invariant internal energy of the system $\mathcal{E}$, which in the presence of an external electromagnetic field and with the minimal coupling prescription is written as,

$$\mathcal{E} = H - eV - \frac{1}{2m}(\boldsymbol{P} - e\boldsymbol{A})^2, \tag{6.28}$$

where V and $\boldsymbol{A}$ are the external scalar and vector potentials, respectively.

In our system $\boldsymbol{A} = 0$, and V is only a function of the x variable. It turns out that because of the structure of the above operators we can find simultaneous eigenfunctions of the following observables: the Casimir operator (6.28), H, P_y, P_z, Z^2 and Z_z. The particle moves along the OX axis, with the spin pointing in the OZ direction, and we look for solutions which are eigenfunctions of the above operators in the form:

$$\left(H - eV(x) - \frac{1}{2m}\boldsymbol{P}^2\right)\psi = \mathcal{E}\psi,\ H\psi = E\psi,\ P_y\psi = 0,\ P_z\psi = 0, \tag{6.29}$$

$$Z^2\psi = s(s+1)\hbar^2\psi, \quad Z_z\psi = \pm s\hbar\psi, \tag{6.30}$$

so that ψ is independent of y and z, and its time dependence is of the form $\exp(-iEt/\hbar)$. Since the spin operators produce derivatives only with respect to the velocity variables, we can look for solutions with the variables separated in the form:

$$\psi(t, x, \boldsymbol{u}) = e^{-iEt/\hbar}\phi(x)\chi(\boldsymbol{u}),$$

and thus

$$\left(\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + E - eV(x) - \mathcal{E}\right)\phi(x) = 0, \tag{6.31}$$

$$Z^2\chi(\boldsymbol{u}) = s(s+1)\hbar^2\chi(\boldsymbol{u}), \quad Z_z\chi(\boldsymbol{u}) = \pm s\hbar\chi(\boldsymbol{u}), \tag{6.32}$$

where the spatial part $\phi(x)$, is uncoupled with the spin part $\chi(\boldsymbol{u})$, and $E - eV(x) - \mathcal{E}$ represents the kinetic energy of the system. The spatial part satisfies the one-dimensional Schroedinger equation, and the spin part is independent of the interaction, so that the probability of quantum tunneling is contained in the spatial part and does not depend on the particular value of the spin. If the particle is initially on the left-hand side of the barrier, with an initial kinetic energy $E_0 = E - \mathcal{E}$, then we can determine the quantum probability for crossing for $a = 1$ and different values of the potential width b.

The one-dimensional quantum mechanical problem of the spatial part for the same one-dimensional potential depicted in Fig. 6.15 is: [21]

$$\phi(x) = \begin{cases} e^{ikx} + Re^{-ikx}, & x \le -a, \\ C_1\mathrm{Ai}(D(1 - G + \frac{x}{a}) + C_2\mathrm{Bi}(D(1 - G + \frac{x}{a}), & -a \le x \le 0, \\ C_3\mathrm{Ai}(L(1 - G - \frac{x}{b})) + C_4\mathrm{Bi}(L(1 - G - \frac{x}{b})), & 0 \le x \le b, \\ Te^{ikx}, & x \ge b, \end{cases} \tag{6.33}$$

where x is the same dimensionless position variable as before, and the constants

$$k = \sqrt{\frac{E}{2mc^2}},\ D = \sqrt[3]{\frac{eV_0a^2}{2mc^2}},\ L = \sqrt[3]{\frac{eV_0b^2}{2mc^2}},\ G = \frac{E}{eV_0}. \tag{6.34}$$

[20] J.M. Levy-Leblond, *Galilei Group and Galilean Invariance*, in E.M. Loebl, *Group Theory and its applications*, Acad. Press, NY (1971), vol. 2, p. 221.

[21] L. Landau and E. Lifchitz, *Mécanique quantique*, Mir Moscow (1988), 3rd. edition.

Functions Ai(x) and Bi(x) are the Airy functions of x. The six integration constants R, T, and $C_i, i = 1, 2, 3, 4$, can be obtained by assuming continuity of the functions and their first order derivatives at the separation points of the different regions. The coefficient $|R|^2$ represents the probability of the particle to be reflected by the potential and $|T|^2$ its probability of crossing.

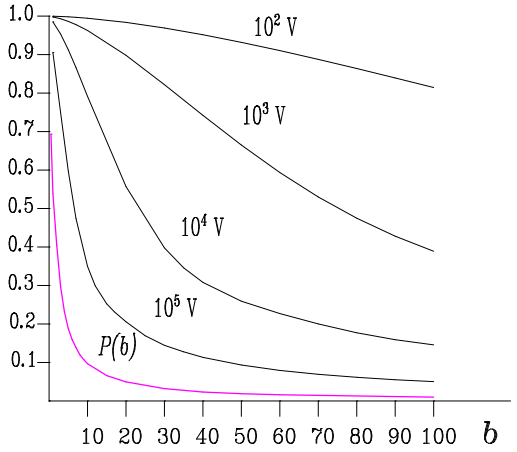


Figure 6.19: **Classical and Quantum Probability of crossing for different potentials.**

Computing the T amplitude for $a = 1$ and different values of the potential width b, and for energies below the top of the barrier eV_0, we show in Fig. 6.19, the average probability for quantum tunneling for four different potentials V_0 of 10^2, 10^3, 10^4 and 10^5 Volts. This average probability has been computed by assuming that on the left of the barrier there is a uniform distribution of particles of energies below eV_0.

If we consider for the classical spinning particle the same uniform distribution of particles, then, the function $P(b) = 1 - K_c(b)$, where $K_c(b)$ is the minimum dimensionless kinetic energy for crossing computed before, represents the ratio of the particles that with kinetic energy below the top of the potential cross the barrier because of the spin contribution.

This function $P(b)$, is also depicted in Fig. 6.19. We see that for the different potentials shown in that figure the classical average probability of crossing is smaller than the quantum one, but for stronger potentials this classical probability, coming from the spin contribution, becomes relatively important.

Because the tunnel effect is a function of $\hbar$ and the spin of elementary particles is also of order of $\hbar$ it is very difficult to separate from the outcome of a real experiment involving elementary particles, which part is due to a pure quantum effect and which is the contribution to crossing coming from the spin structure. From (6.31) and (6.32) it is clear that the quantum probability of tunneling is independent of the spin.

To test experimentally this contribution, it will be necessary to perform separate experiments with particles of the same mass and charge but with different values of the spin. Thus, the difference in the outcome will be related to the spin contribution. This can be accomplished for instance, by using ions of the type A^{++} that could be either in a singlet, ($s = 0$) state or in a triplet ($s = 1$) state.

But if there exists a contribution to crossing not included in the usual quantum mechanical analysis we have to modify the quantum mechanical equations. To be consistent with the above analysis the Schroedinger-Pauli equation should be modified to include the additional electric dipole term. A term of the form $-eER\cos\omega t$, where E is the external electric field and R the radius of the zitterbewegung, should be considered to solve the corresponding quantum wave

function. This term is of the order of the separation R between the center of mass and center of charge, which is responsible for the classical crossing. This additional electric dipole term is already included in Dirac's equation but is suppressed when taking the low velocity limit, as it corresponds to this low energy example. Nevertheless, although this is a low energy process and the time average value of the electric dipole vanishes, there are very high field gradients.

We see that the separation between the center of mass and center of charge that gives rise to the spin structure of this particle model justifies that this system can cross a potential barrier even if its kinetic energy is below the top of the potential.

6.5.1 Spin polarized tunneling

I like to point out the following ideas to discuss whether they can be useful in connection with the interpretation of the giant magnetoresistance of polycrystaline films[22]. This is known in the literature as the **spin polarized tunneling**. [23]

The main feature of the "classical" spin polarized tunneling we have seen in the previous section is not a matter of whether tunneling is classical or not, because this is a nonsense question. Matter at this scale is interpreted under quantum mechanical rules. But if we use a model of a classical spinning particle that, when polarized orthogonal to the direction of motion, produces a crossing that is not predicted by the Schroedinger-Pauli equation, it means that this quantum mechanical equation is lacking some term. The coupling term $-\boldsymbol{\mu}\cdot\boldsymbol{B}$, between the magnetic moment and magnetic field that gives rise to the Pauli equation, is inherited from Dirac's electron theory. But Dirac's equation also predicts another term $-\boldsymbol{d}\cdot\boldsymbol{E}$, of the coupling of an instantaneous electric dipole with the electric field. It is this oscillating electric dipole term that we believe is lacking in quantum mechanical wave equations. In general, the average value of this term in an electric field of smooth variation is zero. But in high intensity fields or in intergranular areas in which the effective potentials are low, but their gradients could be very high, this average value should not be negligible.

The conduction of electrons in synterized materials is completely different than the conduction on normal conductors. The material is not a continuous crystal. It is formed by small grains that are bound together by the action of some external pressure. If we can depict roughly the electric current flow, this is done by the jumping of electrons from grain to grain, through a tunneling process in which there is some estimated effective potential barrier confined in the gap between grains. Therefore these materials show in general a huge resistivity when compared with true conductors.

The form of this potential is unknown. The simplest one is to assume a wall of thickness d, the average separation between grains, and height h. But it can also be estimated as one of the potentials of the former example. What we have shown previously is that for every potential barrier, there is always a minimum energy, below the top of the potential, that electrons above that energy cross with probability 1 when polarized orthogonal to the motion, even within a classical interpretation. But this effect is not predicted by "normal" quantum mechanics because tunneling is spin independent.

Now, let us assume that we are able to estimate some average effective potential barrier in the intergranular zone of this polycristaline material. If the corresponding minimum crossing energy of this barrier for polarized electrons is below the Fermi level, then, when we introduce a magnetic field in the direction of the film and the magnetic domains in the grains become polarized, all electrons above that minimum energy of crossing will flow from grain to grain as

[22] 2007 Nobel Prize of Physics to Albert Fert and Peter Grünberg for the discovery in 1988 of Giant Magnetoresistence.

[23] V.N. Dobrovolsky, D.I. Sheka and B.V. Chernyachuk, Surface Science **397**, 333 (1998); P. Raychaudhuri, T.K. Nath, A.K. Nigam and R. Pinto, cond-mat/9805258, preprint.

in a good conductor, with a classical probability 1. That's all. Here the difficulty is to estimate properly this potential barrier and therefore the corresponding classical crossing energy.

It can be argued that the presence of the magnetic field to polarize electrons produces a change in the energy of particles. Nevertheless, even for a magnetic field of the order of 1 Tesla and in a potential barrier of 1 Volt, the magnetic term $-\boldsymbol{\mu}\cdot\boldsymbol{B}$ contributes with an energy of order of $\pm 5.7\times 10^{-5}$eV, which does not modify the quantum probability of crossing.

6.6 Formation of a bound state of two electrons

We have seen in section 2.6.2 that the dynamical equation of a free Dirac particle and for any inertial observer is a fourth-order differential equation for the position of the charge $\boldsymbol{r}$ which can be separated into a system of coupled second order differential equations for the centre of mass $\boldsymbol{q}$ and centre of charge $\boldsymbol{r}$ in the form (2.201):

$$\ddot{\boldsymbol{q}}=0,\quad \ddot{\boldsymbol{r}}=\frac{1-\dot{\boldsymbol{q}}\cdot\dot{\boldsymbol{r}}}{(\boldsymbol{q}-\boldsymbol{r})^2}(\boldsymbol{q}-\boldsymbol{r}),$$

where now the dot means time derivative. The first equation represents the free motion of the centre of mass and the second a kind of relativistic harmonic oscillation of point $\boldsymbol{r}$ around point $\boldsymbol{q}$ which preserves the constant absolute value c of the velocity $\dot{\boldsymbol{r}}$. In fact, if $\dot{q}\ll\dot{r}=1$, $|\boldsymbol{q}-\boldsymbol{r}|\sim 1$ and the equation is just the harmonic motion $\ddot{\boldsymbol{r}}+\boldsymbol{r}\simeq\boldsymbol{q}$, of point $\boldsymbol{r}$ around $\boldsymbol{q}$. The factor $(1-\dot{\boldsymbol{q}}\cdot\dot{\boldsymbol{r}})/(\boldsymbol{q}-\boldsymbol{r})^2$ prevents that when we take the boundary value $\dot{r}(0)=1$, the solution does not modify this absolute value of the velocity of the charge.

In the case of interaction this second equation remains the same because it corresponds to the definition of the centre of mass position which is unchanged by the interaction, because it only involves the $\boldsymbol{U}$ and $\boldsymbol{W}$ functions. The first equation for particle a is going to be replaced by $d\boldsymbol{p}_a/dt=\boldsymbol{F}_a$ where $\boldsymbol{p}_a$ is the corresponding linear momentum of each particle expressed as usual in terms of the centre of mass velocity

$$\boldsymbol{p}_a=\gamma(\dot{\boldsymbol{q}}_a)m\dot{\boldsymbol{q}}_a,\qquad \gamma(\dot{\boldsymbol{q}}_a)=(1-\dot{\boldsymbol{q}}_a^2)^{-1/2},$$

and the force $\boldsymbol{F}_a$ is computed from the interaction Lagrangian (4.78)

$$\boldsymbol{F}_a=\frac{\partial L_I}{\partial\boldsymbol{r}_a}-\frac{d}{dt}\left(\frac{\partial L_I}{\partial\boldsymbol{u}_a}\right)$$

For particle 1 it takes the form:

$$\boldsymbol{F}_1=-g\frac{\boldsymbol{r}_1-\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}\sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}+\frac{d}{dt}\left(\frac{g\boldsymbol{u}_2}{2|\boldsymbol{r}_1-\boldsymbol{r}_2|\sqrt{1-\boldsymbol{u}_1\cdot\boldsymbol{u}_2}}\right)\tag{6.35}$$

where it contains velocity terms which behave like $1/r^2$ and acceleration terms which go as $1/r$ in terms of the separation of the charges $r=|\boldsymbol{r}_1-\boldsymbol{r}_2|$. In this new notation $\boldsymbol{u}_a=\dot{\boldsymbol{r}}_a$.

Then the system of second order differential equations to be solved are

$$\ddot{\boldsymbol{q}}_a = \frac{\alpha}{\gamma(\dot{\boldsymbol{q}}_a)}\left(\boldsymbol{F}_a-\dot{\boldsymbol{q}}_a(\boldsymbol{F}_a\cdot\dot{\boldsymbol{q}}_a)\right)\tag{6.36}$$

$$\ddot{\boldsymbol{r}}_a = \frac{1-\dot{\boldsymbol{q}}_a\cdot\dot{\boldsymbol{r}}_a}{(\boldsymbol{q}_a-\boldsymbol{r}_a)^2}(\boldsymbol{q}_a-\boldsymbol{r}_a),\quad a=1,2\tag{6.37}$$

where α is the fine structure constant once all the variables are taken dimensionless. For that, we take the space scale factor $R=\hbar/2mc$ and the time scale as $T=\hbar/2mc^2$. All terms of equation (6.36) which depend on the acceleration of the charges have to be replaced by the expressions of (6.37).

It would be desirable to find analytical solutions of the above equations (6.36-6.37). Nevertheless we have not succeded in finding such a goal. However we shall analyse different solutions obtained by numerical integration. We are going to use the computer program Dynamics Solver [24]. The quality of the numerical results is tested by using the different integration schemes this program allows, ranging from the very stable embedded Runge-Kutta code of eighth order, due to Dormand and Prince, to very fast extrapolation routines. All codes have adaptive step size control and we check that smaller tolerances do not change the results. Another advantage is that it can be prepared to analyse solutions corresponding to a wide range of boundary conditions, automatically.

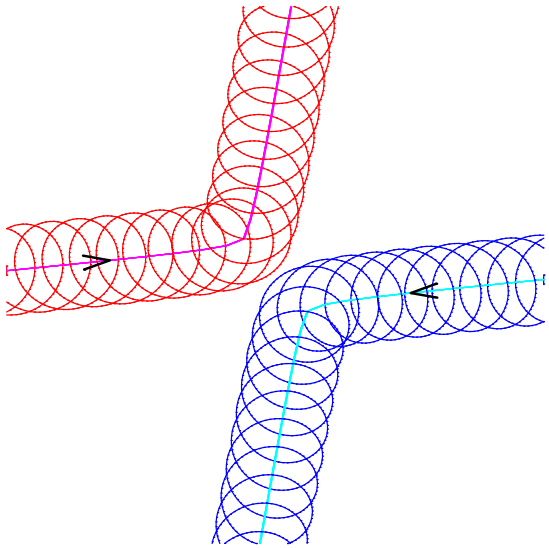


Figure 6.20: **The trajectories of the centres of mass and charge of two spinning particles with an initial centre of mass velocity $\dot{q}_a = 0.1$ and a small impact parameter.**

See in figure 6.20 the scattering of two equal charged particles with parallel spins. The centre of mass motion of each particle is depicted with an arrow. If the two particles do not approach each other too much these trajectories correspond basically to the trajectories of two spinless point particles interacting through an instantaneous Coulomb force. By too much we mean that their relative separation between the corresponding centres of mass is always much greater than Compton's wavelength. This can be understood because of the above discussion about the Coulomb behaviour of the averaged interaction Lagrangian, if the average position of each charge is far from the other. For high energy interaction the two particles approach each other below that separation and therefore the average analysis no longer works because the charges approach each other to very small distances where the interaction term and the exact position of both charges, becomes important. In this case new phenomena appear. We can have, for instance, a forward scattering like the one depicted on figure **6.21**, which is not described in the classical spinless case, or even the formation of bound pairs for particles of the same charge, which we shall analyse in what follows.

In figure **6.22** we represent an initial situation for two equal charged particles with parallel spins such that the corresponding centres of mass are separated by a distance below Compton's wavelength. Remember that the radius of the internal motion is half Compton's wavelength. We locate the charge labels e_a at the corresponding points $\boldsymbol{r}_a$ and the corresponding mass labels m_a to the respective centre of mass $\boldsymbol{q}_a$. We depict in part (a) the situation when the two particles have the same phase $\beta_1 = \beta_2$. The forces $\boldsymbol{F}_a$, on each particle $a = 1, 2$, are computed in terms of the positions, velocities and accelerations of both charges, according to

[24] See reference19

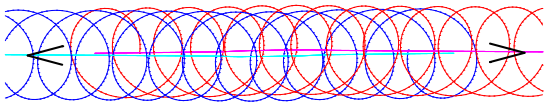


Figure 6.21: **Forward scattering of two spinning particles of the same charge with an initial separation $2q_a(0) = 10$, centre of mass velocity $|\dot{q}_a(0)| = 0.18$ and a very small impact parameter. The two centres of mass cross very close to each other, with a small deviation.**

(6.35), and are also depicted on the corresponding centres of mass as a consequence of the structure of the equations (6.36). We see that a repulsive force between the charges produces also a repulsive force between the centres of mass in this situation. However, in part (b) both charges have opposite phases $\beta_1 = -\beta_2$, and now the repulsive force between the charges implies an atractive force between the corresponding centres of mass. If the initial situation is such that the centres of mass separation is greater than Compton's wavelength, the force is always repulsive irrespective of the internal phases of the particles.

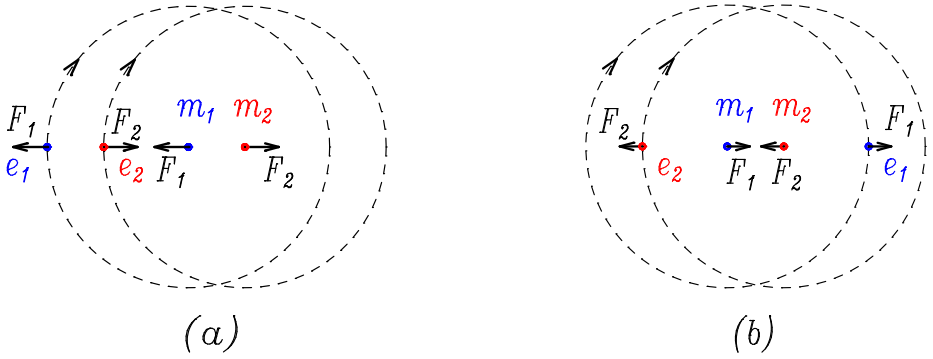


Figure 6.22: **Boundary values for two Dirac particles with parallel spins and with a separation between the centres of mass below Compton's wavelength. The dotted lines represent the previsible clockwise motion of each charge. In (a) both particles have the same phase and the repulsive force between charges produces a repulsive force between their centres of mass, while in (b), with opposite phases, the force between the centres of mass is atractive.**

In figure 6.23 we have another situation of opposite phases and where the initial separation between the centres of mass is larger but still smaller than Compton's wavelength.

To analyse this situation, which is going to produce bound motions, we proceed as follows: We start the numerical integration by imposing the boundary condition that both centres of

Figure 6.23: **(a) Another situation of two charges with opposite phases which produce an atractive force between the centres of mass provided they are separated below Compton's wavelength. In part (b), after half a cycle of the motion of the charges, the force becomes repulsive between the centres of mass, but its intensity is much smaller than the atractive force in (a) so that the resulting motion is also a bound motion.**

mass are at rest and located at the origin of the reference frame $\boldsymbol{q}_a(0) = \dot{\boldsymbol{q}}_a(0) = 0$. For particle 2 we take the initial phase $\beta_2(0) = 0$ and for β_1 we start with $\beta_1(0) = 0$ and, will be increased step by step in one degree in the automatic process, up to reach the whole range of 2π radians. The boundary values of the variables $\boldsymbol{r}_a(0)$ and $\dot{\boldsymbol{r}}_a(0)$, with the constraint $|\dot{r}_a(0)| = 1$, are taken as the corresponding values compatible with these phases. The whole system is analysed in its centre of mass frame, so that for subsequent boundary values these variables are restricted to $\boldsymbol{q}_1(0) = -\boldsymbol{q}_2(0)$ and $\dot{\boldsymbol{q}}_1(0) = -\dot{\boldsymbol{q}}_2(0)$. The automatic integration is performed in such a way that when the two particles separate, i.e., when their centre of mass separation is above Compton's wavelength, the integration stops and starts again with a new boundary value of the phase $\beta_1(0)$ of one degree more, and the new values of the variables $\boldsymbol{r}_a(0)$ and $\dot{\boldsymbol{r}}_a(0)$. If the two particles do not separate we wait until the integration time corresponds to 10^6 turns of the charges around their corresponding centre of mass, stop the process, keep record of the phases and initial velocities, and start again with new boundary values. This corresponds, in the case of electrons, to a bound state living during a time greater than 10^{-15} seconds. For some particular boundary values, with opposite phases, we have left the program working during a whole week and the bound state prevails. This represents a time of life of the bound state greater than 10^{-9} seconds. Leaving the computation program running for a year will only increase this lower bound in two orders of magnitude. The general feeling is that the bound states are sufficiently stable, because even the possible numerical integration errors do not destroy the stability. This process is repeated again and again by changing slightly the initial values of the centre of mass variables $\boldsymbol{q}_a(0)$ and $\dot{\boldsymbol{q}}_a(0)$, in steps of 0.0001 in these dimensionless units and with $\beta_2(0) = 0$, and the same procedure with $\beta_1(0)$, as above. To test the acuracy of the integration method, we check every 10^3 integration steps that the velocities of the charges of both particles remain of absolute value 1, within a numerical error smaller than 10^{-20}.

The whole process is repeated by changing the initial $\beta_2(0)$ phase to any other arbitrary value. We are interested to see whether different results are produced depending on the values of the phase difference $\beta_2(0) - \beta_1(0)$ and of the centre of mass variables $\boldsymbol{q}_a(0)$ and $\dot{\boldsymbol{q}}_a(0)$. We collect all data which produce bound motions, and find the following results:

1. The initial velocity of their centres of mass must be $|\dot{q}_a(0)| < 0.01c$. Otherwise the bound motion is not stable and the two particles, after a few turns, go off.

2. For each velocity $|\dot{q}_a(0)| < 0.01c$ there is a range Δ of the pase $\beta_1(0) = \beta_2(0) + \pi \pm \Delta$ for which the bound motion is stable. The greater the centre of mass velocity of each particle the narrower this range, so that the bound motion is more likely when the phases are opposite to each other.

3. We have found bound motions for an initial separation between the centres of mass up to

0.8 times Compton's wavelength, like the one depicted in figure 6.23, provided the above phases and velocities are kept within the mentioned ranges.

In figure 6.24 we show the bound motion of both particles when their centres of mass are initially separated $q_{1x} = -q_{2x} = 0.2\times$Compton's wavelength, $\dot{q}_{1x} = -\dot{q}_{2x} = 0.008$ and $\dot{q}_{1y} = -\dot{q}_{2y} = 0.001$, $\beta_2 = 0$ and $\beta_1 = \pi$. Now the force between the charges is repulsive but nevertheless, if the internal phases β_1 and β_2 are opposite to each other, it becomes an atractive force between their centres of mass in accordance to the mechanism shown in figure 6.22 (b).

This possibility of formation of low energy metastable bound pairs of particles of the same charge is not peculiar of this interaction Lagrangian. By using the electromagnetic interaction or even the instantaneous Coulomb interaction between the charges of two spinning Dirac particles we found in [25] also this behaviour. This bound motion is not destroyed by external electric fields and also by an external magnetic field along the spin direction. Nevertheless, a transversal magnetic field destroys this bound pair system.

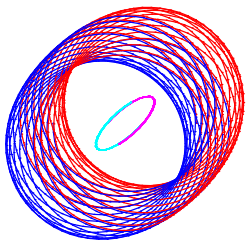


Figure 6.24: **Bound motion of the centres of mass and charge of two spinning particles with parallel spins and with a centre of mass velocity $v \simeq 0.0082$, for an initial separation between the centres of mass of $0.2\times$Compton's wavelength.**

When we make the average of the position $\boldsymbol{r}_a$ it becomes the centre of mass $\boldsymbol{q}_a$ and the repulsive force between the charges is also a repulsive force between the corresponding centres of mass and therefore when we suppress the zitterbewegung spin content of the particles there is no possibility of formation of bound pairs.

Although this result produces a classical mechanism for the formation of a spin 1 bound system from two equal charged fermions we must be careful about its conclusions. First, it is a classical description and although the range of energies which produce this phenomenon is a wide one it does not mean that two electrons can reach that binding energy. This Dirac particle is a system of 7 degrees of freedom: 3 represent the position $\boldsymbol{r}$, another 3 the orientation $\boldsymbol{\alpha}$ and finally the phase β. If we accept the equipartition theorem for the energy, then for the maximum kinetic energy which produces a bound motion $mv^2/2 = 7\kappa T/2$, where κ is Boltzmann's constant and $v = 0.01c$ the maximum velocity of the center of mass of each particle, then it means that a gas of polarized electrons (like the conducting electrons in a quantum Hall effect) could form bound states up to a temperature below $T = 8.47 \times 10^5$K, which is a very high temperature. In a second place, matter at this level behaves according to quantum mechanical rules and

[25] M. Rivas *J. Phys. A: Math. Gen.* **36** 4703 (2003), (*Preprint* physics/0112005)

therefore we must solve the corresponding quantum mechanical bound state to establish the proper energies and angular momenta at which these bound states would be stationary. This problem has not been solved yet, but the existence of this classical possibility of formation of bound pairs justifies an effort in this direction. If the phases of the two particles are the same (or almost the same) there is no possibility of formation of a bound state. The two fermions of the bound state have the same spin and energy. They differ that their phases and linear momenta are opposite to each other. Is this difference in the phase a way to overcome at the classical level, the Pauli exclusion principle?

6.6.1 The Positronium

If the two particles have opposite charges, the force among them is atractive, and if their spins had the orientation opposite to each other, the motion of the corresponding centers of charge of both particles will be the same as the ones we have calculated here. An example of this interaction is the case of the **Positronium**, a bound system of an electron-positron pair. It is a system with bound states although the probability of collision particle-antiparticle is high and therefore they annihilate each other. The ground state is a state of zero spin and zero magnetic moment, and zero orbital angular momentum, which stresses the prediction of this kinematical theory that the spin and magnetic moment of particles and antiparticles have the same relative orientation. It has a mean life of 1.24×10^{-10}s and the system annihilates into two photons with opposite spins and linear momenta and of energy corresponding to the energy of an electron at rest, in the laboratory frame. Another stationary state, with a greater mean life 1.4×10^{-7}s corresponds to the state of total spin 1, which is either the hyperfine transition of the ground state where one of the two particles has reversed its spin or the state of orbital angular momentum 1 with the spins still opposite to each other. This excited state annihilates into three photons.

From the classical point of view these stationary bound states admit the same approximated description as the Bohr atom, with the usual quantization of the orbital angular momentum.

6.7 Hall effect

This effect was discovered by Edwin Hall in 1879. It deals with the influence of an external magnetic field on the current distribution on a conductor. It produces a transversal static difference of potential perpendicular to the direction of the current. If in a conductor we establish a direct current and introduce an external magnetic field orthogonal to the current, then when the electrons interact with the magnetic field they start rotating and will be displaced to the border. A certain amount of charge will be accumulated at the sides of the conductor, negative where the electrons are accumulated and positive in the opposite side, and a static difference of potential will appear between both sides.

Let us consider a flat conductor lying on the XOY plane, under the influence of a magnetic field in the direction of OZ. The motion of the pointlike electrons, of charge $-e$, moving with velocity $\boldsymbol{v}$ in an external magnetic field is given by the differential equation

$$m\frac{d\boldsymbol{v}}{dt} = -e\boldsymbol{v}\times\boldsymbol{B}.$$

If the motion takes place on the plane XOY, $\boldsymbol{v} = (\dot{x}, \dot{y}, 0)$ and $\boldsymbol{B} = (0, 0, B)$ and these equations reduce to two

$$m\ddot{x} = -eB\dot{y}, \quad m\ddot{y} = eB\dot{x},$$

and the general solution is

$$x(t) = X - R\sin(\omega_B t + \phi), \quad y(t) = Y - R\cos(\omega_B t + \phi).$$

These trajectories are circles, of center at the point (X, Y) with an arbitrary phase ϕ and radius R. These four parameters are the four constants of integration. The angular frequency $\omega_B = eB/m$, known as cyclotronic frequency, is constant. The electrons move in circles of different radii, but with the same angular velocity ω_B.

The motion of the spinless electrons in an Ohmic conductor under the influence of an external electromagnetic field can be described by the Drude model,

$$m\frac{d(\gamma(v)\boldsymbol{v})}{dt} = -e\boldsymbol{E} - e\boldsymbol{v}\times\boldsymbol{B} - \frac{1}{\tau}m\boldsymbol{v},$$

where the last term represents a breaking force, opposite to the velocity, which depends on a phenomenological parameter τ, with dimensions of time, called *dispersion time*, that can be interpreted as the average time between consecutive collisions. If this time is large represents that there is quite a few resitence to the motion. Depends on the internal structure of the conducting material and on the fermionic character of the charge carriers.

The stationary solutions are those where the velocity $\boldsymbol{v}$ is time independent,

$$\boldsymbol{v} + \frac{e\tau}{m}\boldsymbol{v}\times\boldsymbol{B} = -\frac{e\tau}{m}\boldsymbol{E}.$$

If $\boldsymbol{j} = -ne\boldsymbol{v}$ is the current density vector, where n represents the density of conducting electrons of the material, the above expression gives a linear relationship between $\boldsymbol{j}$ and $\boldsymbol{E}$, which, in the particular case of a flat conductor, reduces to

$$\frac{m}{ne^2\tau}j_x + \frac{B}{ne}j_y = E_x, \quad \frac{m}{ne^2\tau}j_y - \frac{B}{ne}j_x = E_y.$$

Written in matrix form is

$$\begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix}\boldsymbol{j} = \boldsymbol{E}, \quad \rho_{ik}j_k = E_i, \quad j_i = \sigma_{ik}E_k.$$

It is *Ohm's Law*, where ρ_{ik} is the resistivity tensor and its inverse $\sigma = \rho^{-1}$, the conductivity tensor. The resistivity tensor ρ is diagonal when the magnetic field vanishes, and the only essential component is the Ohmic resistivity of the conductor ρ_0.

$$\rho = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix}, \quad \rho_{xx} = \rho_{yy} \equiv \rho_0 = \frac{m}{ne^2\tau}, \quad \rho_{xy} = \frac{B}{ne}.$$

If in the stationary situation the current flows along the OX axis, $j_y = 0$, and although we have established a difference of potential between the end points to produce the current, a nonvanishing transversal electric field E_y, must exist due to the accumulation of charges in the borders of the conductor. This transversal field prevents the existence of a component of the current vector in the transversal direction.

$$E_x = \rho_{xx}\, j_x, \quad E_y = -\rho_{xy}j_x, \quad \rho_{xx} = \frac{m}{ne^2\tau}, \quad \rho_{xy} = \frac{B}{ne}.$$

The $\rho_{xx} = \rho_0$ depends on the microscopic structure, of the temperature, through the parameter τ, while ρ_{xy} is a linear function of the external magnetic field.

6.7.1 Quantum Hall Effect

If we represent the components of the resistivity tensor ρ_{xx} and ρ_{xy} in the stationary situation as a function of the external magnetic field we get the graphics depicted in the figure **6.25**. This analysis has been done under the assumption of a pointlike structure of the electron. The spin of the electron has played no role in this analysis.

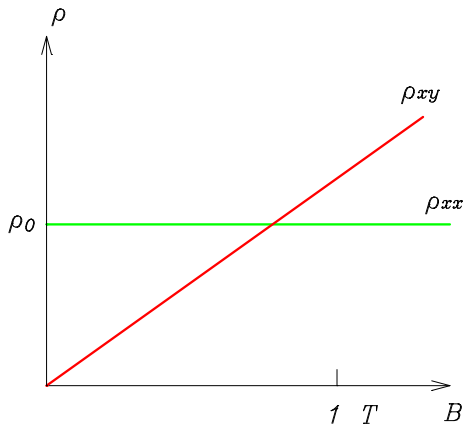


Figure 6.25: **Variation of the resistivity with the external magnetic field in the classical Hall effect, for small magnetic fields below 1 Tesla.**

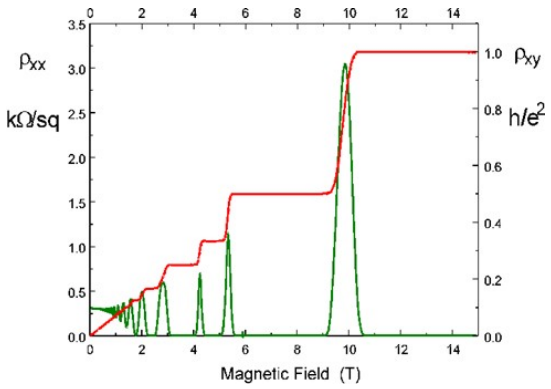


Figure 6.26: **Variation of the resistivity with the external magnetic field, ρ_{xx} (green), ρ_{xy} (red), in the integer quantum Hall effect.**

However, for huge magnetic fields the above figure has the form shown on the figure described in **6.26** which was determined experimentally by Von Klitzing, Dorda And Pepper[26]

The component ρ_{xy}, on average, varies linearly with B but in some places it makes sudden jumps while remaining constant in between. The component ρ_{xx} begins at the constant value ρ_0, but it starts to oscillate. In those ranges where ρ_{xy} is constant, the ρ_{xx} component, of green colour, vanishes, producing a peak when ρ_{xy} is no longer constant and starts rising. At the horizontal gaps the following condition is fulfilled

$$\rho_{xy} = \frac{h}{e^2}\frac{1}{\nu}, \quad \nu = 1, 2, 3, \ldots \tag{6.38}$$

The fine structure constant is $\alpha = e^2/2\epsilon_0 hc$. If the universal constant c and the permitivity of the vacuum ϵ_0, are known, an accurate measurement of the ρ_{xy} in the constant ranges, is an accurate measurement of this universal constant α.

The average magnetic field where these constant gaps are produced takes the value

$$B_\nu = \frac{n}{\nu}\frac{h}{e} = \frac{n}{\nu}\Phi_0,$$

where Φ_0 is known as the *flux quantum* and the magnitude h/e^2 as the *quantum of resistivity*. This is known in the literature as the **integer** quantum Hall effect, to distinguish from the **fractional** quantum Hall efect, where in the resitivity (6.38) the parameter ν, instead of an integer number, is a fractional number, as was shown by using larger magnetic fields [27] and which is represented in the figure **6.27**.

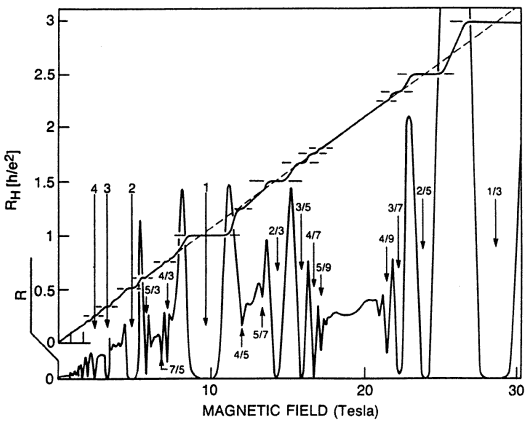


Figure 6.27: **Variation of the resistivity with the external magnetic field in the fractional quantum Hall effect, for very large magnetic fields.**

The Hall effect is analyzed for very flat distributions of electrons, confined between layers of different materials. The first integer effect was discovered in a MOSFET (Metal-Oxide-Semiconductor-Field-Effect-Transistor). It is a sandwhich of a three layers transistor (metal-insulator-semiconductor) with the electrons trapped in a layer of around 30Å of thickness, between the insulator and the semiconductor. The fractional effect was discoverd in the structure GaAs-GaAlAs of *Gallium arseniure*. In both cases the surface density of electrons is of the order of $n \approx 10^{11} - 10^{12}$ cm^{-2}. The temperature of the samples is of 4 K for the integer effect and of

[26] K. v. Klitzing, G. Dorda and M. Pepper, *New method for high accuracy determination of the fine structure constant based on quantized Hall resistance*, Phys. Rev. Lett. **45**, 494 (1980)

[27] D.C. Tsui, H.L. Stormer and A.C. Gossard, *Two-dimensional magnetotransport in the extreme quantum limit*, Phys. Rev. Lett. **48**, 1559 (1982)

2 K for the fractional effect. Recently, the fractional quantum Hall effect has been observed in the graphene [28], at temperatures around 20 K.

The discrete transversal Hall potential remains constant for magnetic fields within finite ranges. In these situations the longitudinal resistivity goes to zero. The current flows longitudinally without resistence, because $\rho_{xx} = 0$. As we have seen in previous sections, electrons with parallel spins can form metastable bound states of spin 1, and behave as bosons, and therefore the electric current associated to the flow of these pairs it is in a superconducting phase. This would mean that in those ranges for the magnetic field, the electrons are polarized along the external magnetic field and with their spins parallel a formation of bound pairs of conducting electrons appears. Some of these pairs would be deflected to the border thus producing a sudden jump in the lateral Hall potential. The remaining bound pairs of electrons would move forward, without resistence, because they are bosons in the superconducting phase. Once these electrons are polarized and the bound states are formed, these pairs are not destroyed by the external electric field and its number is increased when a new magnetic field is introduced, the polarization is reinforced and a sudden number of new bound pairs arises.

When the bound pairs of electrons are formed, the electric field does not destroy the pair and both particles move like a single object of spin 1 and charge $2e$. If we take the sample pointer of a tunnel effect mycroscope and connect it to the plane surface of the Hall conductor, we can try to extract electric carriers and see whether they are single electrons or bound pairs of electrons. The same can be tried when applying the pointer to the negative side of the Hall conductor and check if the accumulation of charge carriers corresponds to free electrons or bound pairs of them. These experiments, to our knowledge, have not been performed.

6.7.2 Spin Hall Effect

The spin Hall effect is the accumulation of electrons with opposite spins, at the lateral edges of a conductor when a current is produced. In this case it is not necessary the presence of an external magnetic field. This was discovered by Mikhail I. Dyakonov and Vladimir I. Perel in 1971 [29]. It is a transport phenomenon where the conducting electrons with the spin of the same orientation are moved towards a border of the conductor, while those electrons with their spins oriented in the opposite direction are moved to the other side. Here there is no accumulation of electric charge at the border, but rather an accumulation of angular momentum of opposite direction. In the presence of an external magnetic field, the electrons would have the same spin orientation and therefore the accumulation would be in only one of the sides. This would produce an accumulation of positive charge in the opposite side, as is confirmed by the usual Hall effect. However the justification of the Hall effect is not related to the existence of the spin of the electron.

The spin Hall effect could be related with the magnetic force, created by the average magnetic field produced by the current, acting on the two possible orientations of the magnetic moment of the conducting electrons $\boldsymbol{F} = -\nabla(\boldsymbol{\mu} \cdot \boldsymbol{B})$. Electrons at the same location, and thus under the same average magnetic field, with their respective magnetic moments in the opposite direction, would start moving with opposite lateral forces.

In some twodimensional semiconductors where an integer Hall effect is observed, when the external magnetic field is not very high, it also has been observed this effect of accumulation of spin at the lateral edges of the conductor.

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[28] X. Du, et al. *Nature* **462** 192 (2009).

[29] M. I. Dyakonov and V. I. Perel, *Possibility of orientating electron spins with current.* Sov. Phys. JETP Lett. **13**, 467 (1971)

6.8 Laser cooling

We are going to consider something which, at first glance, seems to be paradoxical. To cool a gas of free atoms by supplying energy to them. The final result is that the set of atoms looses energy, and then it cools.

When an object with magnetic moment is oriented in an external magnetic field, there are two possible stationary orientations: one, where the magnetic moment is oriented along the external magnetic field and the other where the orientation is the opposite. The first stationary orientation is stable while the second is unstable. Any perturbation alters this stationary state and the magnetic moment is reversed and the system goes into the stable state, which is a state of lower potential energy, and thus emitting outside the extra energy during the transition. To come back to the initial stationary situation of equilibrium it would be necessary to supply energy to the material system.

To understand the difference of these two stationary equilibrium states let us consider the example of a pencil on the surface of a table: There are only two possible equilibrium situations: One, the pencil is at rest lying on the table, and the other the pencil is upright, in a static vertical position. The first is a stable equilibrium situation while the second, of greater potential energy, is unstable. Any perturbation falls down the pencil bringing it to the stable situation and liberating energy in form of heat. To come back to the previous unstable equilibrium situation it would require the waste of some extra energy to rise the pencil.

In an atom, the electrons are under the action of the electric field of the nucleus, but also under the magnetic field created by the magnetic moment of the nucleus. The electrons of the different layers become oriented of both ways under this magnetic field, and in complete shells we have as many electrons oriented in one way and in the other. Let us assume an atom with a single electron at the outer layer, while the remaining layers are complete.

The orientation of the magnetic moment of this electron, if the atom is in the ground state, would be with the magnetic moment along the resultant magnetic field at that place. If we want to reverse the magnetic moment orientation of this electron we have to supply energy to the atom, and therefore to the outer electron. This process of inversion of the magnetic moment of an electron, and therefore of its spin, is known as the **hyperfine transition**.

The usual way to supply energy to an atom is by means of a beam of photons. The inner electrons of the complete shells cannot absorb any photon except to jump to some upper excited state. In that case the energy of the photons have to be sufficent for this task. The possibility that a low energy photon could be absorbed and make a hyperfine transition is also forbiden by the exclusion principle, because the inner levels are complete and an electron cannot do that transition because the new level is already ocupied. The energy to produce a transition between levels is of the order of eV. For the hyperfine transition is of order of 10^{-5} eV, in the microwave range [30].

Let us assume that the energy of the photons of the beam is appropriate to produce a hyperfine transition. If a photon of the beam is captured by the atom, the electron of the outer shell will jump into the new stationary unstable situation by changing its spin orientation. Later, this electron will come back to the previous stable level by the emission of a photon. It seems that what we have given to the atom is recovered later.

Since photons are spin 1 particles and electrons are of spin 1/2, the transition is produced when the electron captures a photon with the spin in the opposite direction to its own spin, changing the orientation of the spin and therefore of its magnetic moment. In this process angular momentum and energy are conserved and the electron goes into another state of higher energy. But in this process the linear momentum is also conserved, so that the linear momentum

[30] The hyperfine transition of ^{133}Cs is of a frecuency $\nu = 9\,192\,631\,770$ Hz, in the microwave range of wavelength $\lambda = 3.261$ cm, which corresponds to photons of energy $h\nu = 3.801 \times 10^{-5}$ eV.

of the atom is increased in the amount of the linear momentum of the absorbed photon.

The idea of the laser cooling method is to use photons of energy a litle lower than the energy for the hyperfine transition in the laboratory reference frame. If the atom gas is not at 0 K, it means that every atom has a nonvanishing kinetic energy and therefore it is moving with certain velocity in the laboratory reference frame.

The above photons will not be able to produce the hyperfine transition on atoms at rest, but for atoms in motion, due to the Doppler effect, the energy of the photon of the beam is a litle bit greater if the atom velocity is directed to the beam and lower if the atom moves in the direction of the beam.

If the atom is moving towards the beam and the velocity is the appropriate such that by the Doppler shift the energy of the photon is sufficient to produce the hyperfine transition, the electron will invert its magnetic moment. But at the same time the component of the velocity of the atom in the direction of the beam will decrease because the absorbed linear momentum of the photon. Because the other two components of the velocity of the center of mass of the atom have not changed, the kinetic energy of the atom in the laboratory frame, is lower than before the transition. But now the atom is in an unstable excited state. Some time later, the excited electron will come back to the previous stable stationary state, by the emission of the corresponding photon. But this photon is emitted from a moving atom and therefore, with respect to the laboratory frame of an energy a litle bit greater than the transition energy.

The net result is that in the absortion process we have lost a photon of energy lower than the energy of transition and we have recovered another photon of greater energy during the emission process. The difference in energy of both photons is the loss of kinetic energy of the atom. The atom is cooling.

In 1997, Chu, Cohen-Tanoudji and Phillips were awarded Nobel Prize of physics by the development of techniques for laser cooling of a gas. The idea is to hold a monoatomic gas in a cavity at very low temperature, of order of mK, under the action of a system of laser beams in the three orthogonal spatial directions in both ways. The energy of the photons of the beam has to be smaller than the energy for the hyperfine transition. In this way, every transition produces a decrease of the velocity of the center of mass of the corresponding atom, by the mentioned procedure. When acting with lasers in opposite directions they will have available photons of the appropriate energy for each atom in one way or the other.

When the gas is cooling, the average velocity of the atoms decrease and the increase effect in energy by Doppler shift is smaller, so that it is necessary to fine tunning the laser frequency by increasing it continuously to reach the corresponding energy of transition. By this cooling method samples are cooled at temperatures of order of μK.

6.9 The spin of the proton

Let us assume, as suggested by the standard model, that the proton is a bound system of three quarks that in the ground state every quark has orbital angular momentum $L = 0$. If we assume that the quarks are Dirac particles of spin 1/2 and gyromagnetic ratio $g = 2$, we can apply to them the same model as the one analyzed for the electron in this formalism and depicted in the front page.

There is in the literature some controversy with respect to the spin of the proton. If we add the three Dirac spin operators of the three quarks it is shown experimentally that this sum does not produce the total angular momentum of the proton. It is known as the *proton spin crisis* [31]. According to this experiment performed at CERN, the sum of the three quarks spins only contributes to the expected spin of the proton around between the 4% and the 24%. This

[31] Ashman, J.; European Muon Collaboration, Phys. Lett. **B 206**, 364 (1988), *A measurement of the spin asymmetry and determination of the structure function $g(1)$ in deep inelastic muon-proton scattering.*

is considerd as one of the unsolved problem of physics [32]. It is argued that we need to include in the spin of the proton the contribution of the spins of the virtual gluons, but this suggestion does not give the right answer. We have to take into account that Dirac spin operator represents the angular momentum of the quark with respect to its center of charge and not with respect to the center of mass.

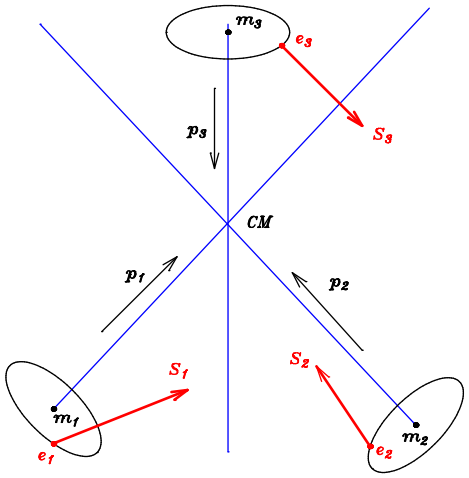

Figure 6.28: **Model of a proton as a bound system of three quarks as three Dirac particles in a state of orbital angular momentum $L = 0$, and in the CM reference frame of the proton. The motion of the CM of each quark necessarily is a straight trajectory (in blue) passing through the center of mass of the proton, located at the origin in this frame. Dirac spin operator of each quark is defined (and depicted) with respect to the corresponding center of charge, such hat the addition of the three Dirac operators $\hbar\boldsymbol{\sigma}/2$, can never give rise to the angular momentum of the proton with respect to its center of mass. It is necessary to add also the three orbital angular momenta of each quark with respect to the common center of mass.**

If the quarks are moving in a state of orbital angular momentum $L = 0$, this means, literally that if we make the analysis in the center of mass of the proton, located at the origin of the reference frame, each quark has always its linear momentum pointing to the location of its common center of mass. Therefore the center of mass of each quark is moving along a straight trajectory passing through the center of mass of the proton, and the three trajectories of the centers of mass of all three quarks are contained on a plane, because the sum $\sum \boldsymbol{p}_i = 0$, in this reference frame.

Let us consider that the spin of the proton represents the angular momentum of the system of these three quarks with respect to the common center of mass at rest, CM. As we see in the figure **6.28**, the three Dirac spin operators $\boldsymbol{S}_i$ of each quark represent the corresponding angular momenta of each quark with respect to the corresponding center of charge, and not with respect to the corresponding center of of mass. They have been depicted at the corresponding point. This means the the sum of the three Dirac spin operators can never give us the total angular momentum of the proton. We need to consider also for each quark i, the corresponding orbital angular momentum $(\boldsymbol{r}_i - \boldsymbol{q}_i) \times \boldsymbol{p}_i$, $i = 1, 2, 3$.

If we take into account that the electric dipole moment of a Dirac particle is defined by $\boldsymbol{d} = e(\boldsymbol{r} - \boldsymbol{q})$, as we have seen in (6.7), this determines the separation operator between the CC

[32] https:/ /en.wikipedia.org/wiki/Proton_spin_crisis

and CM of each quark, and thus this relative position operator is

$$\boldsymbol{r}-\boldsymbol{q}=-\frac{i\hbar}{2mc}\boldsymbol{\alpha},$$

in terms of the $\boldsymbol{\alpha}$ Dirac matrices.

We see that there is a lacking term in the computation of the spin of the proton. In addition to the sum of the three Dirac spin operators, we also need the sum of the three orbital angular momentum for each quark

$$\frac{i\hbar}{2mc}\boldsymbol{\alpha}\times\boldsymbol{p}=\frac{i\hbar}{2mc}\boldsymbol{\alpha}\times\frac{\hbar}{i}\nabla=\frac{\hbar^2}{2mc}\boldsymbol{\alpha}\times\nabla.$$

These terms are not negligible because when the CM of each quark reaches the common CM of the proton at the same time, the potential energy is zero but the kinetic energy takes its maximum value. The average value of the linear momentum of each quark at that moment is around 325 MeV/c. If q represents the espinor field of a quark, the angular momentum of the proton must contain, at least, the following terms:

$$\sum_{i=1}^{3}q_i^{\dagger}\left(\frac{\hbar}{2}\boldsymbol{\sigma}_i\right)q_i-\sum_{i=1}^{3}q_i^{\dagger}\left(\frac{\hbar^2}{2mc}\boldsymbol{\alpha}_i\times\nabla_i\right)q_i,$$

where $q_i^{\dagger}$ represents the hermitian conjugate spinor field of q_i. We have not considered other possible contributions coming from the angular momentum of the gluon plasma, which have been also suggested phenomenologically in other works.

6.10 The kinematical group

Let us assume that to describe the evolution in space of a localized material system it would be sufficient to describe the evolution of a single point. For an elementary particle we have seen that this point represents the center of charge, i.e., the point where we locate the interacting properties of the particle. The most general differential equation satisfied by a point in three-dimensional space is of fourth order and given in (6). Its general solution involves 12 integration constants. If this family of solutions corresponds to the evolution of the point by the different inertial observers, this implies that the kinematical group of spacetime transformations associated to the Restricted Relativity Principle is a 12-parameter Lie group. If what we are describing is the center of charge of the elementary particle, we have seen that this point necessarily moves at the speed c, and this velocity is not changed by any interaction. The constraint $|\boldsymbol{r}^{(1)}|=c$, for the physical solutions holds, so that only 11 parameters are necessary to describe its allowed solutions, so that the family of allowed motions which define the relative situation of the equivalent observers is a 11-parameter family. The kinematical group also has to contain the existence of a velocity unreachable for all inertial observers.

This suggests that the 11-parameter group of spacetime transformations between inertial observers is the Weyl group, which is compound of the 10 transformations of the Poincaré group (4 translations+3 rotations+3 pure Lorentz transformations or boosts), and the spacetime dilations which preserve the speed of light. They are the scale transformations of normal or canonical parameter λ, $t'=e^{\lambda}t$, $\boldsymbol{r}'=e^{\lambda}\boldsymbol{r}$.

6.10.1 The kinematical space

For the variational description of mechanical arbitrary systems the relevant manifold where the evolution of the mechanical system is described, is the **kinematical space**. It is the

manifold which describes the initial and final states of the mechanical system. If what we want to describe is an elementary particle, the atomic principle requires that this manifold is necessarily a homogeneous space of the kinematical group associated to the restricted relativity principle.

If we consider that the kinematical group is the Poincaré group, $\mathcal{P}$, then the kinematical space have at most dimension 10. But if three of these variables represent the velocity of the center of charge and this point moves at the speed c, the maximum dimension of the kinematical space is 9. The nine kinematical variables are: three spatial variables which describe the location of the CC; a temporal variable and five compact dimensionless variables. The interpretation of these five variables is clear: Three represent the orientation of a Cartesian frame linked to the motion of the CC, which rotates with respect to some arbitrary initial orientation. This rotation is described by the rotation axis (two compact variables, the zenithal angle $\theta \in [0, \pi]$, and the azimuthal angle $\phi \in [0, 2\pi]$) and the third is the total rotated angle $\alpha \in [0, \pi]$, which in the quantum version is extended to $\alpha \in [0, 2\pi]$, to obtain the simply connected representation of the rotation group, i.e., the group $SU(2)$.) The remaining two compact variables correspond to the spatial orientation of the velocity vector of the CC motion. One is the zenithal angle $\beta \in [0, \pi]$, and the other is the azimuthal angle $\psi \in [0, 2\pi]$, of the velocity vector of constant absolute value c.

If we admit as kinematical group the Weyl group, $\mathcal{W}$, we also have an extra, dimensionless and non compact variable λ, associated to the dimensionless variable of change of scale.

In this way an elementary particle has a kinematical space of dimension 10. It has 7 degrees of freedom, i.e., the position of a point $\boldsymbol{r}$, a cartesian frame $\boldsymbol{\alpha}$, linked to that point, with no physical reality and that can be chosen arbitrarily at our will at any time, and also a non compact dimensionless variable λ, which represents an internal *gauge* or internal phase. Therefore the symmetry group of the Lagrangian of the free elementary particle is at least the Weyl group, together with the following commuting groups: the local rotation group $SO(3)_L$ or its quantum counterpart, its simply connected representation $SU(2)_L$, which describes the arbitary local frame; and finally the group $U(1)$, the gauge or phase group $\{\mathbb{R}, +\}$. In the quantum case this symmetry group is at least $\mathcal{W} \otimes SU(2) \otimes U(1)$.

The Casimir operators of this complete group of symetries are S^2_{CM}, the angular momentum of the particle with respect to the CM, which is the unique Casimir operator of the Weyl group $\mathcal{W}$; T^2 which is the absolute value of the angular momentum of the rotative part of the local cartesian frame linked to the point, which is the Casimir operator of the local rotation group $SO(3)_L$; finally, the generator Q of the group $U(1)$, with dimension of action with an unknown interpretation. All these three commuting Casimir operators have dimension of action.

In string theory it is postulated that an elementary particle is a point of a 10-dimensional space (or dimension 11 in the M-theory) but that some of the variables have been compactified, thus remaining as non-compact variables a time varible and three spatial variables. With the inclusion of the universal constant c these four variables have dimension of length. The meaning of the remaining compact variables is unknown, but it is suggested that they are spatial variables that when compactified have become unobservable.

It is not clear if this manifold postulated by string theory represents the configuration space, if the number of these variables is restricted or there are some constraints among them. If this manifold represents the kinematical space of a Lagrangian system it is clear what kind of variables are compact and what others do not, and even its geometrical meaning.

But even more, the kinematical formalism developed what clarifies is that the kinematical space is also a Finsler metric space, with a positive definite metric tensor for points causally connected. In this way the variational formalism also contains the Causility Principle. The dynamical evolution of an elementary particle is always a geodesic on its kinematical space.

6.10.2 The Weyl Group

The Weyl group is Poincaré group given by the space-time transformations (2.254) and (2.255) together a space-time scale transformation

$$t' = e^{\lambda} t, \quad \boldsymbol{r}' = e^{\lambda} \boldsymbol{r}. \tag{6.39}$$

In this way, if t and $\boldsymbol{r}$ are kinematical variables of an elementary particle, their derivatives transform under the scale transformation:

$$\dot{t}' = e^{\lambda} \dot{t}, \quad \dot{\boldsymbol{r}}' = e^{\lambda} \dot{\boldsymbol{r}},$$

so that under space-time dilations the velocity transforms as

$$\boldsymbol{u}' = \boldsymbol{u}.$$

It is invariant and therefore the constant velocity c is conserved.

The generators of the Lie algebra of the group in this realization are:

$$H = \partial/\partial t,\ P_i = \partial/\partial r_i,\ K_i = ct\partial/\partial r_i + (r_i/c)\partial/\partial t,\ J_k = \varepsilon_{kli} r_l \partial/\partial r_i,\ Q = t\partial/\partial t + \boldsymbol{r}\cdot\nabla.$$

The generators $\boldsymbol{K}$, $\boldsymbol{J}$ and Q are dimensionless and the commutation rules are:

$$[\boldsymbol{J}, \boldsymbol{J}] = -\boldsymbol{J},\ [\boldsymbol{J}, \boldsymbol{P}] = -\boldsymbol{P},\ [\boldsymbol{J}, \boldsymbol{K}] = -\boldsymbol{K},\ [\boldsymbol{J}, H] = 0, \tag{6.40}$$

$$[H, \boldsymbol{P}] = 0,\ [H, \boldsymbol{K}] = c\boldsymbol{P},\ [\boldsymbol{P}, \boldsymbol{P}] = 0,\ [\boldsymbol{K}, \boldsymbol{K}] = \boldsymbol{J},\ [\boldsymbol{K}, \boldsymbol{P}] = -H/c, \tag{6.41}$$

$$[Q, H] = -H,\ [Q, \boldsymbol{P}] = -\boldsymbol{P},\ [Q, \boldsymbol{K}] = 0,\ [Q, \boldsymbol{J}] = 0. \tag{6.42}$$

If we call $x^0 = ct$, $p^0 = H/c$, $p^i = P_i$, $K_i = J_{0i} = -J_{i0}$ y $J_k = -\frac{1}{2}\epsilon_{klr} J_{lr}$, $x_\mu = \eta_{\mu\nu} x^\nu$, $\mu = 0, 1, 2, 3$ y $\partial_\nu \equiv \partial/\partial x^\nu$, $\partial_\sigma x_\nu = \eta_{\nu\sigma}$, $\partial_\sigma x^\nu = \delta^\nu_\sigma$, then,

$$Q = x^\mu \partial_\mu, \quad p_\mu = \partial_\mu, \quad J_{\mu\nu} = -J_{\nu\mu} = x_\mu \partial_\nu - x_\nu \partial_\mu.$$

In covariant notation the commutation rules become:

$$\begin{aligned}
[Q, p_\nu] &= -p_\nu, \quad [Q, J_{\mu\nu}] = 0, \\
[p_\mu, p_\nu] &= 0, \\
[J_{\mu\nu}, p_\sigma] &= -\eta_{\mu\sigma} p_\nu + \eta_{\nu\sigma} p_\mu, \\
[J_{\mu\nu}, J_{\rho\sigma}] &= -\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\mu\sigma} J_{\nu\rho}.
\end{aligned}$$

The Poincaré group has two Casimir operators funcionally independent. One is interpreted as the mass squared of the particle,

$$C_1 = p^\mu p_\mu = (H/c)^2 - \boldsymbol{P}^2 = m^2 c^2, \tag{6.43}$$

and the other is the squared of the Pauli-Lubanski four-vector, w^μ, defined by

$$w^\mu = \frac{1}{2} \varepsilon^{\mu\nu\sigma\lambda} p_\nu J_{\sigma\lambda} \equiv (\boldsymbol{P}\cdot\boldsymbol{J}, H\boldsymbol{J}/c - \boldsymbol{K}\times\boldsymbol{P}) \equiv (\boldsymbol{P}\cdot\boldsymbol{S}_{CM}, H\boldsymbol{S}_{CM}/c). \tag{6.44}$$

It is expressed, for the free particle, in terms of the generators which are constants of the motion, and by construction it is orthogonal to p_μ, i.e., $w^\mu p_\mu = 0$.

It is related to the spin with respect to the CM, $\boldsymbol{S}_{CM}$, through the expression:

$$\boldsymbol{S}_{CM} = \boldsymbol{J} - \boldsymbol{q}\times\boldsymbol{P}, \quad H\boldsymbol{S}_{CM}/c = H\boldsymbol{J}/c - \boldsymbol{K}\times\boldsymbol{P}. \tag{6.45}$$

We have written $\boldsymbol{K} = H\boldsymbol{q}/c - \boldsymbol{P}t$, such that the temporal component $w^0 = \boldsymbol{P}\cdot\boldsymbol{S} = \boldsymbol{P}\cdot\boldsymbol{J} = \boldsymbol{P}\cdot\boldsymbol{S}_{CM}$, represents the helicity of the particle, and the spatial part is the vector (6.45). In this way the other Casimir operator of the Poincaré group is:

$$-C_2 = w^\mu w_\mu = (\boldsymbol{P}\cdot\boldsymbol{S}_{CM})^2 - H^2\boldsymbol{S}_{CM}^2/c^2 = -m^2c^2S_{CM}^2. \tag{6.46}$$

It depends on the value S_{CM}^2, the squared absolute value of the spin with respect to the center of mass of the particle and of the mass m.

The operators w^μ satisfy the commutation rules:

$$[w^\mu, w^\nu] = \epsilon^{\mu\nu\sigma\rho}w_\sigma p_\rho, \tag{6.47}$$

where the tensor component $\epsilon^{0123} = +1$, and

$$[p^\mu, w^\nu] = 0, \qquad [J_{\mu\nu}, w_\sigma] = -\eta_{\mu\sigma}w_\nu + \eta_{\nu\sigma}w_\mu. \tag{6.48}$$

The Weyl group only has a unique Casimir operator

$$C_W = C_2C_1^{-1} \equiv C_2/C_1 = S_{CM}^2, \tag{6.49}$$

which implies that C_1 has to be invertible, and therefore necessarily the mass observable $m \neq 0$. The unique Casimir operator of the Weyl group is the angular momentum with respect to the center of mass, independently of the mass of the particle. If we admit this group as the kinematical group of our theory, the only intrinsic property of elementary matter is the spin. According to the standard model quarks and leptons are Dirac particles, i.e., spin 1/2 particles of arbitary mass.

The Weyl group has no exponents, and therefore the gauge functions on its kinematical spaces can be taken as zero. The Lagrangians of the elementary particles defined under this kinematical group can be taken strictly invariant.

Since the scale transformation generated by Q does not commute with the generators p_μ, this transformation transforms states of particles of different masses.

6.10.3 Passive and active transformations

Let x_i be the kinematical variables of a mechanical system measured by an inertial observer O. A change to another inertial reference frame, express the kinematical variables x_i' in the new frame O' in terms of the x_j and of the parameters of the kinematical group G

$$x_i' = f_i(x_j, g_\alpha), \quad g : O \to O', \quad g \in G.$$

These variables x_i have been transformed among inertial observers and we say we have performed a **passive** transformation of the kinematical variables. This denomination is to distinguish that transformation of the kinematical variables from an **active** transformation, such that for a fixed inertial observer we produce a physical modification of the mechanical system. For instance, when rotating, moving or deforming the material system, its kinematical variables would change for a fixed inertial observer. If this active transformation corresponds to a transformation of the kinematical group, the modification of the kinematical variables is expressed in the same way as the passive transformation, but instead of using the parameters g_α of the group element g we have to use the parameters of the inverse element g^{-1}. For instance to rotate a material system and express in the same reference frame the kinematical variables after rotation, it is the same that to rotate the Cartesian frame in the opposite direction, while remaining the mechanical system unaltered.

When we state as a fundamental principle the Restricted Relativity Principle, we admit the existence of a kinematical group of space-time transformations among inertial observers. These

tranformations can always be interpreted in two ways, active and passive. In any of these two interpretations the material system is not deformed by the transformations. If we include as a transformation the change to an accelerated frame, from the active point of view we have to apply a force to the material system, and this would produce necessarily a deformation, except if the material system were undeformable, a property which cannot be admited for general material systems. Therefore this kind of transformation in which there is no symmetry between the active and passive interpretation, have to be discarded. The above passive transformation does not deform the material system while the active one does. From the passive interpretation an extended gravitational field appears everywhere, which is not present in the original inertial reference frame. This kind of transformations with a different behaviour between the active and passive interpretation must be kept outside the kinematical group.

We can displace the material system o we can consider a displaced reference frame with the axis translated in the opposite direction. The same happens for rotations or pure inertial transformations with constant velocity. All this kind of transformations have a symmetrical behaviour in the active and passive interpretation. From the passive point of view to admit the invariance of the laws of physics under time translations is to assume that these laws are the same at any time. They are the same in the past and also will be valid in the future.

From the active point of view, a time translation consists in waiting the passing of time to make the physical analysis later. But we cannot make an active time translation to the past. There is no symmetry in the active and passive interpretation of time translations. It is only possible an active time translation to the future. There is the arrow of time. This seems that from the active point of view we have only one way to proceed. The active generator of this group has to have not both signs but only one. The energy or temporal momentum associated to this symmetry is chosen to be positive definite. This selection of a sign does not happen to the other constants of the motion associated to the other transformations of the kinematical group. They can take any sign, like the different components of the linear or angular momentum.

The same thing happens for the discrete symmetry transformations. A passive space reversal can be done by simply reversing the unit vectors of the coordinate axis, but it is not possible to make the active transformation which implies to transform physically the material system by replacing every portion of matter by the equivalent portion spatially inverted. It is possible that given some material system its spatial inversion cannot be physically realizable.

This kind of transformations cannot be taken as part of the symmetry group of the theory. If the physical system seems to be invariant under inversions, this is a property which has to be checked experimentally, but which cannot be admited as a valid symmetry, a priori.

If that symmetry exists can give rise to a conservation law which does not exist in the opposite case. Weak interaction is not invariant under space reversal, and therefore parity is not conserved, while parity is conserved under electromagnetic interactions.

The same can be said about scale transformations. The passive interpetation does not affect the material system. But the active transformation deforms the material system. Since the scale factor can be interpreted as the separation between the center of mass and center of charge $R = \hbar/2mc$, written in terms of two universal constants $\hbar$ and c, and of the observable mass m, this means that to modify the scale is equivalent to modify the mass of the particle and the transformed system is an elementary particle of a different mass.

The interaction Lagrangian we found in section **6.10.3** for two Dirac particles is Poincaré invariant and also invariant under space-time scale transformation. This implies that if it describes the interaction between two electrons, the same interaction is possible between two other particles of the same charge and spin, but of different masses. When we make the passive transformation we are transforming the electron into the muon or tau-particle. The active change of scale represents to substitute the electrons by the other particles. The same can be said for quarks when they are replaced by the corresponding ones of the same charge.

This analysis is suggesting that from the active point of view we have not a complete kinematical group, as the Poincaré or Weyl group, but as far as the time translations are concerned we only have a semigroup: the semigroup of time translations to the future. The generator of these transformations can only have a unique sign.

References

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Didactic Web

http://www.spinningparticles.com

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